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Research Article

Mittag-Leffler Stability Theorem for Fractional Nonlinear Systems with Delay

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Fractional calculus started to play an important role for analysis of the evolution of the nonlinear dynamical systems which are important in various branches of science and engineering. In this line of taught in this paper we studied the stability of fractional order nonlinear time-delay systems for Caputo's derivative, and we proved two theorems for Mittag-Leffler stability of the fractional nonlinear time delay systems.

1. Introduction

During the last decade the fractional calculus [1–5] has gained importance in both theoretical and applied aspects of several branches of science and engineering [6–15]. However there are several open problems in this area. One of them is related to the stability of the fractional systems in the presence of the delay. Both delay and fractional systems are describing separately the evolution of the dynamical systems involving memory effect. Particularly, for the time delay case we mention the seminal works on the applicability of Lyapunov's second method [16, 17].

Since 1950s different types of the Lyapunov functions have been proposed for the stability analysis of delay systems, see the pioneering works of Razumikhin [16] and Krasovski [17]. Whereas Razumikhin [16] used the Lyapunov-type functions $V(x(t))$ depending on the current value $x(\mathrm{t})$ of the solution, Krasovski [17] proposed to use functionals $V(x_t)$ depending on the whole solution segment x_t, that is, the true state of the delay system. The reader can see [18] for more details.

Some literatures published about stability of fractional-order linear time delay systems can be found in [19, 20]. In the base of Lyapunov's second method, some work has been done in the field of stability of fractional order nonlinear systems without delay [21–23]. Razumikhin theorem for the fractional nonlinear time-delay systems was extended recently in [24]. However few attempts were done in order to combine these two powerful concepts and to observe what the benefits of this combination are.

The main aim of this paper is to establish the Mittag-Leffler stability theorem for fractional order nonlinear time-delay systems.

The organization of the manuscript is given belo.

In Section 2 some basic definitions of fractional calculus used in this paper are mentioned. Section 3 introduces briefly the fractional nonlinear time-delay systems. Section 4 deals with the Mittag-Leffler stability theorem when both fractional derivatives and delay are taken into account.

Finally, Section 5 is devoted to our conclusions.

2. Fractional Calculus

2.1. Caputo and Riemann-Liouville Fractional Derivatives

In the fractional calculus the Riemann-Liouville and the Caputo fractional derivatives are defined, respectively [1, 2],

$$\begin{gathered}
{}_{t_0}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^{q+1-n}} ds \right) \quad (n-1 < q \leq n), \\
{}_{t_0}^{c}D_t^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{q+1-n}} ds \quad (n-1 < q \leq n),
\end{gathered} \tag{2.1}$$

where $x(t)$ is an arbitrary differentiable function, $n \in \mathbb{N}$, and ${}_{t_0}D_t^q$ and ${}_{t_0}^{c}D_t^q$ are the Riemann-Liouville and Caputo fractional derivatives of order q on $[t_0, t]$, respectively, and $\Gamma(\cdot)$ denotes the Gamma function.

For $0 < q \leq 1$ we have

$$\begin{gathered}
{}_{t_0}D_t^q x(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \left(\int_{t_0}^{t} \frac{x(s)}{(t-s)^{q}} ds \right) \quad (0 < q \leq 1), \\
{}_{t_0}^{c}D_t^q x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{x'(s)}{(t-s)^{q}} ds \quad (0 < q \leq 1).
\end{gathered} \tag{2.2}$$

Some properties of the Riemann -Liouville and the Caputo derivatives are recalled below [1, 2].

When $0 < q < 1$, we have

$$ {}_{t_0}^{c}D_t^q x(t) = {}_{t_0}D_t^q x(t) - \frac{x(t_0)}{\Gamma(1-q)} (t-t_0)^{-q}. \tag{2.3}$$

In particular, if $x(t_0) = 0$, we obtain

$$ {}_{t_0}^{c}D_t^q x(t) = {}_{t_0}D_t^q x(t) . \tag{2.4}$$

2.2. *Mittag-Leffler Function*

Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solutions of fractional-order systems is the Mittag-Leffler function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \tag{2.5}$$

where $\alpha > 0$ and $z \in \mathbb{C}$. The Mittag-Leffler function with two parameters appears most frequently and has the following form:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \tag{2.6}$$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$. For $\beta = 1$ we obtain $E_{\alpha,1}(z) = E_\alpha(z)$. Also we mention that $E_{1,1}(z) = e^z$.

3. Fractional Nonlinear Time-Delay System

Let $C([a,b], \mathbb{R}^n)$ be the set of continuous functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many situations, one may wish to identify a *maximum time delay* r of a system. In this case, we are often interested in the set of continuous function mapping $[-r,0]$ to $\mathbb{R}^n$, for which we simplify the notation to $C = C([-r,0], \mathbb{R}^n)$. For any $A > 0$ and any continuous function of time $\psi \in C([t_0 - r, t_0 + A], \mathbb{R}^n)$, $t_0 \le t \le t_0 + A$, let $\psi_t(\theta) \in C$ be a segment of function ψ defined as $\psi_t(\theta) = \psi(t + \theta)$, $-r \le \theta \le 0$.

Consider the Caputo fractional nonlinear time-delay system

$$ {}_{t_0}^{c}D_t^\alpha x(t) = f(t, x_t), \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $0 < \alpha \le 1$, and $f : \mathbb{R} \times C \to \mathbb{R}^n$. Equation (3.1) indicates the Caputo derivatives of the state variable x on $[t_0, t]$ and $x(\xi)$ for $t - r \le \xi \le t$. As such, to determine the future evolution of the state, it is necessary to specify the initial state variables $x(t)$ in a time interval of length r, say, from $t_0 - r$ to t_0, that is,

$$x_{t_0} = \varphi, \tag{3.2}$$

where $\phi \in C$ is given. In other words we have $x(t_0 + \theta) = \phi(\theta), -r \le \theta \le 0$.

Throughout the paper we will use the Euclidean norm for vectors denoted by $\|\cdot\|$. The space of continuous initial functions $C([-r,0],\mathbb{R}^n)$is provided with the supremum norm

$$\|\varphi\|_\infty = \max_{\theta\in[-r,0]}\|\varphi(\theta)\|. \tag{3.3}$$

4. Mittag-Leffler Stability of Fractional Nonlinear Time Delay Systems

As in the study of systems without delay, an effective method for determining the stability of a time-delay system is the Lyapunov method. Since in a time-delay system the "state" at time t required the value of $x(t)$ in the interval $[t-r,t]$, that is, x_t, it is natural to expect that for a time-delay system, corresponding the Lyapunov function be a functional $V(t,x_t)$ depending on x_t, which also should measure the deviation of x_t from the trivial solution 0.

Definition 4.1. Let $V(t,\phi)$ be differentiable, and let $x_t(\tau,\varphi)$ be the solution of (3.1) at time t with initial condition $x_\tau = \varphi$. Then, we calculate the Riemann-Liouville and the Caputo derivatives of $V(t,x_t)$ with respect to t and evaluate it at $t=\tau$ as follows, respectively,

$$\begin{aligned} {}_{t_0}D_t^q V(\tau,\varphi) &= {}_{t_0}D_t^q V(t,x_t(\tau,\varphi))\Big|_{t=\tau,x_t=\varphi} = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\left(\int_{t_0}^t \frac{V(s,x_s)}{(t-s)^q}ds\right)\Bigg|_{t=\tau,x_t=\varphi} \\ {}_{t_0}^{c}D_t^q V(\tau,\varphi) &= {}_{t_0}^{c}D_t^q V(t,x_t(\tau,\varphi))\Big|_{t=\tau,x_t=\varphi} = \frac{1}{\Gamma(1-q)}\int_{t_0}^t \frac{V'(s,x_s)}{(t-s)^q}ds\Bigg|_{t=\tau,x_t=\varphi}. \end{aligned} \tag{4.1}$$

where $0<q\le 1$.

Definition 4.2 (exponential stability [25]). The solution of (3.1) is said to be exponential stable if there exist $b>0$ and $a\ge 0$ such that for every solution $x(t,\varphi)$, $\varphi\in C([-r,0],\mathbb{R}^n)$ the following exponential estimate holds:

$$\|x(t,\varphi)\| \le a\|\varphi\|_\infty e^{(-bt)}. \tag{4.2}$$

Definition 4.3 (Mittag-Leffler stability). The solution of (3.1) is said to be Mittag-Leffler stable if

$$\|x(t,\varphi)\| \le \left\{m(\|\varphi\|_\infty)E_\alpha(-\lambda(t-t_0)^\alpha)\right\}^b, \tag{4.3}$$

where $\alpha\in(0,1)$, $\lambda\ge 0$, $b>0$, $m(0)=0$, $m(x)\ge 0$, and $m(x)$ is locally the Lipschitz on $x\in B\subset\mathbb{R}^n$ with the Lipschitz constant m_0.

Theorem 4.4. *Suppose that α_1,α_2, and β are positive constants and $V,w : C([-r,0],\mathbb{R}^n)\to\mathbb{R}$ are continuous functionals. If the following conditions are satisfied for all $\varphi\in C([-r,0],\mathbb{R}^n)$:*

(1) $\alpha_1\|\varphi(0)\|^2 \le V(\varphi)\le \alpha_2\|\varphi\|_\infty^2$,

(2) $\beta V(\varphi)\le w(\varphi)$,

(3) $V(x_t(\varphi))$ *has fractional derivative of order α for all $\phi\in C([-r,0],\mathbb{R}^n)$,*

(4) ${}_{t_0}^{c}D_t^{\gamma} V(x_t(\varphi)) \leq -w(x_t(\varphi))$ *for all* $t \geq t_0$ *and* $0 < \gamma \leq 1$.

then

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{\infty}\left(E_{\gamma}(-\beta t^{\gamma})\right)^{1/2}, \quad t \geq t_0,\ \varphi \in C([-r,0],\mathbb{R}^n). \tag{4.4}$$

That is the solution of (3.1) *is Mittag-Leffler stable.*

Proof. Given any $\varphi \in C([-r,0],\mathbb{R}^n)$, condition (2) implies that

$$-w(\varphi) \leq -\beta V(\varphi). \tag{4.5}$$

From (4.5) and condition (4), we have

$${}_{t_0}^{c}D_t^{\gamma} V(x_t(\varphi)) + \beta V(x_t(\varphi)) \leq 0, \tag{4.6}$$

or

$${}_{t_0}^{c}D_t^{\gamma} V(x_t(\varphi)) + \beta V(x_t(\varphi)) + M(t) = 0, \tag{4.7}$$

where $M(t) \geq 0$ for $t \geq 0$.

From (3.2), we have $V(x_{t_0}(\varphi)) = V(\phi)$. Then, the solution of (4.7) with initial condition $V(x_{t_0}(\varphi)) = V(\varphi)$ is given by

$$\begin{aligned} V(x_t(\varphi)) &= V(\varphi)E_{\gamma}(-\beta t^{\gamma}) - \int_{t_0}^{t}(t-\tau)^{\gamma-1}E_{\gamma,\gamma}(-\beta(t-\tau)^{\gamma})M(\tau)d\tau \\ &= V(\varphi)E_{\gamma}(-\beta t^{\gamma}) - M(t) * \left(t^{\gamma-1}E_{\gamma,\gamma}(-\beta t^{\gamma})\right), \end{aligned} \tag{4.8}$$

where $*$ is convolution operator. Since both $t^{\gamma-1}$and $E_{\gamma,\gamma}(-\beta t^{\gamma})$ are nonnegative functions, it follows that

$$V(x_t(\varphi)) \leq V(\varphi)E_{\gamma}(-\beta t^{\gamma}), \quad t \geq t_0. \tag{4.9}$$

Then conditions (1) and (2) yield

$$\alpha_1\|x(t,\varphi)\|^2 \leq V(x_t(\varphi)) \leq V(\varphi)E_{\gamma}(-\beta t^{\gamma}) \leq \alpha_2\|\varphi\|_{\infty}^2 E_{\gamma}(-\beta t^{\gamma}). \tag{4.10}$$

Comparing the left and the right hand sides, we have

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{\infty}\left(E_{\gamma}(-\beta t^{\gamma})\right)^{1/2}, \quad t \geq t_0,\ \varphi \in C([-r,0],\mathbb{R}^n). \tag{4.11}$$

□

Lemma 4.5. *Let* $\gamma \in (0,1)$ *and* $V(0) \geq 0$*, then*

$$ {}_{t_0}^{c}D_t^{\gamma} V(t) \leq {}_{t_0}D_t^{\gamma} V(t). \tag{4.12}$$

Proof. By using (2.3) we have ${}_{t_0}^{c}D_t^{\gamma} V(t) = {}_{t_0}D_t^{\gamma} V(t) - V(t_0)(t-t_0)^{-\gamma}/\Gamma(1-\gamma)$. Because $\gamma \in (0,1)$ and $V(t_0) \geq 0$, we get ${}_{t_0}^{c}D_t^{\gamma} V(t) \leq {}_{t_0}D_t^{\gamma} V(t)$. □

Theorem 4.6. *Assume that the assumptions in Theorem 4.4 are satisfied except replacing* ${}_{t_0}^{c}D_t^{\gamma}$ *by* ${}_{t_0}D_t^{\gamma}$*, then one has*

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{\infty}\left(E_{\gamma}(-\beta t^{\gamma})\right)^{1/2}, \quad t \geq t_0,\ \varphi \in C([-r,0],\mathbb{R}^n). \tag{4.13}$$

Proof. It follows from Lemma 4.5 and $V(x_t(\varphi)) \geq 0$ that ${}_{t_0}^{c}D_t^{\gamma} V(x_t(\varphi)) \leq {}_{t_0}D_t^{\gamma} V(x_t(\varphi))$ which implies ${}_{t_0}^{c}D_t^{\gamma} V(x_t(\varphi)) \leq {}_{t_0}D_t^{\gamma} V(x_t(\varphi)) \leq -w(x_t(\varphi))$ for all $t \geq t_0$. Following the same proof as in Theorem 4.4 yields

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{\infty}\left(E_{\gamma}(-\beta t^{\gamma})\right)^{1/2}, \quad t \geq t_0,\ \varphi \in C([-r,0],\mathbb{R}^n). \tag{4.14}$$

□

Corollary 4.7. *For* $\gamma = 1$ *one has exponential stability [25]*

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{\infty}\left(E_{1}(-\beta t^{\gamma})\right)^{1/2} = \sqrt{\frac{\alpha_2}{\alpha_1}}\|\varphi\|_{\infty} e^{(-(1/2)\beta t)}, \quad t \geq t_0,\ \varphi \in C([-r,0],\mathbb{R}^n). \tag{4.15}$$

5. Conclusions

Some complex systems which appear in several areas of science and engineering involve delay and they have memory effect. The combined use of the fractional derivative and delay may lead to a better description of such systems. From these reasons in this paper we obtained the Mittag-Leffler stability theorem in the presence of both the Riemann-Liouville or the Caputo fractional derivatives and delay. The obtained theorems contain particular cases of the fractional calculus versions as well as the time-delay ones.

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