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Research Article

On the Continuity Properties of the Attainable Sets of Nonlinear Control Systems with Integral Constraint on Controls

Khalik G. Guseinov and Ali S. Nazlipinar

Department of Mathematics, Anadolu University, Eskisehir 26470, Turkey

Correspondence should be addressed to Khalik G. Guseinov, kguseynov@anadolu.edu.tr

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The attainable sets of the nonlinear control systems with integral constraint on the control functions are considered. It is assumed that the behavior of control system is described by a differential equation which is nonlinear with respect to phase-state vector and control vector. The admissible control functions are chosen from the closed ball centered at the origin with radius μ_0 in $L_p([t_0,\theta];\mathbb{R}^m)$ $(p \in (1,+\infty))$. Precompactness of the solutions set is specified, and dependence of the attainable sets on the initial conditions and on the parameters of the control system is studied.

1. Introduction

Control problems with integral constraints on control arise in various problems of mathematical modeling. For example, the motion of flying apparatus with variable mass is described in the form of controllable system, where the control function has integral constraints (see, e.g., [1–3]). One of the important constructions of the control systems theory is the attainable set notion. Attainable set is the set of all points to which the system can be steered at the instant of given time. Attainable sets of control systems are very useful tools in the study of various problems of optimization, dynamical systems and differential game theory.

In [4–10], topological properties and numerical construction methods of the attainable sets of linear control systems with integral constraint on control functions are investigated. The attainable sets of affine control systems, that is, the attainable sets of control systems which are nonlinear with respect to the phase-state vector, but are linear with respect to the control vector have been considered in [11–14]. The properties of the attainable sets of the nonlinear control systems have been studied in [15–18].

Approximation method for the construction of attainable sets of affine control systems with integral constraints on the control is given in [11, 13]. In [14], using the topological

properties of attainable sets of affine control systems, the continuity properties of minimum time and minimum energy functions are discussed.

The dependence of the attainable set on p is studied in [8, 12, 15]. In [15], it is proved that attainable set of affine control system depends on p continuously. In [15], the same property is shown for nonlinear control systems.

In [17], if the control resource is sufficiently small, then under some suitable assumptions on the right-hand side of the system, it is proved that the attainable set of the nonlinear control system with integral constraints on control is convex.

The value function of nonlinear optimal control problem with generalized integral constraints on control and phase-state vectors is investigated in [16, 18].

In this article, we consider the attainable sets of the control systems the behavior of which is described by nonlinear differential equations. It is assumed that the admissible control functions are chosen from the closed ball centered at the origin with radius μ_0 in $L_p([t_0,\theta];\mathbb{R}^m)$ $(p \in (1,+\infty))$.

In Section 2, it is illustrated that, in general, the attainable set is not closed (Example 2.5) and it is shown that the set of solutions generated by all possible admissible control functions is precompact in the space of continuous functions (Corollary 2.4). In Section 3, the diameter of the attainable set is evaluated (Proposition 3.1) and it is proved that the attainable set is Hölder continuous with respect to time variable (Proposition 3.3). In Section 4, it is shown that the attainable set of the control system is continuous with respect to initial condition (Proposition 4.1). In Section 5, it is proved that the attainable set is Lipschitz continuous with respect to a parameter of the system which define the resource of the control effort (Proposition 5.1).

Consider the control system the behavior of which is described by the differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) \in X_0, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the phase-state vector of the system, $u \in \mathbb{R}^m$ is the control vector, $t \in [t_0,\theta]$ is the time, and $X_0 \subset \mathbb{R}^n$ is a compact set.

For $p \in (1,\infty)$ and $\mu_0 > 0$, we set

$$\mathcal{U}_p^{\mu_0} = \{u(\cdot) \in L_p([t_0,\theta],\mathbb{R}^m) : \|u(\cdot)\|_p \le \mu_0\}, \tag{1.2}$$

where $\|u(\cdot)\|_p = \left(\int_{t_0}^{\theta} \|u(t)\|^p dt\right)^{1/p}$ and $\|\cdot\|$ denotes the Euclidian norm.

A function $u(\cdot) \in \mathcal{U}_p^{\mu_0}$ is said to be an admissible control function. It is obvious that the set of all admissible control functions $\mathcal{U}_p^{\mu_0}$ is the closed ball centered at the origin with the radius μ_0 in $L_p([t_0,\theta];\mathbb{R}^m)$.

It is assumed that the right-hand side of the system (1.1) satisfies the following conditions.

(a) The function $f(\cdot) : [t_0,\theta] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous.

(b) For any bounded set $D \subset [t_0,\theta] \times \mathbb{R}^n$, there exist constants $L_1 = L_1(D) > 0$, $L_2 = L_2(D) > 0$, and $L_3 = L_3(D) > 0$ such that

$$\|f(t,x_1,u_1) - f(t,x_2,u_2)\| \le (L_1 + L_2\|u_2\|)\|x_1 - x_2\| + L_3\|u_1 - u_2\| \tag{1.3}$$

for any $(t,x_1) \in D$, $(t,x_2) \in D$, $u_1 \in \mathbb{R}^m$, and $u_2 \in \mathbb{R}^m$.

(c) There exists a constant $c > 0$ such that

$$\|f(t,x,u)\| \le c(1+\|x\|)\,(1+\|u\|) \tag{1.4}$$

for every $(t,x,u) \in [t_0,\theta] \times \mathbb{R}^n \times \mathbb{R}^m$.

If the right-hand side of the system (1.1) is affine, that is, if $f(t,x,u) = \varphi(t,x) + B(t,x)u$ and the functions $\varphi(\cdot) : [t_0,\theta] \times \mathbb{R}^n \to \mathbb{R}^n$, $B(\cdot) : [t_0,\theta] \times \mathbb{R}^n \to \mathbb{R}^m$ satisfy the assumptions given in [11–14], then, under these assumptions, the conditions (a), (b), and (c) are also fulfilled.

Let $u_*(\cdot) \in U_p^{\mu_0}$. The absolutely continuous function $x_*(\cdot) : [t_0,\theta] \to \mathbb{R}^n$, which satisfies the equation $\dot{x}_*(t) = f(t,x_*(t),u_*(t))$ a.e. in $[t_0,\theta]$, and the initial condition $x_*(t_0) = x_0 \in X_0$ is said to be a solution of the system (1.1) with initial condition $x_*(t_0) = x_0$, generated by the admissible control function $u_*(\cdot)$. By the symbol $x(\cdot;t_0,x_0,u(\cdot))$, we denote the solution of the system (1.1) with initial condition $x(t_0) = x_0$, which is generated by the admissible control function $u(\cdot)$. Note that the conditions (a)–(c) guarantee the existence, uniqueness, and extendability of the solutions up to the instant of time θ for every given $u_*(\cdot) \in U_p^{\mu_0}$ and $x_0 \in X_0$.

Let us define the sets

$$\begin{aligned} X_p(t_0,X_0,\mu_0) &= \{x(\cdot;t_0,x_0,u(\cdot)) : [t_0,\theta] \longrightarrow \mathbb{R}^n \mid x_0 \in X_0,\ u(\cdot) \in U_p^{\mu_0}\}, \\ X_p(t;t_0,X_0,\mu_0) &= \{x(t) \in \mathbb{R}^n : x(\cdot) \in X_p(t_0,X_0,\mu_0)\}, \end{aligned} \tag{1.5}$$

where $t \in [t_0,\theta]$.

The set $X_p(t;t_0,X_0,\mu_0)$ is called the attainable set of the system (1.1) at the instant of time t. It is obvious that the set $X_p(t;t_0,X_0,\mu_0)$ consists of all $x \in \mathbb{R}^n$ to which the system (1.1) can be steered at the instant of time $t \in [t_0,\theta]$.

The Hausdorff distance between the sets $A \subset \mathbb{R}^n$ and $E \subset \mathbb{R}^n$ is denoted by $h(A,E)$ and is defined as

$$h(A,E) = \max\left\{\sup_{x\in A} d(x,E),\ \sup_{y\in E} d(y,A)\right\}, \tag{1.6}$$

where $d(x,E) = \inf\{\|x-y\| : y \in E\}$.

By $C([t_0,\theta];\mathbb{R}^n)$, we denote the space of continuous functions $x(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ with norm

$$\|x(\cdot)\|_C = \max_{t\in[t_0,\theta]} \|x(t)\|. \tag{1.7}$$

Also, $h_C(U,V)$ denotes the Hausdorff distance between the sets $U \subset C([t_0,\theta];\mathbb{R}^n)$ and $V \subset C([t_0,\theta];\mathbb{R}^n)$.

2. Precompactness of the set of solutions

The following proposition asserts that the set of solutions and the attainable sets of the control system (1.1) with constraint (1.2) are bounded.

Proposition 2.1. *Let $p \in (1,\infty)$, $\mu_* \in (0,\mu_0+1)$, $h(X_0,X_*) \le 1$. Then for any $x_*(\cdot) \in X_p(t_0,X_*,\mu_*)$, the inequality*

$$\|x_*(\cdot)\|_C \le r_* \tag{2.1}$$

holds, where

$$r_* = d_1 \exp(k), \tag{2.2}$$

$$\begin{aligned} d_1 &= 1 + d_* + k, \\ d_* &= \max\{\|x\| : x \in X_0\}, \\ k &= c\left[(\theta - t_0) + l_*(\mu_0 + 1)\right], \end{aligned} \tag{2.3}$$

$$l_* = \max\{(\theta - t_0), 1\}, \tag{2.4}$$

and $c > 0$ is the constant given in condition (c).

The proof of the proposition follows from condition (c) and Gronwall's inequality.

For given $\gamma > 0$, we set

$$\begin{aligned} D(\gamma) &= \{(t, x) \in [t_0, \theta] \times \mathbb{R}^n : \|x\| \le \gamma\}, \\ B_n(\gamma) &= \{x \in \mathbb{R}^n : \|x\| \le \gamma\}. \end{aligned} \tag{2.5}$$

We get from Proposition 2.1 that $(t, x(t)) \in D(r_*)$ for every $p \in (1, \infty)$, $x(\cdot) \in X_p(t_0, X_*, \mu_*)$, $t \in [t_0, \theta]$, $\mu_* \in (0, \mu_0 + 1)$, and compact $X_* \subset \mathbb{R}^n$ such that $h(X_0, X_*) \le 1$. So, we have the validity of the following corollary.

Corollary 2.2. *The set $X_p(t_0, X_0, \mu_0)$ is uniformly bounded, and consequently $X_p(t; t_0, X_0, \mu_0) \subset B_n(r_*)$ for every $t \in [t_0, \theta]$, where r_* is defined by* (2.2).

Here and henceforth, we will have in mind the cylinder $D(r_*)$ as the set D in condition (b). We set also

$$k_* = c(1 + r_*)(l_* + \mu_0), \tag{2.6}$$

where r_* is defined by (2.2), l_* is defined by (2.4).

Proposition 2.3. *The set $X_p(t_0, X_0, \mu_0)$ is equicontinuous.*

Proof. Let $\varepsilon > 0$ be an arbitrarily given number. Now, let us choose an arbitrary $x(\cdot) \in X_p(t_0, X_0, \mu_0)$ and $t_1, t_2 \in [t_0, \theta]$. Without loss of generality, we assume that $t_1 \le t_2$. Then from condition (c), we have

$$\|x(t_1) - x(t_2)\| \le \int_{t_1}^{t_2} c(1 + \|x(\tau)\|)\,(1 + \|u(\tau)\|)\,d\tau. \tag{2.7}$$

According to Proposition 2.1, $\|x(\cdot)\|_C \le r_*$, where r_* is defined by (2.2). Then we get from (2.4), (2.6), (2.7), and Hölder's inequality that

$$\begin{aligned} \|x(t_1) - x(t_2)\| &\le c(1 + r_*)\left(|t_2 - t_1| + \mu_0 |t_2 - t_1|^{(p-1)/p}\right) \\ &\le |t_2 - t_1|^{(p-1)/p} c(1 + r_*)\left((\theta - t_0)^{1/p} + \mu_0\right) \\ &\le |t_2 - t_1|^{(p-1)/p} c(1 + r_*)(l_* + \mu_0) = k_* |t_2 - t_1|^{(p-1)/p}. \end{aligned} \tag{2.8}$$

Thus for given $\varepsilon > 0$, setting $\delta(\varepsilon) = (\varepsilon/k_*)^{p/(p-1)}$, we obtain $\|x(t_1) - x(t_2)\| < \varepsilon$ for $|t_1 - t_2| < \delta(\varepsilon)$. Since $x(\cdot) \in X_p(t_0, X_0, \mu_0)$ is arbitrarily chosen, the equicontinuity of the set $X_p(t_0, X_0, \mu_0)$ follows. □

From Corollary 2.2 and Proposition 2.3, we get the validity of the following corollary.

Corollary 2.4. *The set* $X_p(t_0, X_0, \mu_0)$ *is a precompact subset of the space* $C([t_0, \theta], \mathbb{R}^n)$.

Note that if the right-hand side of the system (1.1) is affine with respect to the control vector u, then the weak compactness of the set of admissible control functions $U_p^{\mu_0}$ guaranties the closeness of the attainable sets; but the attainable sets of the control system (1.1) with constraint (1.2), in general, are not closed. In [19, 20], the example is given which illustrates that the attainable set of nonlinear control system with geometric constraint on control is not closed. We use that example to show that the attainable set of nonlinear control system with integral constraint on control is not also closed.

Example 2.5. Let us consider the control system

$$\begin{aligned} \dot{x} &= -y^2 + u^2, \qquad x(0) = 0, \\ \dot{y} &= u, \qquad y(0) = 0, \end{aligned} \tag{2.9}$$

where $(x, y) \in \mathbb{R}^2$ is the phase-state vector of the system, $u \in \mathbb{R}$ is the control vector, $t \in [0, 1]$. It is assumed that $\mu_0 = 1$ and the control function $u(\cdot) \in L_2([0, 1]; \mathbb{R})$ of the system (2.9) satisfies the integral constraint

$$\int_0^1 u^2(t)dt \le 1, \tag{2.10}$$

that is, $u(\cdot) \in U_2^1$. Let us denote

$$\begin{aligned} X_2(0, (0,0), 1) &= \{(x(\cdot; 0, (0,0), u(\cdot)), y(\cdot; 0, (0,0), u(\cdot))) : u(\cdot) \in U_2^1\}, \\ X_2(t; 0, (0,0), 1) &= \{(x(t), y(t)) \in \mathbb{R}^2 : (x(\cdot), y(\cdot)) \in X_2(0, (0,0), 1)\}. \end{aligned} \tag{2.11}$$

Thus $X_2(0, (0,0), 1)$ is the set of solutions, $X_2(t; 0, (0,0), 1)$ is the attainable set of the control system (2.9) at the instant of time $t \in [0, 1]$, generated by control functions $u(\cdot) \in U_2^1$.

Now, let us prove that the solution set $X_2(0, (0,0), 1)$ is bounded. Let $(x(\cdot), y(\cdot)) \in X_2(0, (0,0), 1)$ be an arbitrarily chosen solution of the system (2.9) with integral constraint (2.10). Then there exists $u(\cdot) \in U_2^1$ such that

$$x(t) = \int_0^t -y^2(\tau)d\tau + \int_0^t u^2(\tau)d\tau, \tag{2.12}$$

$$y(t) = \int_0^t u(\tau)d\tau \tag{2.13}$$

for any $t \in [0, 1]$. From (2.10), (2.13), and Hölder's inequality, the inequality

$$|y(t)| \le \int_0^t |u(\tau)|d\tau \le \left(\int_0^t 1^2 d\tau\right)^{1/2} \left(\int_0^t |u(\tau)|^2 d\tau\right)^{1/2} \le \sqrt{t} \le 1 \tag{2.14}$$

holds for all $t \in [0,1]$. Then we get from (2.10), (2.12), and (2.14) that

$$|x(t)| \leq \int_0^t |y(\tau)|^2 d\tau + \int_0^t |u(\tau)|^2 d\tau \leq \int_0^t \tau d\tau + 1 = 1 + \frac{t^2}{2} \leq \frac{3}{2} \tag{2.15}$$

for all $t \in [0,1]$.

However, (2.14) and (2.15) imply that

$$\|(x(\cdot), y(\cdot))\|_C = \max_{t\in[0,1]} \|(x(t), y(t))\| = \max_{t\in[0,1]} \sqrt{x^2(t) + y^2(t)} \leq \sqrt{1 + \frac{9}{4}} < 2. \tag{2.16}$$

Since $(x(\cdot), y(\cdot)) \in X_2(0,(0,0),1)$ is arbitrarily chosen, we get that the set $X_2(0,(0,0),1)$ is bounded.

Now we prove that $(1,0) \notin X_2(1;0,(0,0),1)$. Let us assume the contrary, that is, let $(1,0) \in X_2(1;0,(0,0),1)$. Then there exists $(x_*(\cdot), y_*(\cdot)) \in X_2(0,(0,0),1)$ such that

$$x_*(1) = 1, \qquad y_*(1) = 0. \tag{2.17}$$

Since $(x_*(\cdot), y_*(\cdot)) \in X_2(0,(0,0),1)$, then there exists $u_*(\cdot) \in U_2^1$ such that

$$x_*(t) = \int_0^t -y_*^2(\tau) d\tau + \int_0^t u_*^2(\tau) d\tau, \tag{2.18}$$

$$y_*(t) = \int_0^t u_*(\tau) d\tau \tag{2.19}$$

for all $t \in [0,1]$.

From (2.17), (2.18), and (2.19), it follows that

$$x_*(1) = \int_0^1 u_*^2(\tau) d\tau - \int_0^1 \left(\int_0^\tau u_*(s) ds \right)^2 d\tau = 1. \tag{2.20}$$

Since $u_*(\cdot) \in U_2^1$, then it follows from (2.20) that $u_*(t) = 0$ for almost all $t \in [0,1]$. Then we have from (2.18) and (2.19) that $x_*(t) = 0$, $y_*(t) = 0$ for every $t \in [0,1]$, which contradicts (2.17). Thus

$$(1,0) \notin X_2(1;0,(0,0),1). \tag{2.21}$$

Let us show that $(1,0) \in \mathrm{cl}(X_2(1;0,(0,0)),1)$.

Let $\Gamma^k = \{0, 1/2k, 2/2k, \ldots, (2k-1)/2k, 1\}$ be a uniform partition of the closed interval $[0,1]$, where $k = 1,2,\ldots$. Now we define a sequence of functions $\{u_k(\cdot)\}_{k=1}^\infty$, setting

$$u_k(t) = \begin{cases} 1, & t \in \left[\dfrac{2i}{2k}, \dfrac{2i+1}{2k}\right), \\ -1, & t \in \left[\dfrac{2i+1}{2k}, \dfrac{2i+2}{2k}\right), \end{cases} \tag{2.22}$$

where $i = 0,1,\ldots,k-1$.

It is obvious that $u_k(\cdot) \in U_2^1$ for all $k = 1, 2, \ldots$. Let $(x_k(\cdot), y_k(\cdot)) \in X_2(0, (0,0), 1)$ be the solution of the system (2.9) generated by the admissible control function $u_k(\cdot) \in U_2^1$. Then it follows from (2.9) that

$$x_k(t) = \int_0^t -y_k^2(\tau)d\tau + \int_0^t u_k^2(\tau)d\tau, \tag{2.23}$$

$$y_k(t) = \int_0^t u_k(\tau)d\tau \tag{2.24}$$

for every $t \in [0,1]$.

We get from (2.22) and (2.24) that

$$y_k(t) = \begin{cases} t - \dfrac{2i}{2k}, & t \in \left[\dfrac{2i}{2k}, \dfrac{2i+1}{2k}\right) \\ -t + \dfrac{2i+2}{2k}, & t \in \left[\dfrac{2i+1}{2k}, \dfrac{2i+2}{2k}\right) \end{cases} \tag{2.25}$$

for every $t \in [0,1]$ where $i = 0, 1, \ldots, k-1$.

Then, from (2.25) we have

$$0 \le y_k(t) \le \frac{1}{2k} \tag{2.26}$$

for every $t \in [0,1]$, and consequently

$$0 \le y_k^2(t) \le \frac{1}{4k^2} \tag{2.27}$$

for every $t \in [0,1]$.

According to (2.22), $u_k^2(t) = 1$ for all $t \in [0,1]$. Then, from (2.23) and (2.27) we obtain that

$$1 - \frac{1}{4k^2} \le \dot{x}_k(t) \le 1 \tag{2.28}$$

for almost all $t \in [0,1]$, and consequently

$$\left(1 - \frac{1}{4k^2}\right)t \le x_k(t) \le t \tag{2.29}$$

for every $t \in [0,1]$, where $k = 1, 2, \ldots$.

We conclude from the last inequality that

$$\left(1 - \frac{1}{4k^2}\right) \le x_k(1) \le 1 \tag{2.30}$$

for every $k = 1, 2, \ldots$.

It follows from (2.26) and (2.30) that

$$\big(x_k(1), y_k(1)\big) \longrightarrow (1,0) \quad \text{as } k \longrightarrow \infty. \tag{2.31}$$

Since $(x_k(1), y_k(1)) \in X_2(1; 0, (0,0), 1)$ for every $k = 1, 2, \ldots$, from (2.31) we obtain that

$$(1,0) \in \mathrm{cl}\big(X_2(1; 0, (0,0), 1)\big). \tag{2.32}$$

However, (2.21) and (2.32) imply that $X_2(1; 0, (0,0), 1)$ is not a closed set.

3. Diameter of the attainable set and continuity with respect to t

In this section we will give an upper estimation for the diameter of the attainable set $X_p(t;t_0,X_0,\mu_0)$ and will show that the set-valued map $t \to X_p(t;t_0,X_0,\mu_0)$ is Hölder continuous with respect to t.

We denote the diameter of a set $A \subset \mathbb{R}^n$ by $\operatorname{diam}(A)$ and define it as

$$\operatorname{diam} A = \sup_{x,y\in A} \|x-y\|. \tag{3.1}$$

The following proposition characterizes the diameter of the attainable set $X_p(t;t_0,X_0,\mu_0)$.

Proposition 3.1. *For every $p \in (1,+\infty)$, the inequality*

$$\operatorname{diam} X_p(t;t_0,X_0,\mu_0) \le [d_0 + r_1(t,p)] \exp(r_0(t,p)) \tag{3.2}$$

holds for any $t \in [t_0,\theta]$, where

$$d_0 = \operatorname{diam} X_0, \tag{3.3}$$

$$r_0(t,p) = L_1(t-t_0) + L_2(t-t_0)^{(p-1)/p}, \tag{3.4}$$

$$r_1(t,p) = 2L_3\mu_0(t-t_0)^{(p-1)/p}. \tag{3.5}$$

Proof. Let $t \in [t_0,\theta]$ and $x_1(t) \in X_p(t;t_0,X_0,\mu_0)$, $x_2(t) \in X_p(t;t_0,X_0,\mu_0)$ be arbitrarily chosen. Then there exist $x_1 \in X_0$, $x_1(\cdot) \in X_p(t_0,x_0,\mu_0)$, $u_1(\cdot) \in U_p^{\mu_0}$, $x_2 \in X_0$, $x_2(\cdot) \in X_p(t_0,x_0,\mu_0)$, $u_2(\cdot) \in U_p^{\mu_0}$ such that

$$\begin{aligned} x_1(t) &= x_1 + \int_{t_0}^{t} f(\tau,x_1(\tau),u_1(\tau))d\tau, \\ x_2(t) &= x_2 + \int_{t_0}^{t} f(\tau,x_2(\tau),u_2(\tau))d\tau. \end{aligned} \tag{3.6}$$

Since $\|x_1 - x_2\| \le d_0$, then from (3.6), and the condition (b), we get

$$\begin{aligned} \|x_1(t)-x_2(t)\| \le d_0 &+ \int_{t_0}^{t} (L_1 + L_2\|u_2(\tau)\|)\|x_1(\tau)-x_2(\tau)\|d\tau \\ &+ \int_{t_0}^{t} L_3\|u_1(\tau)-u_2(\tau)\|d\tau. \end{aligned} \tag{3.7}$$

Since $u_1(\cdot),u_2(\cdot) \in U_p^{\mu_0}$, then the Hölder and Minkowski inequalities imply that

$$L_3\int_{t_0}^{t} \|u_1(\tau)-u_2(\tau)\|d\tau \le 2L_3\mu_0(t-t_0)^{(p-1)/p} = r_1(t,p), \tag{3.8}$$

where $r_1(t,p)$ is defined by (3.5). Since $t \in [t_0,\theta]$ is arbitrarily chosen, we obtain from (3.7), (3.8), and Gronwall's inequality that

$$\begin{aligned}\|x_1(t)-x_2(t)\| &\le [d_0+r_1(t,p)]\exp\left(\int_{t_0}^{t}(L_1+L_2\|u_2(\tau)\|)d\tau\right)\\ &\le [d_0+r_1(t,p)]\exp\left(L_1(t-t_0)+L_2(t-t_0)^{(p-1)/p}\mu_0\right)\\ &= [d_0+r_1(t,p)]\exp\left(r_0(t,p)\right),\end{aligned} \tag{3.9}$$

where $r_0(t,p)$ is defined by (3.4). □

Note that an estimation for diameter of the attainable set can be obtained from Proposition 2.1; but the estimation given by Proposition 3.1 is more precise.

Corollary 3.2. $\operatorname{diam} X_p(t;t_0,X_0,\mu_0) \to \operatorname{diam} X_0$ *as* $t \to t_0$.

The following proposition asserts that the attainable set $X_p(t;t_0,X_0,\mu_0)$ is Hölder continuous with respect to t.

Proposition 3.3. *Let* $t_1 \in [t_0,\theta]$, $t_2 \in [t_0,\theta]$. *Then*

$$h\left(X_p(t_1;t_0,X_0,\mu_0),\ X_p(t_2;t_0,X_0,\mu_0)\right) \le k_*|t_1-t_2|^{(p-1)/p}, \tag{3.10}$$

where $k_* > 0$ *is defined by* (2.6).

Proof. Without loss of generality, let us assume that $t_1 < t_2$. Let $y_1 \in X_p(t_1;t_0,X_0,\mu_0)$ be arbitrarily chosen. Then there exist $x_0 \in X_0$, $x_0(\cdot) \in X_p(t_0,X_0,\mu_0)$ and $u_0(\cdot) \in U_p^{\mu_0}$ such that

$$y_1 = x_0(t_1) = x_0 + \int_{t_0}^{t_1} f(\tau,x_0(\tau),u_0(\tau))d\tau. \tag{3.11}$$

Let

$$y_2 = x_0(t_2) = x_0 + \int_{t_0}^{t_2} f(\tau,x_0(\tau),u_0(\tau))d\tau. \tag{3.12}$$

It is obvious that $y_2 \in X_p(t_2;t_0,X_0,\mu_0)$. From Proposition 2.1, relations (2.4), (2.6), (3.11), (3.12), and the condition (c), we have

$$\begin{aligned}\|y_1-y_2\| &\le \int_{t_1}^{t_2} c(1+\|x_0(\tau)\|)\,(1+\|u_0(\tau)\|)d\tau\\ &\le c(1+r_*)\int_{t_1}^{t_2}(1+\|u_0(\tau)\|)d\tau\\ &\le c(1+r_*)\left(|t_2-t_1|+\mu_0|t_2-t_1|^{(p-1)/p}\right)\\ &\le c(1+r_*)\left(l_*+\mu_0\right)|t_2-t_1|^{(p-1)/p}) = k_*|t_2-t_1|^{(p-1)/p},\end{aligned} \tag{3.13}$$

where $c > 0$ is the constant given in condition (c).

Since $y_1 \in X_p(t_1, t_0, X_0, \mu_0)$ is arbitrarily chosen, then (3.13) implies that

$$X_p(t_1; t_0, X_0, \mu_0) \subset X_p(t_2; t_0, X_0, \mu_0) + k_* |t_2 - t_1|^{(p-1)/p} B_n(1). \tag{3.14}$$

Analogously, it is possible to show that

$$X_p(t_2; t_0, X_0, \mu_0) \subset X_p(t_1; t_0, X_0, \mu_0) + k_* |t_2 - t_1|^{(p-1)/p} B_n(1). \tag{3.15}$$

In fact, (3.14) and (3.15) yield the proof. □

From Proposition 3.3, we obtain the following corollary.

Corollary 3.4. *The set-valued map $t \to X_p(t; t_0, X_0, \mu_0)$, $t \in [t_0, \theta]$, is $(p-1)/p$-Hölder continuous.*

4. Dependence of the attainable sets on parameters t_0 and X_0

The following proposition characterizes the continuity of the set-valued map $(t_0, X_0) \to X_p(t; t_0, X_0, \mu_0)$ in the Hausdorff metric.

Let us denote

$$\omega_* = (L_1 + L_2 \mu_0) l_*, \tag{4.1}$$

where l_* is defined by (2.4), L_1 and L_2 are the constants given in condition (b).

Proposition 4.1. *Let $t_1 \geq t_0$ and $X_0, X_1 \subset \mathbb{R}^n$ be compact sets. Then the inequality*

$$h(X_p(t; t_0, X_0, \mu_0), X_p(t; t_1, X_1, \mu_0)) \leq [h(X_0, X_1) + (t_1 - t_0)^{(p-1)/p} k_*] \exp(\omega_*) \tag{4.2}$$

holds for all $t \in [t_1, \theta]$, where k_* is defined by (2.6), *ω_* is defined by* (4.1).

Proof. Let us choose arbitrary $t \in [t_1, \theta]$ and $x_0(t) \in X_p(t; t_0, X_0, \mu_0)$, where $x_0(\cdot) \in X_p(t_0, X_0, \mu_0)$. Then there exist $x_0 \in X_0$ and $u_0(\cdot) \in U_p^{\mu_0}$ such that

$$x_0(t) = x_0 + \int_{t_0}^{t} f(\tau, x_0(\tau), u_0(\tau)) d\tau \tag{4.3}$$

holds. According to the definition of Hausdorff distance, there exists $x_1 \in X_1$ such that $\|x_1 - x_0\| \leq h(X_0, X_1)$. Let $x_1(\cdot)$ be a solution of the control system (1.1), generated by the admissible control function $u_0(\cdot)$ with initial condition $x_1(t_1) = x_1 \in X_1$. Then

$$x_1(t) = x_1 + \int_{t_1}^{t} f(\tau, x_1(\tau), u_0(\tau)) d\tau \tag{4.4}$$

and $x_1(t) \in X_p(t; t_1, X_1, \mu_0)$.

From (4.3), (4.4), and conditions (b) and (c), we have

$$\begin{aligned} \|x_0(t) - x_1(t)\| \leq h(X_0, X_1) &+ c \int_{t_0}^{t_1} (1 + \|u_0(\tau)\|)(1 + \|x_0(\tau)\|) d\tau \\ &+ \int_{t_1}^{t} (L_1 + L_2 \|u_0(\tau)\|)(\|x_0(\tau) - x_1(\tau)\|) d\tau. \end{aligned} \tag{4.5}$$

Proposition 2.1 implies that

$$c\int_{t_0}^{t_1}\left(1+\|u_0(\tau)\|\right)\left(1+\|x_0(\tau)\|\right)d\tau \le (t_1-t_0)^{(p-1)/p}k_*, \tag{4.6}$$

where k_* is defined by (2.6). Since $t \in [t_1,\theta]$ is arbitrarily chosen, from (4.5), (4.6), and Gronwall's inequality, we get

$$\begin{aligned}\|x_0(t)-x_1(t)\| &\le \left[h(X_0,X_1)+(t_1-t_0)^{(p-1)/p}k_*\right]\exp\left(\int_{t_1}^{t}\left(L_1+L_2\|u_0(\tau)\|\right)d\tau\right)\\ &\le \left[h(X_0,X_1)+(t_1-t_0)^{(p-1)/p}k_*\right]\exp(\omega_*).\end{aligned} \tag{4.7}$$

Hence, we obtain from (4.7) that

$$X_p(t;t_0,X_0,\mu_0) \subset X_p(t;t_1,X_1,\mu_0)+\left[h(X_0,X_1)+(t_1-t_0)^{(p-1)/p}k_*\right]\exp(\omega_*)B_n(1). \tag{4.8}$$

Similarly, one can prove that

$$X_p(t;t_1,X_1,\mu_0) \subset X_p(t;t_0,X_0,\mu_0)+\left[h(X_0,X_1)+(t_1-t_0)^{(p-1)/p}k_*\right]\exp(\omega_*)B_n(1). \tag{4.9}$$

Finally, (4.8) and (4.9) complete the proof. □

From Proposition 4.1, the validity of the following corollaries follow.

Corollary 4.2. *The inequality*

$$h\left(X_p(t;t_0,X_0,\mu_0),\, X_p(t;t_0,X_1,\mu_0)\right) \le h(X_0,X_1)\exp(\omega_*) \tag{4.10}$$

holds for all $t \in [t_0,\theta]$, where $\omega_* > 0$ is defined by (4.1).

Corollary 4.3. *The inequality*

$$h_C\left(X_p(t_0,X_0,\mu_0),\, X_p(t_0,X_1,\mu_0)\right) \le h(X_0,X_1)\exp(\omega_*) \tag{4.11}$$

holds.

Corollary 4.4. *Let $X_0 \subset \mathbb{R}^n$ and $X_n \subset \mathbb{R}^n$ be compact sets for all $n = 1,2,\ldots$. Assume that $h(X_n,X_0) \to 0$ and $t_n \to t_0+0$ as $n \to \infty$. Then for all $t \in (t_0,\theta]$,*

$$h\left(X_p(t;t_n,X_n,\mu_0),\, X_p(t;t_0,X_0,\mu_0)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.12}$$

5. Dependence of the attainable sets on μ_0

In this section we specify dependence of the set $X_p(t_0,X_0,\mu_0)$ on the constraint parameter μ_0. Let

$$r_1 = L_3 l_* \exp(\omega_*), \tag{5.1}$$

where ω_* is defined by (4.1).

The following proposition characterizes the relation between the solutions sets $X_p(t_0, X_0,\mu_*)$ and $X_p(t_0,X_0,\mu_0)$.

Proposition 5.1. *The inequality*

$$h_C(X_p(t_0, X_0, \mu_*), X_p(t_0, X_0, \mu_0)) \le r_1|\mu_* - \mu_0| \tag{5.2}$$

is satisfied, where r_1 is defined by (5.1).

Proof. Let $x_0(\cdot) \in X_p(t_0, X_0, \mu_0)$ be an arbitrarily chosen solution. Then there exist $x_0 \in X_0$ and $u_0(\cdot) \in U_p^{\mu_0}$ such that

$$x_0(t) = x_0 + \int_{t_0}^{t} f(\tau, x_0(\tau), u_0(\tau))d\tau \tag{5.3}$$

for every $t \in [t_0, \theta]$.

We define a new control function $u_*(\cdot) : [t_0, \theta] \to \mathbb{R}^m$, setting

$$u_*(t) = \frac{\mu_*}{\mu_0} u_0(t), \quad t \in [t_0, \theta]. \tag{5.4}$$

It is not difficult to verify that $u_*(\cdot) \in U_p^{\mu_*}$. Let $x_*(\cdot)$ be a solution of the control system (1.1), generated by $u_*(\cdot) \in U_p^{\mu_*}$ from the initial point (t_0, x_0). Then $x_*(\cdot) \in X_p(t_0, X_0, \mu_*)$ and

$$x_*(t) = x_0 + \int_{t_0}^{t} f(\tau, x_*(\tau), u_*(\tau))d\tau \tag{5.5}$$

for every $t \in [t_0, \theta]$. From (5.3), (5.4), (5.5), and condition (b), we get

$$\begin{aligned} \|x_*(t) - x_0(t)\| &\le L_3\left|\frac{\mu_*}{\mu_0} - 1\right| \int_{t_0}^{t} \|u_0(\tau)\|d\tau + \int_{t_0}^{t} (L_1 + L_2\|u_0(\tau)\|)\|x_*(\tau) - x_0(\tau)\|d\tau \\ &\le L_3 l_*|\mu_* - \mu_0| + \int_{t_0}^{t} (L_1 + L_2\|u_0(\tau)\|)\|x_*(\tau) - x_0(\tau)\|d\tau \end{aligned} \tag{5.6}$$

for every $t \in [t_0, \theta]$, where l_* is defined by (2.4).

The Gronwall inequality, (5.1), and (5.6) yield that

$$\begin{aligned} \|x_*(t) - x_0(t)\| &\le L_3 l_*|\mu_* - \mu_0| \exp\left(L_1(\theta - t_0) + L_2\mu_0(\theta - t_0)^{(p-1)/p}\right) \\ &\le L_3 l_* \exp(\omega_*)|\mu_* - \mu_0| = r_1|\mu_* - \mu_0| \end{aligned} \tag{5.7}$$

for all $t \in [t_0, \theta]$.

Thus from (5.7) we get that for any fixed $x_0(\cdot) \in X_p(t_0, X_0, \mu_0)$ there exists $x_1(\cdot) \in X_p(t_0, X_0, \mu_*)$ such that

$$\|x_0(\cdot) - x_*(\cdot)\|_C \le r_1|\mu_* - \mu_0|, \tag{5.8}$$

and consequently

$$X_p(t_0, X_0, \mu_0) \subset X_p(t_0, X_0, \mu_*) + r_1|\mu_* - \mu_0|B_C, \tag{5.9}$$

where B_C is the closed unit ball centered at the origin in the space $C([t_0, \theta], \mathbb{R}^n)$.

Analogously, it is possible to prove that

$$X_p(t_0, X_0, \mu_*) \subset X_p(t_0, X_0, \mu_0) + r_1|\mu_* - \mu_0|B_C. \tag{5.10}$$

Hence, from (5.9) and (5.10), we obtain the proof of the proposition. □

From Proposition 5.1, it follows that the following corollaries are satisfied.

Corollary 5.2. *The inequality*

$$h\left(X_p(t;t_0,X_0,\mu_*),\ X_p(t;t_0,X_0,\mu_0)\right) \le r_1\left|\mu_* - \mu_0\right| \tag{5.11}$$

is satisfied for any $t \in [t_0,\theta]$, *where* $r_1 > 0$ *is defined by* (5.1).

Corollary 5.3. *Let* $\mu_n \to \mu_0$ *as* $n \to \infty$. *Then*

$$\begin{gathered} h_C\left(X_p(t_0,X_0,\mu_n),\ X_p(t_0,X_0,\mu_0)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ h\left(X_p(t;t_0,X_0,\mu_n),\ X_p(t;t_0,X_0,\mu_0)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{gathered} \tag{5.12}$$

for every $t \in [t_0,\theta]$.

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