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Research Article

Bounded Motions of the Dynamical Systems Described by Differential Inclusions

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The boundedness of the motions of the dynamical system described by a differential inclusion with control vector is studied. It is assumed that the right-hand side of the differential inclusion is upper semicontinuous. Using positionally weakly invariant sets, sufficient conditions for boundedness of the motions of a dynamical system are given. These conditions have infinitesimal form and are expressed by the Hamiltonian of the dynamical system.

1. Introduction

Consider the dynamical system, the behavior of which is described by the differential inclusion

$$\dot{x} \in F(t, x, u), \tag{1.1}$$

where $x \in R^n$ is the phase state vector, $u \in P$ is the control vector, $P \subset R^p$ is a compact set, and $t \in [0, \theta] = T$ is the time.

It will be assumed that the right-hand side of system (1.1) satisfies the following conditions:

(a) $F(t, x, u) \subset R^n$ is a nonempty, convex and compact set for every $(t, x, u) \in T \times R^n \times P$;

(b) the set valued map $(t, x) \to F(t, x, u), (t, x) \in T \times R^n$, is upper semicontinuous for every fixed $u \in P$;

(c) $\max\{\|f\| : f \in F(t, x, u),\ u \in P\} \leq c(1 + \|x\|)$ for every $(t, x) \in T \times R^n$ where $c = \text{const}$, and $\|\cdot\|$ denotes Euclidean norm.

Note that the study of a dynamical system described by an ordinary differential equation with discontinuous right-hand side, can be carried out in the framework of systems, given in the form (1.1) (see, e.g., [1–3] and references therein). The investigation of a conflict control system the dynamic of which is given by an ordinary differential equation, can also be reduced to a study of system in form (1.1) (see, e.g., [3–5] and references therein). The tracking control problem and its applications for uncertain dynamical systems, the behavior of which is described by differential inclusion with control vector, have been studied in [6].

In Section 2 the feedback principle is chosen as control method of the system (1.1). The motion of the system generated by strategy $(U_*, \delta_*(\cdot))$ from initial position (t_0, x_0) is defined. Here U_* is a positional strategy and it specifies the control effort to the system for realized position (t_*, x_*).The function $\delta_*(\cdot)$ defines the time interval; along the length of which the control effort, $U_*(t_*, x_*)$ will have an effect on. It is proved that the pencil of motions is a compact set in the space of continuous functions and every motion from the pencil of motions is an absolutely continuous function (Proposition 2.1).

In Section 3 the notion of a positionally weakly invariant set with respect to the dynamical system (1.1) is introduced. The positionally weak invariance of the closed set $W \subset T \times R^n$ means that for each $(t_0, x_0) \in W$ there exists a strategy $(U_*, \delta_*(\cdot))$ such that the graph of all motions of system (1.1) generated by strategy $(U_*, \delta_*(\cdot))$ from initial position (t_0, x_0) is in the set W right up to instant of time θ. Note that this notion is a generalization of the notions of weakly and strongly invariant sets with respect to a differential inclusion (see, e.g., [5, 7–11]) and close to the positional absorbing sets notion in the theory of differential games (see, e.g., [3–5]). In terms of upper directional derivatives, the sufficient conditions for posititionally weak invariance of the sets $W = \{(t, x) \in T \times R^n : c(t, x) \leq 0\}$ with respect to system (1.1) are formulated where $c(\cdot) : T \times R^n \to R$ is a continuous function (Theorems 3.2 and 3.3).

In Section 4, the boundedness of the motions of the system is investigated. Using the Hamiltonian of the system (1.1), the sufficient condition for boundedness of the motions is given (Theorem 4.3 and Corollary 4.4).

2. Motion of the System

Now let us give a method of control for the system (1.1) and define the motion of the system (1.1).

A function $U : T \times R^n \to P$ is called a positional strategy. The set of all positional strategies $U : T \times R^n \to P$ is denoted by symbol U_{pos} (see, e.g., [3–5]).

The set of all functions $\delta(\mu, t, x, u) : (0,1) \times [0, \theta) \times R^n \times P \to (0,1)$ such that $\delta(\mu, t, x, u) \leq \mu$ for every $(\mu, t, x, u) \in (0,1) \times [0, \theta) \times R^n \times P$ is denoted by $\Delta(0,1)$.

A pair $(U, \delta(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$ is said to be a strategy. Note that such a definition of a strategy is closely related to concept of ε-strategy for player E given in [12].

Now let us give a definition of motion of the system (1.1) generated by the strategy $(U_*, \delta_*(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$ from initial position $(t_0, x_0) \in [0, \theta) \times R^n$.

At first we give a definition of step-by-step motion of the system (1.1) generated by the strategy $(U_*, \delta_*(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$ from initial position $(t_0, x_0) \in [0, \theta) \times R^n$. Note that step-by-step procedure of control via strategy $(U_*, \delta_*(\cdot))$ uses the constructions developed in [3, 4, 12].

For $\delta_*(\cdot) \in \Delta(0,1)$ and fixed $\mu_* \in (0,1)$, we set

$$\begin{aligned} \Delta_{\mu_*}(\delta_*(\cdot)) = \{ & h(t, x, u) : [0, \theta) \times R^n \times P \longrightarrow (0,1) : h(t, x, u) \leq \delta_*(\mu_*, t, x, u), \\ & \text{for every } (t, x, u) \in [0, \theta) \times R^n \times P \}. \end{aligned} \tag{2.1}$$

It is obvious that $\delta_*(\mu_*,\cdot,\cdot,\cdot) \in \Delta_{\mu_*}(\delta_*(\cdot))$. Let us choose an arbitrary $h(\cdot) \in \Delta_{\mu_*}(\delta_*(\cdot))$. For given $(t_0, x_0) \in [0,\theta) \times R^n$, $(U_*, \delta_*(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$, $h(\cdot) \in \Delta_{\mu_*}(\delta_*(\cdot))$, we define the function $x(\cdot) : [t_0, \theta] \to R^n$ in the following way.

The function $x_*(\cdot)$ on the closed interval $[t_0, t_0 + h(t_0, x_0, U_*(t_0, x_0))] \cap [t_0, \theta]$ is defined as a solution of the differential inclusion $\dot{x}_*(t) \in F(t, x_*(t), U_*(t_0, x_0))$, $x_*(t_0) = x_0$ (see, e.g., [13]). If $t_0 + h(t_0, x_0, U_*(t_0, x_0)) < \theta$, then setting $t_1 = t_0 + h(t_0, x_0, U_*(t_0, x_0))$, $x_*(t_1) = x_1$, the function $x_*(\cdot)$ on the closed interval $[t_1, t_1 + h(t_1, x_1, U_*(t_1, x_1))] \cap [t_1, \theta]$ is defined as a solution of the differential inclusion $\dot{x}_*(t) \in F(t, x_*(t), U_*(t_1, x_1))$, $x_*(t_1) = x_1$ and so on.

Continuing this process we obtain an increasing sequence $\{t_k\}_{k=1}^{\infty}$ and function $x_*(\cdot) : [t_0, t_*) \to R^n$, where $t_* = \sup t_k$. If $t_* = \theta$, then it can be considered that the definition of the function $x_*(\cdot)$ is completed. If $t_* < \theta$, then to define the function $x_*(\cdot)$ on the interval $[t_0, \theta]$, the transfinite induction method should be used (see, e.g., [14]).

Let ν be an arbitrary ordinal number and $\{t_\lambda\}_{\lambda<\nu}$ are defined for every $\lambda < \nu$, where $t_\lambda \in [t_0, \theta)$ and $t_{\lambda_1} < t_{\lambda_2}$ if $\lambda_1 < \lambda_2$. If $t_* = \sup_{\lambda<\nu} t_\lambda = \theta$, then it can be considered that the definition of the function $x_*(\cdot)$ on the interval $[t_0, \theta]$ is completed. Let $t_* < \theta$. If ν follows after an ordinal number σ, then setting $x_*(t_\sigma) = x_\sigma$, we define the function $x_*(\cdot)$ on the closed interval $[t_\sigma, t_\nu] \cap [t_\sigma, \theta]$, where $t_\nu = t_\sigma + h(t_\sigma, x_\sigma, U_*(t_\sigma, x_\sigma))$, as a solution of the differential inclusion $\dot{x}_*(t) \in F(t, x_*(t), U_*(t_\sigma, x_\sigma))$, $x_*(t_\sigma) = x_\sigma$. If ν has no predecessor, then there exists a sequence $\{t_{\lambda_i}\}_{i=1}^{\infty}$ such that $t_{\lambda_{i_1}} < t_{\lambda_{i_2}} < \cdots$ and $t_{\lambda_i} \to t_\nu - 0$ as $i \to \infty$. Then we set $x_*(t_\nu) = \lim_{i\to\infty} x_*(t_{\lambda_i})$. Note that it is not difficult to prove that via conditions (a)–(c), this limit exists.

Since the intervals $(t_\lambda, t_{\lambda+1})$ are not empty and pairwise disjoint then $t_\nu = \theta$ for some ordinal number ν which does not exceed first uncountable ordinal number (see, e.g., [15, 16]). So, the function $x_*(\cdot)$ is defined on the interval $[t_0, \theta]$.

From the construction of the function $x_*(\cdot)$ it follows that for given $(t_0, x_0) \in [0,\theta) \times R^n$, $(U_*, \delta_*(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$, $\mu_* \in (0,1)$, $h(\cdot) \in \Delta_{\mu_*}(\delta_*(\cdot))$ such a function is not unique. The set of such functions is denoted by $Y_{\mu_*}(t_0, x_0, U_*, h(\cdot))$. Further, we set

$$Z_{\mu_*}(t_0, x_0, U_*, \delta_*(\cdot)) = \bigcup_{h(\cdot)\in\Delta_{\mu_*}(\delta_*(\cdot))} Y_{\mu_*}(t_0, x_0, U_*, h(\cdot)). \tag{2.2}$$

The set $Z_{\mu_*}(t_0, x_0, U_*, \delta_*(\cdot))$ is called the pencil of step-by-step motions and each function $x(\cdot) \in Z_{\mu_*}(t_0, x_0, U_*, \delta_*(\cdot))$ is called step-by-step motion of the system (1.1), generated by the strategy $(U_*, \delta_*(\cdot))$ from the initial position (t_0, x_0).

It is obvious that for each step-by-step motion $x(\cdot) \in Z_{\mu_*}(t_0, x_0, U_*, \delta_*(\cdot))$ there exists an $h_*(\cdot) \in \Delta_{\mu_*}(\delta_*(\cdot))$ such that $x(\cdot) \in Y_{\mu_*}(t_0, x_0, U_*, h_*(\cdot))$.

By $X(t_0, x_0, U_*, \delta_*(\cdot))$ we denote the set of all functions $x(\cdot) : [t_0, \theta] \to R^n$ such that $x(\cdot) = \lim_{k\to\infty} x_k(\cdot)$, where $x_k(\cdot) \in Z_{\mu_k}(t_0, x_0, U_*, \delta_*(\cdot))$, $\mu_k \to 0^+$ as $k \to \infty$. $X(t_0, x_0, U_*, \delta_*(\cdot))$ is said to be the pencil of motions and each function $x(\cdot) \in X(t_0, x_0, U_*, \delta_*(\cdot))$ is said to be the motion of the system (1.1), generated by the strategy $(U_*, \delta_*(\cdot))$ from initial position (t_0, x_0).

For every initial position (θ, x_0) we set $X(\theta, x_0, U, \delta(\cdot)) = \{x_0\}$ for all $(U, \delta(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$.

Using the constructions developed in [3, 4] it is possible to prove the validity of the following proposition.

Proposition 2.1. *For each $(t_0, x_0) \in [0,\theta) \times R^n$, $(U_*, \delta_*(\cdot)) \in U_{pos} \times \Delta(0,1)$ the set $X(t_0, x_0, U_*, \delta_*(\cdot))$ is nonempty compact subset of the space $C([t_0, \theta]; R^n)$ and each motion $x(\cdot) \in X(t_0, x_0, U_*, \delta_*(\cdot))$ is an absolutely continuous function.*

Here $C([t_0,\theta];R^n)$ is the space of continuous functions $x(\cdot):[t_0,\theta]\to R^n$ with norm $|x(\cdot)|=\max\|x(t)\|$ as $t\in[t_0,\theta]$.

3. Positionally Weakly Invariant Set

Let $W\subset T\times R^n$ be a closed set. We set

$$W(t)=\{x\in R^n:(t,x)\in W\}. \tag{3.1}$$

Let us give the definition of positionally weak invariance of the set $W\subset T\times R^n$ with respect to dynamical system (1.1).

Definition 3.1. A closed set $W\subset T\times R^n$ is said to be positionally weakly invariant with respect to a dynamical system (1.1) if for each position $(t_0,x_0)\in W$ it is possible to define a strategy $(U_*,\delta_*(\cdot))\in U_{\text{pos}}\times\Delta(0,1)$ such that for all $x(\cdot)\in X(t_0,x_0,U_*,\delta_*(\cdot))$ the inclusion $x(t)\in W(t)$ holds for every $t\in[t_0,\theta]$.

We will consider positionally weak invariance of the set $W\subset T\times R^n$, described by the relation

$$W=\{(t,x)\in T\times R^n:c(t,x)\le 0\}, \tag{3.2}$$

where $c(\cdot):T\times R^n\to R^1$. For $(t,x)\in[0,\theta)\times R^n$, $f\in R^n$ we denote

$$\frac{\partial^+c(t,x)}{\partial(1,f)}=\limsup_{\delta\to0^+,\|y\|\to0}\left[c(t+\delta,x+\delta f+\delta y)-c(t,x)\right]\delta^{-1}. \tag{3.3}$$

Let us formulate the theorem which characterizes positionally weak invariance of the set W given by relation (3.2) with respect to dynamical system (1.1).

Theorem 3.2 ([17]). *Let $\varepsilon_*>0$, and let the set $W\subset T\times R^n$ be defined by relation* (3.2) *where $c(\cdot):T\times R^n\to R^1$ is a continuous function. Assume that for each $(t,x)\in[0,\theta)\times R^n$ such that $0<c(t,x)<\varepsilon_*$, it is possible to define $u_*\in P$ such that the inequality*

$$\sup_{f\in F(t,x,u_*)}\frac{\partial^+c(t,x)}{\partial(1,f)}\le 0 \tag{3.4}$$

holds.

Then the set W described by relation (3.2) *is positionally weakly invariant with respect to the dynamical system* (1.1).

Theorem 3.3. *Let $\varepsilon_* > 0$, and let the set $W \subset T \times R^n$ be defined by relation* (3.2) *where $c(\cdot) : T \times R^n \to R^1$ is a continuous function. Assume that for each $(t, x) \in [0, \theta) \times R^n$ such that $0 < c(t, x) < \varepsilon_*$, the inequality*

$$\inf_{u \in P} \sup_{f \in F(t,x,u)} \frac{\partial^+ c(t, x)}{\partial(1, f)} \leq 0 \tag{3.5}$$

is verified.

Then for each fixed $(t_0, x_0) \in W$ and $\varepsilon \in (0, \varepsilon_*)$ it is possible to define a strategy $(U_\varepsilon, \delta_\varepsilon(\cdot)) \in U_{pos} \times \Delta(0, 1)$ such that for all $x(\cdot) \in X(t_0, x_0, U_\varepsilon, \delta_\varepsilon(\cdot))$ the inequality $c(t, x(t)) \leq \varepsilon$ holds for every $t \in [t_0, \theta]$.

For $(t, x, s) \in T \times R^n \times R^n$ we denote

$$\xi(t, x, s) = \inf_{u \in P} \sup_{f \in F(t,x,u)} \langle s, f \rangle. \tag{3.6}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n.

The function $\xi(\cdot) : T \times R^n \times R^n \to R$ is said to be the Hamiltonian of the system (1.1).

We obtain from Theorem 3.3 the validity of the following theorem.

Theorem 3.4. *Let $\varepsilon_* > 0$, and let the set $W \subset T \times R^n$ be defined by relation* (3.2) *where $c(\cdot) : T \times R^n \to R^1$ is a differentiable function. Assume that for each $(t, x) \in [0, \theta) \times R^n$ such that $0 < c(t, x) < \varepsilon_*$, the inequality*

$$\frac{\partial c(t, x)}{\partial t} + \xi\left(t, x, \frac{\partial c(t, x)}{\partial x}\right) \leq 0 \tag{3.7}$$

holds.

Then for each fixed $(t_0, x_0) \in W$ and $\varepsilon \in (0, \varepsilon_*)$ it is possible to define a strategy $(U_\varepsilon, \delta_\varepsilon(\cdot)) \in U_{pos} \times \Delta(0, 1)$ such that for all $x(\cdot) \in X(t_0, x_0, U_\varepsilon, \delta_\varepsilon(\cdot))$ the inequality $c(t, x(t)) \leq \varepsilon$ holds for every $t \in [t_0, \theta]$.

4. Boundedness of the Motion of the System

Consider positionally weak invariance of the set $W \subset T \times R^n$ given by relation (3.2) where

$$c(t, x) = \langle E(t)(x - a(t)), (x - a(t)) \rangle - 1, \tag{4.1}$$

$E(\cdot)$ is a differentiable $(n \times n)$ matrix function, $a(\cdot) : T \to R^n$ is a differentiable function. Then the set W is given by relation

$$W = \{(t, x) \in T \times R^n : \langle E(t)(x - a(t)), (x - a(t)) \rangle - 1 \leq 0\}. \tag{4.2}$$

If the matrix $E(t)$ is symmetrical and positive definite for every $t \in T$, then it is obvious that for every $t \in T$ the set $W(t) \subset R^n$ is ellipsoid.

Theorem 4.1. *Let $\varepsilon_* > 0$, and let the set $W \subset T \times R^n$ be defined by relation* (4.2) *where $E(\cdot)$ is a differentiable $(n\times n)$ matrix function, $a(\cdot) : T \to R^n$ is a differentiable function. Assume that for each $(t,x) \in [0,\theta) \times R^n$ such that $0 < \langle E(t)(x-a(t)), (x-a(t))\rangle - 1 < \varepsilon_*$ the inequality*

$$\left\langle \left[\frac{dE(t)}{dt}(x-a(t)) - \left(E(t)+E^T(t)\right)\frac{da(t)}{dt}\right], (x-a(t))\right\rangle + \xi\left(t,x,\left[E(t)+E^T(t)\right](x-a(t))\right) \leq 0 \tag{4.3}$$

holds.

Then for each fixed $(t_0,x_0) \in W$ and $\varepsilon \in (0,\varepsilon_*)$ it is possible to define a strategy $(U_\varepsilon, \delta_\varepsilon(\cdot)) \in U_{pos} \times \Delta(0,1)$ such that for all $x(\cdot) \in X(t_0,x_0,U_\varepsilon,\delta_\varepsilon(\cdot))$ the inequality

$$\langle E(t)(x(t)-a(t)), (x(t)-a(t))\rangle - 1 < \varepsilon \tag{4.4}$$

holds for every $t \in [t_0,\theta]$.

Here $E^T(t)$ means the transpose of the matrix $E(t)$.

Proof. Since the function $c(\cdot)$ given by relation (4.1) is differentiable and

$$\begin{gathered}\frac{\partial c(t,x)}{\partial x} = \left[E(t)+E^T(t)\right](x-a(t)),\\ \frac{\partial c(t,x)}{\partial t} = \left\langle \left[\frac{dE(t)}{dt}(x-a(t)) - E(t)\frac{da(t)}{dt} - E^T(t)\frac{da(t)}{dt}\right], (x-a(t))\right\rangle\end{gathered} \tag{4.5}$$

then the validity of the theorem follows from Theorem 3.4. □

We obtain from Theorem 4.1 the following corollary.

Corollary 4.2. *Let $\varepsilon_* > 0$, and let the set $W \subset T \times R^n$ be defined by relation* (4.2) *where $E(\cdot)$ is a differentiable $(n \times n)$ matrix function, $a(\cdot) : T \to R^n$ is a differentiable function and $E(t)$ is a symmetrical positive definite matrix for every $t \in T$. Assume that for each $(t,x) \in [0,\theta) \times R^n$ for which*

$$0 < \langle E(t)(x-a(t)), (x-a(t))\rangle - 1 < \varepsilon_*, \tag{4.6}$$

the inequality

$$\left\langle \left[\frac{1}{2}\frac{dE(t)}{dt}(x-a(t)) - E(t)\frac{da(t)}{dt}\right], (x-a(t))\right\rangle + \xi(t,x,E(t)(x-a(t))) \leq 0 \tag{4.7}$$

holds.

Then for each fixed $(t_0, x_0) \in W$ *and* $\varepsilon \in (0, \varepsilon_*)$ *it is possible to define a strategy* $(U_\varepsilon, \delta_\varepsilon(\cdot)) \in U_{pos} \times \Delta(0,1)$ *such that for all* $x(\cdot) \in X(t_0, x_0, U_\varepsilon, \delta_\varepsilon(\cdot))$ *the inequality*

$$\langle E(t)(x(t) - a(t)), (x(t) - a(t)) \rangle - 1 < \varepsilon \tag{4.8}$$

holds for every $t \in [t_0, \theta]$.

Now let us give the theorem which characterizes boundedness of the motion of the system (1.1).

For $a \in R^n$, $r > 0$, and $\varepsilon_* > 0$ denote

$$\begin{gathered} S_{\varepsilon_*}(a, r) = \{x \in R^n : r < \|x - a\| < r + \varepsilon_*\}, \\ S_{\varepsilon_*}(r) = \{x \in R^n : r < \|x\| < r + \varepsilon_*\}, \\ B(a, r) = \{x \in R^n : \|x - a\| \leq r\}, \qquad B(r) = \{x \in R^n : \|x\| \leq r\}, \\ \alpha_* = \varepsilon_*^2 + 2r\varepsilon_*. \end{gathered} \tag{4.9}$$

Theorem 4.3. *Let* $\varepsilon_* > 0$ *and let* $r > 0$. *Assume that for each* $(t, x) \in [0, \theta) \times R^n$ *such that* $t \in [0, \theta)$, *and* $x \in S_{\varepsilon_*}(a, r)$ *the inequality*

$$\xi(t, x, (x - a)) \leq 0 \tag{4.10}$$

holds.

Then for each fixed $(t_0, x_0) \in T \times B(a, r)$ *and* $\varepsilon \in (0, \alpha_*)$ *it is possible to define a strategy* $(U_\varepsilon, \delta_\varepsilon(\cdot)) \in U_{pos} \times \Delta(0,1)$ *such that for all* $x(\cdot) \in X(t_0, x_0, U_\varepsilon, \delta_\varepsilon(\cdot))$ *the inequality* $\|x(t) - a\| \leq r + \varepsilon$ *holds for every* $t \in [t_0, \theta]$.

Here $\alpha_* > 0$ *is defined by relation* (4.9).

Proof. Let

$$c(t, x) = \langle x - a, x - a \rangle - r^2. \tag{4.11}$$

Then

$$\frac{\partial c(t, x)}{\partial t} = 0, \qquad \frac{\partial c(t, x)}{\partial x} = 2(x - a), \tag{4.12}$$

and consequently

$$\frac{\partial c(t, x)}{\partial t} + \xi\left(t, x, \frac{\partial c(t, x)}{\partial x}\right) = 2\xi(t, x, x - a). \tag{4.13}$$

Let

$$W = \{(t, x) \in T \times R^n : c(t, x) \leq 0\}, \tag{4.14}$$

where the function $c(\cdot) : T \times R^n \to R$ is defined by (4.11). It is obvious that $(t,x) \in W$ if and only if $t \in T$ and $x \in B(a,r)$.

It is not difficult to verify that

$$\{(t,x) \in T \times R^n : 0 < c(t,x) < \alpha_*\} = \{(t,x) \in T \times R^n : x \in S_{\varepsilon_*}(a,r)\}, \tag{4.15}$$

where $\alpha_* > 0$ is defined by relation (4.9). Then we obtain from (4.10), (4.13) and (4.15) that for every $(t,x) \in [0,\theta) \times R^n$ such that $0 < c(t,x) < \alpha_*$ the inequality

$$\frac{\partial c(t,x)}{\partial t} + \xi\left(t, x, \frac{\partial c(t,x)}{\partial x}\right) \leq 0 \tag{4.16}$$

holds. So we get from Theorem 3.4 and (4.16) the validity of Theorem 4.3. □

Corollary 4.4. *Let $\varepsilon_* > 0$ and let $r > 0$. Assume that for each $(t,x) \in [0,\theta) \times R^n$ such that $t \in [0,\theta)$, and $x \in S_{\varepsilon_*}(r)$ the inequality*

$$\xi(t,x,x) \leq 0 \tag{4.17}$$

holds.

Then for each fixed $(t_0,x_0) \in T \times B(r)$ and $\varepsilon \in (0,\alpha_*)$ it is possible to define a strategy $(U_\varepsilon, \delta_\varepsilon(\cdot)) \in U_{pos} \times \Delta(0,1)$ such that for all $x(\cdot) \in X(t_0,x_0,U_\varepsilon,\delta_\varepsilon(\cdot))$ the inequality $\|x(t)\| \leq r + \varepsilon$ holds for every $t \in [t_0,\theta]$.

Here $\alpha_* > 0$ is defined by relation (4.9).

Using the results obtained above, we illustrate in the following example that the given system has bounded motions.

Example 4.5. Let the behavior of the dynamical system be described by the differential inclusion

$$\dot{x} \in \left[x^{1/3} - \beta|x|, x^{1/3} + \beta|x|\right] + x^{1/5}u, \tag{4.18}$$

where $x \in R$, $u \in R$, $|u| \leq \alpha$, $\alpha > 0$, $\beta \geq 0$, $t \in [0,T]$, and $T > 0$ is sufficiently large number.

Let $\gamma_* > 0$ be such that $\beta x^{4/5} + x^{2/15} - \alpha \leq 0$ for every $x \in [-\gamma_*, \gamma_*]$. Then for every $t \in [0,T]$ and $x \in [-\gamma_*, \gamma_*]$ we get that

$$\begin{aligned}
\xi(t,x,x) &= \inf_{u \in [-\alpha,\alpha]} \max_{f \in \left[x^{1/3} - \beta|x|, x^{1/3} + \beta|x|\right]} \left(xf + xx^{1/5}u\right) \\
&= \inf_{u \in [-\alpha,\alpha]} x^{6/5}u + \max_{f \in \left[x^{1/3} - \beta|x|, x^{1/3} + \beta|x|\right]} xf \\
&= -\alpha x^{6/5} + x^{4/3} + \beta x^2 = x^{6/5}\left[\beta x^{4/5} + x^{2/15} - \alpha\right] \leq 0.
\end{aligned} \tag{4.19}$$

Thus, we get from (4.19) and Corollary 4.4 that for each $x_0 \in R$ such that $|x_0| < \gamma < \gamma_*$ there exists a strategy $(U_\gamma, \delta_\gamma(\cdot)) \in U_{\text{pos}} \times \Delta(0,1)$ such that for all $x(\cdot) \in X(0,x_0,U_\gamma,\delta_\gamma(\cdot))$

the inequality $|x(t)| \leq \gamma$ holds for every $t \in [0, T]$, where $X(0, x_0, U_\gamma, \delta_\gamma(\cdot))$ is the pencil of motions of the system (4.18) generated by the strategy $(U_\gamma, \delta_\gamma(\cdot))$ from initial position $(0, x_0)$.

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