Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 639439, 7 pages
doi:10.1155/2009/639439

Research Article

Some Identities of the Frobenius-Euler Polynomials

Taekyun Kim[1] and Byungje Lee[2]

[1] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, South Korea*
[2] *Department of Wireless Communications Engineering, Kwangwoon University, Seoul 139-701, South Korea*

Correspondence should be addressed to Byungje Lee, bj_lee@kw.ac.kr

Received 5 November 2008; Accepted 5 January 2009

Recommended by Ferhan Atici

By using the ordinary fermionic p-adic invariant integral on $\mathbb{Z}_p$, we derive some interesting identities related to the Frobenius-Euler polynomials.

1. Introduction

Let p be a fixed prime. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. When one talks about q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$; see [1–14]. If $q \in \mathbb{C}$, then we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1-q|_p < 1$. For $x \in \mathbb{Q}_p$, we use the notation $[x]_q = (1-q^x)/(1-q)$, and $[x]_{-q} = (1-(-q)^x)/(1+q)$; see [15, 16]. The normalized valuation in $\mathbb{C}_p$ is denoted by $|\cdot|_p$ with $|p|_p = 1/p$. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = (f(x)-f(y))/(x-y)$ have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \le j < p^N} q^j f(j) = \sum_{0 \le j < p^N} f(j) \mu_q (j + p^N \mathbb{Z}_p), \tag{1.1}$$

representing a q-analogue of Riemann sums for f; see [15, 16]. The integral of f on $\mathbb{Z}_p$ will be defined as a limit $(n \to \infty)$ of those sums, when it exists. The q-deformed bosonic p-adic integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{0 \le x < dp^N} f(x) q^x, \tag{1.2}$$

see [15]. Thus, we note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q} f'(0), \tag{1.3}$$

where $f_1(x) = f(x+1)$, $f'(0) = df(0)/dx$.

The fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-1}} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1.4}$$

see [15].

In this paper, we prove an identity of symmetry for the Frobenius-Euler polynomials. Finally we investigate the several further interesting properties of the symmetry for the fermionic p-adic invariant integral on $\mathbb{Z}_p$ related to the Frobenius-Euler polynomials and numbers.

2. Some Identities of the Frobenius-Euler Polynomials

Let $u(\neq 1) \in \mathbb{C}_p$ (or $\mathbb{C}$) be algebraic. Then the nth Frobenius-Euler numbers $H_n(u)$ are defined as

$$H_0(u) = 1, \quad (H(u)+1)^n - uH_n(u) = 0, \quad \text{if } n \geq 1, \tag{2.1}$$

with the usual convention about replacing $H^n(u)$ by $H_n(u)$.

The nth Frobenius-Euler polynomials $H_n(u, x)$ are also defined as

$$H_n(u,x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} H_l(u). \tag{2.2}$$

From (1.4), we can easily derive

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \quad \text{where } f_1(x) = f(x+1). \tag{2.3}$$

By continuing this process, we see that

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{where } f_n(x) = f(x+n). \tag{2.4}$$

When n is an odd positive integer, we obtain

$$I_{-1}(f_n) + I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^l f(l). \tag{2.5}$$

If $n \in \mathbb{N}$ with $n \equiv 0 \pmod 2$, then we have

$$I_{-1}(f_n) - I_{-1}(f) = 2\sum_{l=0}^{n-1}(-1)^{l-1}f(l). \tag{2.6}$$

From (1.4) and (2.3), we derive

$$\int_{\mathbb{Z}_p} e^{xt}q^x d\mu_{-1}(x) = \frac{2}{[2]_q}\frac{1-(-q)^{-1}}{e^t-(-q)^{-1}} = \frac{2}{[2]_q}\sum_{n=0}^{\infty}H_n(-q^{-1})\frac{t^n}{n!}. \tag{2.7}$$

Thus, we note that

$$\int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x) = \frac{2}{[2]_q}H_n(-q^{-1}), \qquad \int_{\mathbb{Z}_p}(y+x)^n q^y d\mu_{-1}(x) = \frac{2}{[2]_q}H_n(-q^{-1},x). \tag{2.8}$$

Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Then we obtain

$$[2]_q\sum_{l=0}^{n-1}(-1)^l q^l l^m = q^n H_m(-q^{-1},n) + H_m(-q^{-1}). \tag{2.9}$$

For $n \in \mathbb{N}$ with $n \equiv 0 \pmod 2$, we have

$$q^n H_m(-q^{-1},n) - H_m(-q^{-1}) = [2]_q\sum_{l=0}^{n-1}(-1)^{l-1}q^l l^m. \tag{2.10}$$

By substituting $f(x) = q^x e^{xt}$ into (2.5), we can easily see that

$$\int_{\mathbb{Z}_p} q^{n+x}e^{(x+n)t}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x e^{xt}d\mu_{-1}(x) = 2\frac{q^n e^{nt}+1}{qe^t+1} = 2\sum_{l=0}^{n-1}(-1)^l q^l e^{lt}. \tag{2.11}$$

Let $S_{k,q}(n) = \sum_{l=0}^{n}(-1)^l l^k q^l$. Then $S_{k,q}(n)$ is called the alternating sums of powers of consecutive q-integers. From the definition of the fermionic p-adic invariant integral on $\mathbb{Z}_p$, we can derive

$$\int_{\mathbb{Z}_p} q^{x+n}e^{(x+n)t}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x e^{xt}d\mu_{-1}(x) = \frac{2\int_{\mathbb{Z}_p} q^x e^{xt}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{nxt}q^{nx}d\mu_{-1}(x)}. \tag{2.12}$$

By (2.12), we easily see that

$$\int_{\mathbb{Z}_p} q^{nx}e^{nxt}d\mu_{-1}(x) = \frac{2}{q^n e^{nt}+1}. \tag{2.13}$$

Let $w_1, w_2 (\in \mathbb{N})$ be odd. By using double fermionic p-adic invariant integral on $\mathbb{Z}_p$, we obtain

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e^{(w_1x_1+w_2x_2)t} q^{w_1x_1+w_2x_2} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{w_1w_2x} d\mu_{-1}(x)} = \frac{2\left(q^{w_1w_2} e^{w_1w_2t}+1\right)}{\left(q^{w_1}e^{w_1t}+1\right)\left(q^{w_2}e^{w_2t}+1\right)}. \tag{2.14}$$

Now we also consider the following fermionic p-adic invariant integral on $\mathbb{Z}_p$ associated with Frobenius-Euler polynomials:

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e^{(w_1x_1+w_2x_2+w_1w_2x)t} q^{w_1x_1+w_2x_2} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{w_1w_2x} d\mu_{-1}(x)} = \frac{2e^{w_1w_2xt}\left(q^{w_1w_2} e^{w_1w_2t}+1\right)}{\left(q^{w_1}e^{w_1t}+1\right)\left(q^{w_2}e^{w_2t}+1\right)}. \tag{2.15}$$

From (2.15) and (2.12), we can derive

$$\begin{aligned}
\frac{2\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{w_1xt} q^{w_1x} d\mu_{-1}(x)} &= 2\sum_{l=0}^{w_1-1}(-1)^l q^l e^{lt} \\
&= \sum_{k=0}^{\infty}\left(2\sum_{l=0}^{w_1-1}(-1)^l q^l l^k\right)\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} 2S_{k,q}(w_1-1)\frac{t^k}{k!}.
\end{aligned} \tag{2.16}$$

Let

$$M^{(w_1,w_2)}(t,x) = \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{w_1x_1+w_2x_2} e^{(w_1x_1+w_2x_2+w_1w_2x)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1w_2x_3t} q^{w_1w_2x_3} d\mu_{-1}(x_3)}. \tag{2.17}$$

By (2.15), (2.16), and (2.17), we see that

$$M^{(w_1,w_2)}(t,x) = \frac{e^{w_1w_2xt}\left(q^{w_1w_2} e^{w_1w_2t}+1\right)}{\left(q^{w_1}e^{w_1t}+1\right)\left(q^{w_2}e^{w_2t}+1\right)}. \tag{2.18}$$

From (2.17) we derive

$$M^{(w_1,w_2)}(t,x) = \left(\frac{1}{2}\int_{\mathbb{Z}_p} e^{w_1(x_1+w_2x)t} q^{w_1x_1} d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p} e^{w_2x_2t} q^{w_2x_2} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{w_1w_2x} d\mu_{-1}(x)}\right). \tag{2.19}$$

By (2.16) and (2.19), we see that

$$M^{(w_1,w_2)}(t,x) = \left(\frac{1}{1+q^{w_1}}\sum_{i=0}^{\infty}H_i\left(-q^{-w_1},w_2x\right)\frac{w_1^i}{i!}t^i\right)\left(\sum_{l=0}^{\infty}S_{l,q^{w_2}}(w_1-1)\frac{w_2^l}{l!}t^l\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\frac{H_i\left(-q^{-w_1},w_2x\right)}{1+q^{w_1}}S_{n-i,q^{w_2}}(w_1-1)w_1^i w_2^{n-i}\right)\frac{t^n}{n!}. \tag{2.20}$$

By the symmetry of p-adic invariant integral on $\mathbb{Z}_p$, we also see that

$$M^{(w_1,w_2)}(t,x) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\frac{H_i(-q^{-w_2},w_1x)}{1+q^{w_2}}S_{n-i,q^{w_1}}(w_2-1)w_2^i w_1^{n-i}\right)\frac{t^n}{n!}, \tag{2.21}$$

where $H_n(-q^{-1},x)$ are the nth Frobenius-Euler polynomials.

By comparing the coefficients on the both sides of (2.20) and (2.21), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2, n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$, $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, one has*

$$\sum_{i=0}^{n}\binom{n}{i}\frac{H_i\left(-q^{-w_1},w_2x\right)}{1+q^{w_1}}S_{n-i,q^{w_2}}(w_1-1)w_1^i w_2^{n-i}$$
$$= \sum_{i=0}^{n}\binom{n}{i}\frac{H_i\left(-q^{-w_2},w_1x\right)}{1+q^{w_2}}S_{n-i,q^{w_1}}(w_2-1)w_2^i w_1^{n-i}, \tag{2.22}$$

where $H_n(q,x)$ are the nth Frobenius-Euler polynomials.

If we take $w_2 = 1$ in Theorem 2.1, then we have

$$\frac{H_n\left(-q^{-1},w_1x\right)}{1+q} = \sum_{i=0}^{n}\binom{n}{i}\frac{H_i\left(-q^{-w_1},x\right)}{1+q^{w_1}}S_{n-i,q}(w_1-1)w_1^i. \tag{2.23}$$

From (2.11) and (2.12), we derive

$$M^{(w_1,w_2)}(t,x) = \left(\frac{e^{w_1w_2xt}}{2}\int_{\mathbb{Z}_p}e^{w_1x_1t}q^{w_1x_1}d\mu_{-1}(x_1)\right)\left(\frac{2\int_{\mathbb{Z}_p}e^{w_2x_2t}q^{w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}e^{w_1w_2xt}q^{w_1w_2x}d\mu_{-1}(x)}\right)$$
$$= \left(\frac{e^{w_1w_2xt}}{2}\int_{\mathbb{Z}_p}e^{w_1x_1t}q^{w_1x_1}d\mu_{-1}(x_1)\right)\left(2\sum_{l=0}^{w_1-1}(-1)^l q^{w_2l}e^{w_2lt}\right)$$
$$= \sum_{l=0}^{w_1-1}(-1)^l q^{w_2l}\int_{\mathbb{Z}_p}e^{(x_1+w_2x+(w_2/w_1)l)tw_1}q^{x_1w_1}d\mu_{-1}(x_1)$$
$$= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{w_1-1}(-1)^l\frac{H_n\left(-q^{-w_1},w_2x+(w_2/w_1)l\right)}{1+q^{w_1}}q^{w_2l}\right)\frac{t^n}{n!}. \tag{2.24}$$

From the symmetry of $M^{(w_1,w_2)}(t,x)$, we note that

$$M^{(w_1,w_2)}(t,x) = \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{w_2-1}(-1)^l \frac{H_n\left(-q^{-w_2}, w_1x + (w_1/w_2)l\right)}{1+q^{w_2}} q^{w_1 l}\right)\frac{t^n}{n!}. \tag{2.25}$$

By comparing the coefficients on the both sides of (2.24) and (2.25), we obtain the following theorem.

Theorem 2.2. *Let $w_1, w_2 (\in \mathbb{N})$ be odd, and let $n \in \mathbb{Z}_+$ with $n \equiv 1 \pmod 2$. Then, one has*

$$\sum_{l=0}^{w_1-1}(-1)^l \frac{H_n\left(-q^{-w_1}, w_2x + (w_2/w_1)l\right)}{1+q^{w_1}} q^{w_2 l} = \sum_{l=0}^{w_2-1}(-1)^l \frac{H_n\left(-q^{-w_2}, w_1x + (w_1/w_2)l\right)}{1+q^{w_2}} q^{w_1 l}. \tag{2.26}$$

By setting $w_2 = 1$ in Theorem 2.2, we get the multiplication theorem for the Frobenius-Euler polynomials as follows:

$$\frac{H_n\left(-q^{-1}, w_1x\right)}{1+q} = \sum_{l=0}^{w_1-1}(-1)^l q^l H_n\left(-q^{-w_1}, x + \frac{l}{w_1}\right). \tag{2.27}$$

Remark 2.3. By using the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$, the symmetric properties related to Frobenius-Euler polynomials are studied in [17]. In this paper, we have studied the symmetric properties of Frobenius-Euler polynomials, which are different from the symmetric properties treated in a previous paper [17]. To derive the symmetric properties of Frobenius-Euler polynomials, we used the ordinary fermionic p-adic invariant integrals on $\mathbb{Z}_p$ in this paper.

Acknowledgment

The present research has been conducted by the research grant of the Kwangwoon University in 2008.

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