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Research Article

Weighted Composition Operators from $F(p,q,s)$ Spaces to H_μ^∞ Spaces

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Let $H(B)$ denote the space of all holomorphic functions on the unit ball B. Let $u \in H(B)$ and φ be a holomorphic self-map of B. In this paper, we investigate the boundedness and compactness of the weighted composition operator uC_φ from the general function space $F(p,q,s)$ to the weighted-type space H_μ^∞ in the unit ball.

1. Introduction

Let B be the unit ball of $\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ and let $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$, we write

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \qquad \langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n. \tag{1.1}$$

Thus $B = \{z \in \mathbb{C}^n : |z| < 1\}$. Let dv be the normalized Lebesgue measure of B, that is, $v(B) = 1$. Let $H(B)$ be the space of all holomorphic functions on B. For $f \in H(B)$, let

$$\mathfrak{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) \tag{1.2}$$

represent the radial derivative of $f \in H(B)$. For $a, z \in B$, $a \neq 0$, let φ_a denote the Möbius transformation of B taking 0 to a, which is defined by

$$\varphi_a(z) = \frac{a - P_a(z) - \sqrt{1-|z|^2}Q_a(z)}{1 - \langle z, a\rangle}, \tag{1.3}$$

where $P_a(z)$ is the orthogonal projection of z onto the one dimensional subspace of $\mathbb{C}^n$ spanned by a, and $Q_a(z) = z - P_a(z)$.

A positive continuous function μ on $[0,1)$ is called normal, if there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that

$$\begin{aligned} &\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \qquad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^s} = 0; \\ &\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \qquad \lim_{r\to 1}\frac{\mu(r)}{(1-r)^t} = \infty \end{aligned} \tag{1.4}$$

(see [1]).

Let μ be a normal function on $[0,1)$. An $f \in H(B)$ is said to belong to the weighted-type space $H_\mu^\infty = H_\mu^\infty(B)$, if

$$\|f\|_{H_\mu^\infty} = \sup_{z\in B} \mu(|z|)|f(z)| < \infty, \tag{1.5}$$

where μ is normal on $[0,1)$ (see, e.g., [2–4]). H_μ^∞ is a Banach space with the norm $\|\cdot\|_{H_\mu^\infty}$. We denote by $H_{\mu,0}^\infty$ the subspace of H_μ^∞ consisting of those $f \in H_\mu^\infty$ such that

$$\lim_{|z|\to 1} \mu(|z|)|f(z)| = 0. \tag{1.6}$$

When $\mu(r) = (1-r^2)^\alpha$, the induced spaces H_μ^∞ and $H_{\mu,0}^\infty$ become the (classical) weighted space H_α^∞ and $H_{\alpha,0}^\infty$, respectively.

For $\alpha > 0$, recall that the α-Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$ is the space of all $f \in H(B)$ for which (see [5])

$$b_\alpha(f) = \sup_{z\in B}\left(1-|z|^2\right)^\alpha |\Re f(z)| < \infty. \tag{1.7}$$

Under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, we get the classical Bloch space $\mathcal{B}$. For more information of the Bloch space and the α-Bloch space (see, e.g., [5–8] and the references therein).

For $p \in (0,\infty)$, the weighted Bergman space $A^p(B)$ is the space of all holomorphic functions f on B for which

$$\|f\|_{A^p}^p = \int_B |f(z)|^p dv(z) < \infty. \tag{1.8}$$

The Hardy space $H^p(B)$ $(0 < p < \infty)$ on the unit ball is defined by

$$H^p(B) = \left\{ f \mid f \in H(B),\ \|f\|_{H^p(B)} = \sup_{0\le r<1} M_p(f,r) < \infty \right\}, \tag{1.9}$$

where

$$M_p(f,r) = \left(\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p} \tag{1.10}$$

and $d\sigma$ is the normalized surface measure on ∂B.

For $0 < p < \infty$, the Q_p space is defined by (see [9])

$$Q_p(B) = \left\{ f \in H(B) \,:\, \|f\|_{Q_p}^2 = \sup_{a\in B} \int_B \left|\tilde{\nabla} f(z)\right|^2 G^p(z,a) \frac{dv(z)}{\left(1-|z|^2\right)^{n+1}} < \infty \right\}. \tag{1.11}$$

Here $\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$ denotes the invariant gradient of f, and $G(z,a)$ is the invariant Green function defined by $G(z,a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 \left(1-t^2\right)^{n-1} t^{-2n+1} dt. \tag{1.12}$$

Let $0 < p,\ s < \infty$, $-n-1 < q < \infty$. A function $f \in H(B)$ is said to belong to $F(p,q,s) = F(p,q,s)(B)$ (see [10–12]) if

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in B} \int_B |\nabla f(z)|^p \left(1-|z|^2\right)^q G^s(z,a) dv(z) < \infty. \tag{1.13}$$

$F(p,q,s)$ is called the general function space since we can get many function spaces, such as Hardy space, Bergman space, Bloch space, Q_p space, if we take special parameters of p,q,s. For example, $F(2,1,0) = H^2$, $F(p,p,0) = A^p$, and $F(2,0,s) = Q_s$. If $q+s \le -1$, then $F(p,q,s)$ is the space of constant functions. For the setting of the unit disk, see [13].

Let $u \in H(B)$ and φ be a holomorphic self-map of B. For $f \in H(B)$, the weighted composition operator uC_φ is defined by

$$\left(uC_\varphi f\right)(z) = u(z) f(\varphi(z)), \quad z \in B. \tag{1.14}$$

The weighted composition operator is the generalization of a multiplication operator and a composition operator, which is defined by $(C_\varphi f)(z) = f(\varphi(z))$. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. The book [14] is a good reference for the theory of composition operators. Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure.

In the setting of the unit ball, we studied the boundedness and compactness of the weighted composition operator between Bergman-type spaces and H^∞ in [15]. More general results can be found in [16]. Some necessary and sufficient conditions for the weighted composition operator to be bounded or compact between the Bloch space and H^∞ are given in [17]. In the setting of the unit polydisk D^n, some necessary and sufficient conditions for a weighted composition operator to be bounded or compact between the Bloch space and

$H^\infty(D^n)$ are given in [18, 19] (see, also [20] for the case of composition operators). Some related results can be found, for example, in [2, 3, 6, 21–31].

In the present paper, we are mainly concerned about the boundedness and compactness of the weighted composition operator from $F(p,q,s)$ to the space H_μ^∞. Some necessary and sufficient conditions for the weighted composition operator uC_φ to be bounded and compact are given.

Constants are denoted by C in this paper, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Main Results and Proofs

In order to prove our results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 2.1 (see [12]). *For $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, if $f \in F(p,q,s)$, then $f \in \mathcal{B}^{(n+1+q)/p}$ and*

$$\|f\|_{\mathcal{B}^{(n+1+q)/p}} \le C\|f\|_{F(p,q,s)}. \tag{2.1}$$

The following lemma can be found, for example, in [32].

Lemma 2.2. *If $f \in \mathcal{B}^\alpha$, then*

$$|f(z)| \le C \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1, \\ \|f\|_{\mathcal{B}^\alpha} \ln \dfrac{e}{1-|z|^2}, & \alpha = 1, \\ \dfrac{\|f\|_{\mathcal{B}^\alpha}}{\left(1-|z|^2\right)^{\alpha-1}}, & \alpha > 1, \end{cases} \tag{2.2}$$

for some C independent of f.

Lemma 2.3. *Assume that μ is normal. A closed set K in $H_{\mu,0}^\infty$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(|z|)|f(z)| = 0. \tag{2.3}$$

The proof of Lemma 2.3 is similar to the proof of Lemma 1 of [33]. We omit the details.

Lemma 2.4. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$. Then $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact if and only if $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $F(p,q,s)$ which converges to zero uniformly on compact subsets of B as $k \to \infty$, one has $\|uC_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$.*

Lemma 2.4 follows by standard arguments similar to those outlined in [14, Proposition 3.11] (see, also, the corresponding lemmas in [20, 34, 35]). We omit the details.

Note that when $p > n+1+q$, the function in $F(p,q,s)$ is indeed Lipschitz continuous by Lemmas 2.1 and 2.2. By Arzela-Ascoli theorem, similarly to the proof of Lemma 3.6 of [26], we have the following result.

Lemma 2.5. *Let $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, and $p > n+1+q$. Let (f_k) be a bounded sequence in $F(p,q,s)$ which converges to 0 uniformly on compact subsets of B, then*

$$\lim_{k\to\infty} \sup_{z\in B} \left| f_k(z) \right| = 0. \tag{2.4}$$

2.1. Case $p < n+1+q$

In this subsection, we consider the case $p < n+1+q$. Our first result is the following theorem.

Theorem 2.6. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p < n+1+q$. Then $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded if and only if*

$$\sup_{z\in B} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} < \infty. \tag{2.5}$$

Moreover, when $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded, the following relationship

$$\left\| uC_\varphi \right\|_{F(p,q,s)\to H_\mu^\infty} \asymp \sup_{z\in B} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} \tag{2.6}$$

holds.

Proof. Assume that (2.5) holds. For any $f \in F(p,q,s)$, by Lemmas 2.1 and 2.2,

$$\begin{aligned}
\left\| uC_\varphi f \right\|_{H_\mu^\infty} &= \sup_{z\in B} \mu(|z|) \left| \left(uC_\varphi f\right)(z) \right| \\
&= \sup_{z\in B} \mu(|z|) |f(\varphi(z))||u(z)| \\
&\le C\|f\|_{\mathcal{B}^{(n+1+q)/p}} \sup_{z\in B} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} \\
&\le C\|f\|_{F(p,q,s)} \sup_{z\in B} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}}.
\end{aligned} \tag{2.7}$$

Therefore, (2.5) implies that $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded.

Conversely, suppose that $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded. For $b \in B$, let

$$f_b(z) = \frac{1-|b|^2}{(1-\langle z,b\rangle)^{(n+1+q)/p}}. \tag{2.8}$$

It is easy to see that

$$f_{\varphi(w)}(\varphi(w)) = \frac{1}{\left(1-|\varphi(w)|^2\right)^{(n+1+q)/p-1}}, \qquad \left|\Re f_{\varphi(w)}(\varphi(w))\right| = \frac{(n+1+q)|\varphi(w)|^2}{p\left(1-|\varphi(w)|^2\right)^{(n+1+q)/p}}. \tag{2.9}$$

If $\varphi(w) = 0$, then $f_{\varphi(w)} \equiv 1$ obviously belongs to $F(p,q,s)$. From [12], we know that $f_b \in F(p,q,s)$; moreover, there is a positive constant K such that $\sup_{b\in B}\|f_b\|_{F(p,q,s)} \le K$. Therefore,

$$\begin{aligned}\sup_{z\in B}\mu(|z|)\left|f_{\varphi(w)}(\varphi(z))u(z)\right| &= \sup_{z\in B}\mu(|z|)\left|\left(uC_\varphi f_{\varphi(w)}\right)(z)\right| \\ &= \left\|uC_\varphi f_{\varphi(w)}\right\|_{H_\mu^\infty} \le K\left\|uC_\varphi\right\|_{F(p,q,s)\to H_\mu^\infty},\end{aligned} \tag{2.10}$$

for every $w \in B$, from which we get

$$\sup_{w\in B}\frac{\mu(|w|)|u(w)|}{\left(1-|\varphi(w)|^2\right)^{(n+1+q)/p-1}} \le K\left\|uC_\varphi\right\|_{F(p,q,s)\to H_\mu^\infty} < \infty, \tag{2.11}$$

that is, (2.5) follows. From (2.7) and (2.11), we see that (2.6) holds. The proof of this theorem is finished. □

Theorem 2.7. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p < n+1+q$. Then $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact if and only if $u \in H_\mu^\infty$ and*

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} = 0. \tag{2.12}$$

Proof. First assume that $u \in H_\mu^\infty$ and the condition in (2.12) hold. In order to prove that uC_φ is compact, according to Lemma 2.4 it suffices to show that if $(f_k)_{k\in\mathbb{N}}$ is bounded in $F(p,q,s)$ and converges to 0 uniformly on compact subsets of B as $k \to \infty$, then $\|uC_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$.

Now assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $F(p,q,s)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{F(p,q,s)} \le L$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. From (2.12), we have that for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$ such that

$$\frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} < \varepsilon, \tag{2.13}$$

when $\delta < |\varphi(z)| < 1$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\|uC_\varphi f_k\|_{H_\mu^\infty} &= \sup_{z\in B} \mu(|z|)\left|(uC_\varphi f_k)(z)\right| \\
&= \sup_{z\in B} \mu(|z|)|u(z)|\left|f_k(\varphi(z))\right| \\
&\le \sup_{\varphi(z)\in\overline{B(0,\delta)}} \mu(|z|)|u(z)|\left|f_k(\varphi(z))\right| \\
&\quad + C \sup_{\varphi(z)\in B\setminus\overline{B(0,\delta)}} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} \|f_k\|_{F(p,q,s)} \\
&\le M_1 \sup_{\varphi(z)\in\overline{B(0,\delta)}} \left|f_k(\varphi(z))\right| + CL\varepsilon,
\end{aligned} \tag{2.14}$$

where $M_1 := \sup_{z\in B} \mu(|z|)|u(z)| < \infty$. Using the fact that $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$, we obtain

$$M_1 \limsup_{k\to\infty} \sup_{\varphi(z)\in\overline{B(0,\delta)}} \left|f_k(\varphi(z))\right| = 0. \tag{2.15}$$

Therefore,

$$\limsup_{k\to\infty} \|uC_\varphi f_k\|_{H_\mu^\infty} \le CL\varepsilon. \tag{2.16}$$

Since ε is an arbitrary positive number, we have that $\lim_{k\to\infty}\|uC_\varphi f_k\|_{H_\mu^\infty} = 0$, and, therefore, $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact by Lemma 2.4.

Conversely, suppose $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist that condition (2.12) is vacuously satisfied). Set

$$f_k(z) = \frac{1-\left|\varphi(z_k)\right|^2}{\left(1-\langle z,\varphi(z_k)\rangle\right)^{(q+n+1)/p}}, \quad k\in\mathbb{N}. \tag{2.17}$$

From the proof of Theorem 2.6, we see that $f_k \in F(p,q,s)$ for every $k \in \mathbb{N}$; moreover, $\sup_{k\in\mathbb{N}}\|f_k\|_{F(p,q,s)} \le C$. Beside this, f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since uC_φ is compact, by Lemma 2.4 we have that $\|uC_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}
\left(1-\left|\varphi(z_k)\right|^2\right)^{1-(q+n+1)/p} \mu\left(\left|z_k\right|\right)\left|u(z_k)\right| &= \mu\left(\left|z_k\right|\right)\left|u(z_k)\right|\left|f_k(\varphi(z_k))\right| \\
&\le \sup_{z\in B} \mu(|z|)\left|f_k(\varphi(z))\right||u(z)| \\
&= \sup_{z\in B} \mu(|z|)\left|(uC_\varphi f_k)(z)\right| \\
&= \|uC_\varphi f_k\|_{H_\mu^\infty} \longrightarrow 0,
\end{aligned} \tag{2.18}$$

as $k \to \infty$, from which we obtain (2.12), finishing the proof of the theorem. □

Theorem 2.8. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p < n+1+q$. Then $uC_\varphi : F(p,q,s) \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} = 0. \tag{2.19}$$

Proof. Suppose that (2.19) holds. From Lemma 2.3, we see that $uC_\varphi : F(p,q,s) \to H^\infty_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{F(p,q,s)}\le 1} \mu(|z|)\left|\left(uC_\varphi f\right)(z)\right| = 0. \tag{2.20}$$

On the other hand, by Lemmas 2.1 and 2.2, we have that

$$\mu(|z|)\left|\left(uC_\varphi f\right)(z)\right| \le C\|f\|_{F(p,q,s)} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}}. \tag{2.21}$$

Taking the supremum in (2.21) over the the unit ball in the space $F(p,q,s)$, then letting $|z| \to 1$ and applying (2.19) the result follows.

Conversely, suppose that $uC_\varphi : F(p,q,s) \to H^\infty_{\mu,0}$ is compact. Then $uC_\varphi : F(p,q,s) \to H^\infty_{\mu}$ is compact and hence by Theorem 2.7,

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} = 0. \tag{2.22}$$

In addition, $uC_\varphi : F(p,q,s) \to H^\infty_{\mu,0}$ is bounded. Taking $f(z) = 1$, then employing the boundedness of $uC_\varphi : F(p,q,s) \to H^\infty_{\mu,0}$, we get

$$\lim_{|z|\to 1} \mu(|z|)|u(z)| = 0. \tag{2.23}$$

By (2.22), for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$,

$$\frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} < \varepsilon, \tag{2.24}$$

when $\delta < |\varphi(z)| < 1$. By (2.23), for above chosen ε, there exists $r \in (0,1)$,

$$\mu(|z|)|u(z)| \le \varepsilon\left(1-\delta^2\right)^{(n+1+q)/p-1}, \tag{2.25}$$

when $r < |z| < 1$.

Therefore, when $r < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have that

$$\frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} < \varepsilon. \tag{2.26}$$

If $|\varphi(z)| \le \delta$ and $r < |z| < 1$, we obtain

$$\frac{\mu(|z|)|u(z)|}{\left(1-|\varphi(z)|^2\right)^{(n+1+q)/p-1}} \le \frac{1}{\left(1-\delta^2\right)^{(n+1+q)/p-1}}\mu(|z|)|u(z)| < \varepsilon. \tag{2.27}$$

Combing (2.26) with (2.27), we get (2.19), as desired. □

2.2. *Case* $p = n + 1 + q$

Theorem 2.9. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q + s > -1$, $p = n + 1 + q$. Then $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded if and only if*

$$\sup_{z\in B} \mu(|z|)|u(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty. \tag{2.28}$$

Moreover, the following relationship

$$\left\|uC_\varphi\right\|_{F(p,q,s)\to H_\mu^\infty} \asymp \sup_{z\in B} \mu(|z|)|u(z)| \ln \frac{e}{1-|\varphi(z)|^2} \tag{2.29}$$

holds.

Proof. Suppose that (2.28) holds. For any $f \in F(p,q,s) \subseteq \mathcal{B}$, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\left\|uC_\varphi f\right\|_{H_\mu^\infty} &= \sup_{z\in B} \mu(|z|)\left|\left(uC_\varphi f\right)(z)\right| \\ &= \sup_{z\in B} \mu(|z|)\left|f(\varphi(z))\right||u(z)| \\ &\le C\|f\|_{F(p,q,s)} \sup_{z\in B} \mu(|z|)|u(z)| \ln \frac{e}{1-|\varphi(z)|^2}.\end{aligned} \tag{2.30}$$

Therefore, (2.28) implies that uC_φ is a bounded operator from $F(p,q,s)$ to H_μ^∞.

Conversely, suppose that $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded. For $b \in B$, let

$$f_b(z) = \ln \frac{e}{1-\langle z,b\rangle}, \quad z \in B. \tag{2.31}$$

Then by [12] we see that $f_b \in F(p,q,s)$; moreover, there is a positive constant K such that $\sup_{b\in B}\|f_b\|_{F(p,q,s)} \le K$. Hence

$$\mu(|w|)|u(w)|\ln\frac{e}{1-|\varphi(w)|^2} = \mu(|w|)\left|f_{\varphi(w)}(\varphi(w))\right||u(w)| \le \left\|uC_\varphi f_{\varphi(w)}\right\|_{H_\mu^\infty}, \tag{2.32}$$

for every $w \in B$, that is, we get

$$\begin{aligned}\sup_{w\in B}\mu(|w|)|u(w)|\ln\frac{e}{1-|\varphi(w)|^2} &\le C\left\|uC_\varphi f_{\varphi(w)}\right\|_{H_\mu^\infty}\\ &\le CK\left\|uC_\varphi\right\|_{F(p,q,s)\to H_\mu^\infty} < \infty.\end{aligned} \tag{2.33}$$

From (2.30) and (2.33), (2.29) follows. The proof is finished. □

Theorem 2.10. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p = n+1+q$. Then $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact if and only if $u \in H_\mu^\infty$ and*

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)|u(z)|\ln\frac{e}{1-|\varphi(z)|^2} = 0. \tag{2.34}$$

Proof. Assume that $u \in H_\mu^\infty$ and (2.34) hold, and that $(f_k)_{k\in\mathbb{N}}$ is bounded in $F(p,q,s)$ and converges to 0 uniformly on compact subsets of B as $k \to \infty$. We have that, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\mu(|z|)|u(z)|\ln\frac{e}{1-|\varphi(z)|^2} < \varepsilon, \tag{2.35}$$

when $\delta < |\varphi(z)| < 1$.

In addition

$$\begin{aligned}\left\|uC_\varphi f_k\right\|_{H_\mu^\infty} &= \sup_{z\in B}\mu(|z|)\left|\left(uC_\varphi f_k\right)(z)\right|\\ &= \sup_{z\in B}\mu(|z|)|u(z)|\left|f_k(\varphi(z))\right|\\ &\le \sup_{\varphi(z)\in\overline{B(0,\delta)}}\mu(|z|)\left|u(z)f_k(\varphi(z))\right|\\ &\quad + C\left\|f_k\right\|_{F(p,q,s)}\sup_{\varphi(z)\in B\setminus\overline{B(0,\delta)}}\mu(|z|)|u(z)|\ln\frac{e}{1-|\varphi(z)|^2}\\ &\le C\sup_{\varphi(z)\in\overline{B(0,\delta)}}\left|f_k(\varphi(z))\right| + C\varepsilon.\end{aligned} \tag{2.36}$$

Similar to the proof of Theorem 2.7, we obtain $\|uC_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$. Therefore, $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact by Lemma 2.4.

Conversely, suppose that $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in B such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist that condition (2.34) is vacuously satisfied). Set

$$f_k(z) = \left(\ln \frac{e}{1-|\varphi(z_k)|^2}\right)^{-1} \left(\ln \frac{e}{1-\langle z,\varphi(z_k)\rangle}\right)^2, \quad k \in \mathbb{N}. \tag{2.37}$$

After some calculations or from [12], we see that $\sup_{k\in\mathbb{N}}\|f_k\|_{F(p,q,s)} \le C$ for some positive C independent of k, and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since uC_φ is compact, by Lemma 2.4, we have $\|uC_\varphi f_k\|_{H_\mu^\infty} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}\mu(|z_k|)|u(z_k)| \ln \frac{e}{1-|\varphi(z_k)|^2} &\le \sup_{z\in B} \mu(|z|)|f_k(\varphi(z_k))||u(z_k)| \\ &= \sup_{z\in B} \mu(|z|)|(uC_\varphi f_k)(z)| \\ &= \|uC_\varphi f_k\|_{H_\mu^\infty} \longrightarrow 0,\end{aligned} \tag{2.38}$$

as $k \to \infty$, which is equivalent to (2.34). The proof of this theorem is finished. □

Similarly to the proof of Theorem 2.8, we can obtain the following results. We omit the details of the proof.

Theorem 2.11. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p = n+1+q$. Then $uC_\varphi : F(p,q,s) \to H_{\mu,0}^\infty$ is compact if and only if*

$$\lim_{|z|\to 1} \mu(|z|)|u(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty. \tag{2.39}$$

2.3. Case $p > n+1+q$

Theorem 2.12. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p > n+1+q$. Then the following statements are equivalent:*

(i) *$uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded;*

(ii) *$uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is compact;*

(iii) *$u \in H_\mu^\infty$.*

Proof. (ii)⇒(i) This implication is obvious.

(i)⇒(iii) Taking $f(z) = 1$, then using the boundedness of $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ the implication follows.

(iii)⇒(ii) Suppose that $u \in H_\mu^\infty$. For an $f \in F(p,q,s)$, by Lemma 2.1, we see that f is continuous on the closed unit ball and so is bounded in B. Therefore,

$$\mu(|z|)|(uC_\varphi f)(z)| = \mu(|z|)|f(\varphi(z))||u(z)| \le C\|f\|_{F(p,q,s)}\mu(|z|)|u(z)|. \tag{2.40}$$

From the above inequality, we see that $uC_\varphi : F(p,q,s) \to H_\mu^\infty$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in $F(p,q,s)$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. Employing Lemma 2.5, we have

$$\|uC_\varphi f_k\|_{H_\mu^\infty} = \sup_{z\in B} \mu(|z|)\left|f_k(\varphi(z))u(z)\right| \le \|u\|_{H_\mu^\infty} \sup_{z\in B}\left|f_k(\varphi(z))\right| \longrightarrow 0, \tag{2.41}$$

as $k \to \infty$. Then the result follows from Lemma 2.3. □

Theorem 2.13. *Assume that $u \in H(B)$, φ is a holomorphic self-map of B, μ is a normal function on $[0,1)$, $0 < p,\ s < \infty$, $-n-1 < q < \infty$, $q+s > -1$, $p > n+1+q$. Then the following statements are equivalent:*

(i) $uC_\varphi : F(p,q,s) \to H_{\mu,0}^\infty$ *is bounded;*

(ii) $uC_\varphi : F(p,q,s) \to H_{\mu,0}^\infty$ *is compact;*

(iii) $u \in H_{\mu,0}^\infty$.

Proof. (ii)⇒(i) It is obvious.

(i)⇒(iii) Taking $f(z) = 1$, then using the boundedness of $uC_\varphi : F(p,q,s) \to H_{\mu,0}^\infty$, we get the desired result.

(iii)⇒(ii) Suppose that $u \in H_{\mu,0}^\infty$. For any $f \in F(p,q,s)$ with $\|f\|_{F(p,q,s)} \le 1$, we have

$$\mu(|z|)\left|(uC_\varphi f)(z)\right| \le C\|f\|_{F(p,q,s)}\mu(|z|)|u(z)| \le C\mu(|z|)|u(z)|, \tag{2.42}$$

from which we obtain

$$\lim_{|z|\to 1} \sup_{\|f\|_{F(p,q,s)}\le 1} \mu(|z|)\left|(uC_\varphi f)(z)\right| \le C \lim_{|z|\to 1} \mu(|z|)|u(z)| = 0. \tag{2.43}$$

Using Lemma 2.3, we see that $uC_\varphi : F(p,q,s) \to H_{\mu,0}^\infty$ is compact, as desired. □

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