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Research Article

Products of Composition and Differentiation Operators from $Q_K(p,q)$ Spaces to Bloch-Type Spaces

Weifeng Yang

Department of Mathematics and Physics, Hunan Institute of Engineering, Xiangtan 411104, China

Correspondence should be addressed to Weifeng Yang, yangweifeng09@163.com

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We study the boundedness and compactness of the products of composition and differentiation operators from $Q_K(p,q)$ spaces to Bloch-type spaces and little Bloch-type spaces.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. The α-Bloch space $\mathcal{B}^\alpha$ $(\alpha > 0)$ on $\mathbb{D}$ is the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha |f'(z)| < \infty. \tag{1.1}$$

Under the above norm, $\mathcal{B}^\alpha$ is a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which $(1-|z|^2)^\alpha |f'(z)| \to 0$ as $|z| \to 1$. This space is called the little α-Bloch space.

Assume that μ is a positive continuous function on $[0,1)$, and there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that

$$\begin{gathered} \frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \lim_{r\to 1} \frac{\mu(r)}{(1-r)^s} = 0, \\ \frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \lim_{r\to 1} \frac{\mu(r)}{(1-r)^t} = \infty, \end{gathered} \tag{1.2}$$

then μ is called a normal function (see [1]).

An $f \in H(\mathbb{D})$ is said to belong to the Bloch-type space $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{D})$, if (see, e.g, [2–5])

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f'(z)| < \infty. \tag{1.3}$$

$\mathcal{B}_\mu$ is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\mu}$ (see [3]). When $\mu(r) = (1-r^2)^\alpha$, the induced space $\mathcal{B}_\mu$ becomes the α-Bloch space $\mathcal{B}^\alpha$.

Throughout this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function. Assume that $p > 0$, $q > -2$. A function $f \in H(\mathbb{D})$ is said to belong to $Q_K(p,q)$ (see [6]) if

$$\|f\| = \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q K(g(z,a))\,dA(z)\right)^{1/p} < \infty, \tag{1.4}$$

where dA denotes the normalized Lebesgue area measure in $\mathbb{D}$ (i.e., $A(\mathbb{D}) = 1$) and $g(z,a)$ is the Green function with logarithmic singularity at a, that is, $g(z,a) = \log(1/|\varphi_a(z)|)$ (φ_a is a conformal automorphism defined by $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for $a \in \mathbb{D}$). If $K(x) = x^s$, $s \geq 0$, the space $Q_K(p,q)$ equals to $F(p,q,s)$, which is introduced by Zhao in [7]. Moreover (see [7]) we have that, $F(p,q,s) = \mathcal{B}^{(q+2)/p}$, and $F_0(p,q,s) = \mathcal{B}_0^{(q+2)/p}$ for $s > 1$, $F(p,q,s) \subseteq \mathcal{B}^{(q+2)/p}$, and $F_0(p,q,s) \subseteq \mathcal{B}_0^{(q+2)/p}$ for $0 < s \leq 1$, $F(2,0,s) = Q_s$, and $F_0(2,0,s) = Q_{s,0}$, $F(2,1,0) = H^2$, $F(2,0,1) = BMOA$, and $F_0(2,0,1) = VMOA$. When $p \geq 1$, $Q_K(p,q)$ is a Banach space under the norm

$$\|f\|_{Q_K(p,q)} = |f(0)| + \|f\|. \tag{1.5}$$

From [6], we know that $Q_K(p,q) \subseteq \mathcal{B}^{(q+2)/p}$, $Q_K(p,q) = \mathcal{B}^{(q+2)/p}$ if and only if

$$\int_0^1 \left(1-r^2\right)^{-2} K(-\log r)\, r\, dr < \infty. \tag{1.6}$$

Moreover, $\|f\|_{\mathcal{B}^{(q+2)/p}} \leq C\|f\|_{Q_K(p,q)}$ (see in [6, Theorem 2.1] or [8, Lemma 2.1]). Throughout the paper we assume that (see [6])

$$\int_0^1 \left(1-r^2\right)^{q} K(-\log r)\, r\, dr < \infty, \tag{1.7}$$

since otherwise $Q_K(p,q)$ consists only of constant functions.

Let φ denote a nonconstant analytic self-map of $\mathbb{D}$. Associated with φ is the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. The problem of characterizing the boundedness and compactness of composition operators on many Banach spaces of analytic functions has attracted lots of attention recently, see, for example, [9, 10] and the reference therein.

Let D be the differentiation operator on $H(\mathbb{D})$, that is, $Df(z) = f'(z)$. For $f \in H(\mathbb{D})$, the products of composition and differentiation operators DC_φ and $C_\varphi D$ are defined, respectively, by

$$\begin{aligned} DC_\varphi(f) &= (f \circ \varphi)' = f'(\varphi)\ \varphi', \\ C_\varphi D(f) &= f'(\varphi), \quad f \in H(\mathbb{D}). \end{aligned} \tag{1.8}$$

The boundedness and compactness of DC_φ on the Hardy space were investigated by Hibschweiler and Portnoy in [11] and by Ohno in [12]. The case of the Bergman spaces was studied in [11], while the case of the Hilbert-Bergman space was studied by Stević in [13]. In [14], Li and Stević studied the boundedness and compactness of the operator DC_φ on α-Bloch spaces, while in [15] they studied these operators between H^∞ and α-Bloch spaces. The boundedness and compactness of the operator DC_φ from mixed-norm spaces to α-Bloch spaces was studied by Li and Stević in [16]. Norm and essential norm of the operator DC_φ from α-Bloch spaces to weighted-type spaces were studied by Stević in [17]. Some related operators can be also found in [18–21]. For some other papers on products of linear operators on spaces of holomorphic functions, mostly integral-type and composition operators, see, for example, the following papers by Li and Stević: [5, 22–30].

Motivated basically by papers [14, 15], in this paper, we study the operators DC_φ and $C_\varphi D$ from $Q_K(p,q)$ space to $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu,0}$ spaces. Some sufficient and necessary conditions for the boundedness and compactness of these operators are given.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Main Results and Proofs

In this section we give our main results and proofs. For this purpose, we need some auxiliary results. The following lemma can be proved in a standard way (see, e.g, in [9, Proposition 3.11]). A detailed proof, can be found, for example, in [31].

Lemma 2.1. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that μ is normal, $p > 0$, $q > -2$. Then DC_φ(or $C_\varphi D$) $: Q_K(p,q) \to \mathcal{B}_\mu$ is compact if and only if DC_φ(or $C_\varphi D$) $: Q_K(p,q) \to \mathcal{B}_\mu$ is bounded and for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $Q_K(p,q)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, one has $\|DC_\varphi f_n\|_{\mathcal{B}_\mu} \to 0$ (or $\|C_\varphi D f_n\|_{\mathcal{B}_\mu} \to 0$) as $n \to \infty$.*

The following lemma can be proved similarly as [32], one omits the details (see also [2, 4]).

Lemma 2.2. *A closed set K in $\mathcal{B}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in K} \mu(|z|)\left|f'(z)\right| = 0. \tag{2.1}$$

Now one is in a position to state and prove the main results of this paper.

Theorem 2.3. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that μ is normal, $p > 0$, $q > -2$, and K is a nonnegative nondecreasing function on $[0,\infty)$ such that*

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}\left(\log\frac{1}{1-r}\right)^{\chi_{-1}(q)} r\,dr < \infty, \tag{2.2}$$

where $\chi_O(x)$ denote the characteristic function of the set O. Then $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}} < \infty\,, \qquad \sup_{z\in\mathbb{D}} \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} < \infty. \tag{2.3}$$

Proof. Suppose that the conditions in (2.3) hold. Then for any $z \in \mathbb{D}$ and $f \in Q_K(p,q)$,

$$\begin{aligned}
\mu(|z|)\left|\left(DC_\varphi f\right)'(z)\right| &= \mu(|z|)\left|\left(f'(\varphi)\varphi'\right)'(z)\right| \\
&\le \mu(|z|)\left|\varphi'(z)\right|^2\left|f''(\varphi(z))\right| + \mu(|z|)\left|\varphi''(z)\right|\left|f'(\varphi(z))\right| \\
&\le \frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}}\|f\|_{\mathcal{B}^{(q+2)/p}} + \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}}\|f\|_{\mathcal{B}^{(q+2)/p}} \\
&\le \frac{C\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}}\|f\|_{Q_K(p,q)} + \frac{C\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}}\|f\|_{Q_K(p,q)},
\end{aligned} \tag{2.4}$$

where we have used the fact that $\|f\|_{\mathcal{B}^{(q+2)/p}} \le \|f\|_{Q_K(p,q)}$, as well as the following well-known characterization for α-Bloch functions (see, e.g., [33])

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\left|\varphi'(z)\right| \asymp \left|\varphi'(0)\right| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{(1+\beta)}\left|\varphi''(z)\right|. \tag{2.5}$$

Taking the supremum in (2.4) for $z \in \mathbb{D}$, then employing (2.3) we obtain that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded.

Conversely, suppose that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded, that is, there exists a constant C such that $\|DC_\varphi f\|_{\mathcal{B}_\mu} \le C\|f\|_{Q_K(p,q)}$ for all $f \in Q_K(p,q)$. Taking the functions $f(z) \equiv z$, and $f(z) \equiv z^2$, which belong to $Q_K(p,q)$, we get

$$\sup_{z\in\mathbb{D}}\mu(|z|)\left|\varphi''(z)\right| < \infty, \tag{2.6}$$

$$\sup_{z\in\mathbb{D}}\mu(|z|)\left|\left(\varphi'(z)\right)^2 + \varphi''(z)\varphi(z)\right| < \infty. \tag{2.7}$$

From (2.6), (2.7), and the boundedness of the function $\varphi(z)$, it follows that

$$\sup_{z\in\mathbb{D}}\mu(|z|)\left|\varphi'(z)\right|^2<\infty. \tag{2.8}$$

For $w\in\mathbb{D}$, let

$$f_w(z)=\frac{1-|w|^2}{(1-z\overline{w})^{(q+2)/p}}. \tag{2.9}$$

By some direct calculation we have that

$$\begin{aligned} f_w'(w)&=\frac{q+2}{p}\frac{\overline{w}}{(1-|w|^2)^{((q+2)/p)}},\\ f_w''(w)&=\left(\frac{q+2}{p}\right)\left(\frac{q+2}{p}+1\right)\frac{\overline{w}^2}{(1-|w|^2)^{(q+2)/p+1}}. \end{aligned} \tag{2.10}$$

From [8], we know that $f_w\in Q_K(p,q)$, for each $w\in\mathbb{D}$, moreover there is a positive constant C such that $\sup_{w\in\mathbb{D}}\|f_w\|_{Q_K(p,q)}\le C$. Hence, we have

$$\begin{aligned} C\|DC_\varphi\|_{Q_K(p,q)\to\mathcal{B}_\mu}&\ge\|DC_\varphi f_{\varphi(\lambda)}\|_{\mathcal{B}_\mu}\\ &\ge-\frac{q+2}{p}\frac{q+2+p}{p}\frac{\mu(|\lambda|)\left|\varphi'(\lambda)\right|^2\left|\varphi(\lambda)\right|^2}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q+p)/p}}\\ &\quad+\frac{q+2}{p}\frac{\mu(|\lambda|)\left|\varphi''(\lambda)\right|\left|\varphi(\lambda)\right|}{\left(1-\left|\varphi(\lambda)\right|^2\right)^{(2+q)/p}}, \end{aligned} \tag{2.11}$$

for $\lambda\in\mathbb{D}$. Therefore, we obtain

$$\frac{\mu(|\lambda|)\left|\varphi''(\lambda)\right|\left|\varphi(\lambda)\right|}{\left(1-\left|\varphi(\lambda)\right|^2\right)^{(2+q)/p}}\le C\|DC_\varphi\|_{Q_K(p,q)\to\mathcal{B}_\mu}+\frac{q+2+p}{p}\frac{\mu(|\lambda|)\left|\varphi'(\lambda)\right|^2\left|\varphi(\lambda)\right|^2}{\left(1-\left|\varphi(\lambda)\right|^2\right)^{(2+q+p)/p}}. \tag{2.12}$$

Next, for $w\in\mathbb{D}$, let

$$g_w(z)=\frac{\left(1-|w|^2\right)^2}{(1-z\overline{w})^{(q+2)/p+1}}-\frac{(q+2)/p+1}{(q+2)/p}\frac{1-|w|^2}{(1-z\overline{w})^{(q+2)/p}}. \tag{2.13}$$

Then from [8], we see that $g_w \in Q_K(p,q)$ and $\sup_{w\in\mathbb{D}}\|g_w\|_{Q_K(p,q)} < \infty$. Since

$$g'_{\varphi(\lambda)}(\varphi(\lambda)) = 0, \qquad \left|g''_{\varphi(\lambda)}(\varphi(\lambda))\right| = \frac{q+2+p}{p}\frac{|\varphi(\lambda)|^2}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q+p)/p}}, \tag{2.14}$$

we have

$$\infty > C\|DC_\varphi\|_{Q_K(p,q)\to\mathcal{B}_\mu} \geq \|DC_\varphi g_{\varphi(\lambda)}\|_{\mathcal{B}_\mu} \geq \frac{q+2+p}{p}\frac{\mu(|\lambda|)|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q+p)/p}}. \tag{2.15}$$

Thus

$$\begin{aligned}\sup_{|\varphi(\lambda)|>1/2}\frac{\mu(|\lambda|)|\varphi'(\lambda)|^2}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q+p)/p}} &\leq \sup_{|\varphi(\lambda)|>1/2} 4\frac{\mu(|\lambda|)|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q+p)/p}} \\ &\leq C\|DC_\varphi\|_{Q_K(p,q)\to\mathcal{B}_\mu} < \infty.\end{aligned} \tag{2.16}$$

Inequality (2.8) gives

$$\sup_{|\varphi(\lambda)|\leq 1/2}\frac{\mu(|\lambda|)|\varphi'(\lambda)|^2}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q+p)/p}} \leq \frac{4^{(2+q+p)/p}}{3^{(2+q+p)/p}}\sup_{|\varphi(\lambda)|\leq 1/2}\mu(|\lambda|)|\varphi'(\lambda)|^2 < \infty. \tag{2.17}$$

Therefore, the first inequality in (2.3) follows from (2.16) and (2.17). From (2.12) and (2.15), we obtain

$$\sup_{\lambda\in\mathbb{D}}\frac{\mu(|\lambda|)|\varphi''(\lambda)||\varphi(\lambda)|}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q)/p}} < \infty. \tag{2.18}$$

Equations (2.6) and (2.18) imply

$$\sup_{|\varphi(\lambda)|>1/2}\frac{\mu(|\lambda|)|\varphi''(\lambda)|}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q)/p}} \leq 2\sup_{|\varphi(\lambda)|>1/2}\frac{\mu(|\lambda|)|\varphi''(\lambda)||\varphi(\lambda)|}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q)/p}} < \infty, \tag{2.19}$$

$$\sup_{|\varphi(\lambda)|\leq 1/2}\frac{\mu(|\lambda|)|\varphi''(\lambda)|}{\left(1-|\varphi(\lambda)|^2\right)^{(2+q)/p}} \leq \frac{4^{(2+q)/p}}{3^{(2+q)/p}}\sup_{|\varphi(\lambda)|\leq 1/2}\mu(|\lambda|)|\varphi''(\lambda)| < \infty. \tag{2.20}$$

Inequality (2.19) together with (2.20) implies the second inequality of (2.3). This completes the proof of Theorem 2.3. □

Theorem 2.4. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that μ is normal, $p > 0$, $q > -2$ and K is a nonnegative nondecreasing function on $[0,\infty)$ such that (2.2) holds. Then $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is compact if and only if $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}} = 0\ , \qquad \lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} = 0. \tag{2.21}$$

Proof. Suppose that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded and (2.21) holds. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $Q_K(p,q)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{Q_K(p,q)} < \infty$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\frac{\mu(|z|)\left|\varphi\prime(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+p+q)/p}} < \varepsilon, \qquad \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} < \varepsilon, \tag{2.22}$$

when $\delta < |\varphi(z)| < 1$. Since $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded, then from the proof of Theorem 2.3 we have

$$M_1 := \sup_{z\in\mathbb{D}} \mu(|z|)\left|\varphi''(z)\right| < \infty, \qquad M_2 := \sup_{z\in\mathbb{D}} \mu(|z|)\left|\varphi'(z)\right|^2 < \infty. \tag{2.23}$$

Let $K = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$. Then, we have

$$\begin{aligned}
\left\|DC_\varphi f_k\right\|_{\mathcal{B}_\mu} &= \sup_{z\in\mathbb{D}} \mu(|z|)\left|\left(DC_\varphi f_k\right)'(z)\right| + \left|f_k'(\varphi(0))\right|\left|\varphi'(0)\right| \\
&= \sup_{z\in\mathbb{D}} \mu(|z|)\left|\left(\varphi' f_k'(\varphi)\right)'(z)\right| + \left|f_k'(\varphi(0))\right|\left|\varphi'(0)\right| \\
&\le \sup_{z\in\mathbb{D}} \mu(|z|)\left|\varphi'(z)\right|^2\left|f_k''(\varphi(z))\right| + \sup_{z\in\mathbb{D}} \mu(|z|)\left|\varphi''(z)\right|\left|f_k'(\varphi(z))\right| \\
&\quad + \left|f_k'(\varphi(0))\right|\left|\varphi'(0)\right| \\
&\le \sup_{K} \mu(|z|)\left|\varphi'(z)\right|^2\left|f_k''(\varphi(z))\right| + \sup_{K} \mu(|z|)\left|\varphi''(z)\right|\left|f_k'(\varphi(z))\right| \\
&\quad + \sup_{\mathbb{D}\setminus K} \mu(|z|)\left|\varphi''(z)\right|^2\left|f_k''(\varphi(z))\right| + \sup_{\mathbb{D}\setminus K} \mu(|z|)\left|\varphi''(z)\right|\left|f_k'(\varphi(z))\right| \\
&\quad + \left|f_k'(\varphi(0))\right|\left|\varphi'(0)\right| \\
&\le \sup_{K} \mu(|z|)\left|\varphi'(z)\right|^2\left|f_k''(\varphi(z))\right| + \sup_{K} \mu(|z|)\left|\varphi''(z)\right|\left|f_k'(\varphi(z))\right| \\
&\quad + \left|f_k'(\varphi(0))\right|\ \left|\varphi'(0)\right| + C\sup_{\mathbb{D}\setminus K} \frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+p+q)/p}}\left\|f_k\right\|_{Q_K(p,q)}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\mathbb{D}\setminus K}\frac{\mu(|z|)|\varphi''(z)|}{\left(1-|\varphi(z)|^2\right)^{(2+p)/p}}\|f_k\|_{Q_K(p,q)}\\
&\le M_2\sup_K|f_k''(\varphi(z))|+M_1\sup_K|f_k'(\varphi(z))|+2C\varepsilon\|f_k\|_{Q_K(p,q)}\\
&\quad+|f_k'(\varphi(0))||\varphi'(0)|.
\end{aligned}
\tag{2.24}
$$

The assumption that $f_k \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$ along with Cauchy's estimate give that $f_k' \to 0$ and $f_k'' \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$. Letting $k \to \infty$ in (2.24) and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k\to\infty}\|DC_\varphi f_k\|_{\mathcal{B}_\mu} = 0$. Applying Lemma 2.1, the result follows.

Now, suppose that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is compact. Then it is clear that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist then condition (2.21) is vacuously satisfied). Let

$$
f_k(z) = \frac{1-|\varphi(z_k)|^2}{\left(1-\overline{\varphi(z_k)}z\right)^{(q+2)/p}}. \tag{2.25}
$$

Then, $\sup_{k\in\mathbb{N}}\|f_k\|_{Q_K(p,q)} < \infty$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is compact, by Lemma 2.1 we have $\lim_{k\to\infty}\|DC_\varphi f_k\|_{\mathcal{B}_\mu} = 0$. On the other hand, from (2.11) we have

$$
C\|DC_\varphi f_k\|_{\mathcal{B}_\mu} \ge \left|-\frac{2+q}{p}\frac{2+p+q}{p}\frac{\mu(|z_k|)|\varphi'(z_k)|^2|\varphi(z_k)|^2}{\left(1-|\varphi(z_k)|^2\right)^{(2+q+p)/p}}+\frac{2+q}{p}\frac{\mu(|z_k|)|\varphi''(z_k)||\varphi(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^{(2+q)/p}}\right|, \tag{2.26}
$$

which implies that

$$
\lim_{|\varphi(z_k)|\to 1}\frac{2+q+p}{p}\frac{\mu(|z_k|)|\varphi'(z_k)|^2|\varphi(z_k)|^2}{\left(1-|\varphi(z_k)|^2\right)^{(2+q+p)/p}} = \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)|\varphi''(z_k)||\varphi(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^{(2+q)/p}}, \tag{2.27}
$$

if one of these two limits exists.

Next, for $k \in \mathbb{N}$, set

$$
g_k(z) = \frac{\left(1-|\varphi(z_k)|^2\right)^2}{\left(1-\overline{\varphi(z_k)}z\right)^{(q+2)/p+1}} - \frac{q+2+p}{q+2}\frac{1-|\varphi(z_k)|^2}{\left(1-\overline{\varphi(z_k)}z\right)^{(q+2)/p}}. \tag{2.28}
$$

Then $(g_k)_{k\in\mathbb{N}}$ is a sequence in $Q_K(p,q)$. Notice that $g_k'(\varphi(z_k)) = 0$,

$$\left|g_k''(\varphi(z_k))\right| = \frac{2+q+p}{p}\frac{\left|\varphi(z_k)\right|^2}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(2+q+p)/p}}, \tag{2.29}$$

and g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is compact, we have $\lim_{k\to\infty}\|DC_\varphi g_k\|_{\mathcal{B}_\mu} = 0$. On the other hand, we have

$$\frac{2+q+p}{p}\frac{\mu(|z_k|)\left|\varphi'(z_k)\right|^2\left|\varphi(z_k)\right|^2}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(2+q+p)/p}} \le \|DC_\varphi g_k\|_{\mathcal{B}_\mu}. \tag{2.30}$$

Therefore

$$\lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)\left|\varphi'(z_k)\right|^2}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(2+q+p)/p}} = \lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)\left|\varphi'(z_k)\right|^2\left|\varphi(z_k)\right|^2}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(2+q+p)/p}} = 0. \tag{2.31}$$

This along with (2.27) implies

$$\lim_{|\varphi(z_k)|\to 1}\frac{\mu(|z_k|)\left|\varphi''(z_k)\right|}{\left(1-\left|\varphi(z_k)\right|^2\right)^{(2+p)/p}} = 0. \tag{2.32}$$

From the last two equalities, the desired result follows. □

Theorem 2.5. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that μ is normal, $p > 0$, $q > -2$ and K is a nonnegative nondecreasing function on $[0,\infty)$ such that* (2.2) *holds. Then $DC_\varphi : Q_K(p,q) \to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}} = 0, \qquad \lim_{|z|\to 1}\frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} = 0. \tag{2.33}$$

Proof. Sufficiency. Let $f \in Q_K(p,q)$. By the proof of Theorem 2.3 we have

$$\mu(|z|)\left|\left(DC_\varphi f\right)'(z)\right| \le C\left(\frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}} + \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}}\right)\|f\|_{Q_K(p,q)}. \tag{2.34}$$

Taking the supremum in (2.34) over all $f \in Q_K(p,q)$ such that $\|f\|_{Q_K(p,q)} \leq 1$, then letting $|z| \to 1$, we get

$$\lim_{|z|\to 1} \sup_{\|f\|_{Q_K(p,q)}\leq 1} \mu(|z|)\left|\left(DC_\varphi f\right)'(z)\right| = 0. \tag{2.35}$$

From which by Lemma 2.2 we see that the operator $DC_\varphi : Q_K(p,q) \to \mathcal{B}_{\mu,0}$ is compact. Necessity. Assume that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_{\mu,0}$ is compact. By taking the function given by $f(z) \equiv z$ and using the boundedness of $DC_\varphi : Q_K(p,q) \to \mathcal{B}_{\mu,0}$, we get

$$\lim_{|z|\to 1} \mu(|z|)\left|\varphi''(z)\right| = 0. \tag{2.36}$$

From this, by taking the test function $f(z) \equiv z^2$ and using the boundedness of $DC_\varphi : Q_K(p,q) \to \mathcal{B}_{\mu,0}$ it follows that

$$\lim_{|z|\to 1} \mu(|z|)\left|\varphi'(z)\right|^2 = 0. \tag{2.37}$$

If $\|\varphi\|_\infty < 1$, from (2.36) and (2.37), we obtain that

$$\begin{aligned}
\lim_{|z|\to 1} \frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q+p)/p}} &\leq \frac{1}{\left(1-\|\varphi\|_\infty^2\right)^{(2+q+p)/p}} \lim_{|z|\to 1} \mu(|z|)\left|\varphi'(z)\right|^2 = 0, \\
\lim_{|z|\to 1} \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} &\leq \frac{1}{\left(1-\|\varphi\|_\infty^2\right)^{(2+q)/p}} \lim_{|z|\to 1} \mu(|z|)\left|\varphi''(z)\right| = 0,
\end{aligned} \tag{2.38}$$

from which the result follows in this case.

Assume that$\|\varphi\|_\infty = 1$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$. From the compactness of $DC_\varphi : Q_K(p,q) \to \mathcal{B}_{\mu,0}$ we see that $DC_\varphi : Q_K(p,q) \to \mathcal{B}_\mu$ is compact. From Theorem 2.4 we get

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+p+q)/p}} = 0, \tag{2.39}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} = 0. \tag{2.40}$$

From (2.36) and (2.40), we have that for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} < \varepsilon, \tag{2.41}$$

when $r < |\varphi(z)| < 1$, and there exists a $\sigma \in (0,1)$ such that $\mu(|z|)|\varphi''(z)| \le \varepsilon(1-r^2)^{(2+q)/p}$ when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$, and $r < |\varphi(z)| < 1$, we have

$$\frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} < \varepsilon. \tag{2.42}$$

On the other hand, if $\sigma < |z| < 1$, and $|\varphi(z)| \le r$, we obtain

$$\frac{\mu(|z|)\left|\varphi''(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+q)/p}} < \frac{1}{(1-r^2)^{(2+q)/p}}\mu(|z|)\left|\varphi''(z)\right| < \varepsilon. \tag{2.43}$$

Inequality (2.42) together with (2.43) gives the second equality of (2.33). Similarly to the above arguments, by (2.37) and (2.39) we get the first equality of (2.33). The proof is completed. □

From the above three theorems, we get the following corollary (see [14]).

Corollary 2.6. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements hold.*

(i) *$DC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded if and only if*

$$\operatorname*{Sup}_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^2} < \infty, \qquad \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)\left|\varphi''(z)\right|}{1-\left|\varphi(z)\right|^2} < \infty; \tag{2.44}$$

(ii) *$DC_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $DC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)\left|\varphi'(z)\right|^2}{\left(1-|\varphi(z)|^2\right)^2} = 0\ , \qquad \lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)\left|\varphi''(z)\right|}{1-\left|\varphi(z)\right|^2} = 0; \tag{2.45}$$

(iii) *$DC_\varphi : \mathcal{B} \to \mathcal{B}_0$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left|\varphi'(z)\right|^2}{\left(1-\left|\varphi(z)\right|^2\right)^2} = 0, \qquad \lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left|\varphi''(z)\right|}{1-\left|\varphi(z)\right|^2} = 0. \tag{2.46}$$

Similarly to the proofs of Theorems 2.3–2.5, we can get the following result. We omit the proof.

Theorem 2.7. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that μ is normal, $p > 0$, $q > -2$ and K is a nonnegative nondecreasing function on $[0,\infty)$ such that* (2.2) *holds. Then the following statements hold.*

(i) $C_\varphi D : Q_K(p,q) \to \mathcal{B}_\mu$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+p+q)/p}} < \infty; \tag{2.47}$$

(ii) $C_\varphi D : Q_K(p,q) \to \mathcal{B}_\mu$ *is compact if and only if* $C_\varphi D : Q_K(p,q) \to \mathcal{B}_\mu$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+p+q)/p}} = 0; \tag{2.48}$$

(iii) $C_\varphi D : Q_K(p,q) \to \mathcal{B}_{\mu,0}$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(|z|)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(2+p+q)/p}} = 0. \tag{2.49}$$

From Theorem 2.7 we get the following corollary.

Corollary 2.8. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following statements hold.*

(i) $C_\varphi D : \mathcal{B} \to \mathcal{B}$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^2} < \infty; \tag{2.50}$$

(ii) $C_\varphi D : \mathcal{B} \to \mathcal{B}$ *is compact if and only if* $C_\varphi D : \mathcal{B} \to \mathcal{B}$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^2} = 0; \tag{2.51}$$

(iii) $C_\varphi D : \mathcal{B} \to \mathcal{B}_0$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\left(1-|(z)|^2\right)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^2} = 0. \tag{2.52}$$

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