Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 141376, 17 pages
doi:10.1155/2010/141376

Research Article

Convergence Analysis for a System of Equilibrium Problems and a Countable Family of Relatively Quasi-Nonexpansive Mappings in Banach Spaces

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Received 11 April 2010; Revised 9 July 2010; Accepted 16 July 2010

Academic Editor: Simeon Reich

We introduce a new hybrid iterative scheme for finding a common element in the solutions set of a system of equilibrium problems and the common fixed points set of an infinitely countable family of relatively quasi-nonexpansive mappings in the framework of Banach spaces. We prove the strong convergence theorem by the shrinking projection method. In addition, the results obtained in this paper can be applied to a system of variational inequality problems and to a system of convex minimization problems in a Banach space.

1. Introduction

Let E be a real Banach space, and let E^* be the dual of E. Let C be a closed and convex subset of E. Let $\{f_j\}_{j\in\Lambda}$ be bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers and Λ is an arbitrary index set. The system of equilibrium problems is to find $\hat{x} \in C$ such that

$$f_j(\hat{x}, y) \geq 0, \quad \forall y \in C,\ j \in \Lambda. \tag{1.1}$$

If Λ is a singleton, then problem (1.1) reduces to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of the equilibrium problem (1.2) is denoted by $\mathrm{EP}(f)$.

Combettes and Hirstoaga [1] introduced an iterative scheme for finding a common element in the solutions set of problem (1.1) in a Hilbert space and obtained a weak convergence theorem.

In 2004, Matsushita and Takahashi [2] introduced the following algorithm for a relatively nonexpansive mapping T in a Banach space E: for any initial point $x_0 \in C$, define the sequence $\{x_n\}$ by

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTx_n), \quad n \geq 0, \tag{1.3}$$

where J is the duality mapping on E, Π_C is the generalized projection from E onto C, and $\{\alpha_n\}$ is a sequence in $[0,1]$. They proved that the sequence $\{x_n\}$ converges weakly to fixed point of T under some suitable conditions on $\{\alpha_n\}$.

In 2008, Takahashi and Zembayashi [3] introduced the following iterative scheme which is called the shrinking projection method for a relatively nonexpansive mapping T and an equilibrium problem in a Banach space E:

$$\begin{gathered}
x_0 = x \in C, \quad C_0 = C, \\
y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JTx_n), \\
u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0.
\end{gathered} \tag{1.4}$$

They proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap \mathrm{EP}(f)} x_0$ under some appropriate conditions.

2. Preliminaries and Lemmas

Let E be a real Banach space, and let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is said to be *strictly convex* if, for any $x, y \in U$,

$$x \neq y \text{ implies } \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \text{ implies } \left\| \frac{x+y}{2} \right\| < 1 - \delta. \tag{2.2}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. The function $\delta : [0,2] \to [0,1]$ which is called the *modulus of convexity* of E is defined as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}. \tag{2.3}$$

The space E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.4}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.4) is attained uniformly for $x, y \in U$. The *duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\} \tag{2.5}$$

for all $x \in E$. If E is a Hilbert space, then $J = I$, where I is the identity operator. It is also known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subset of E (see [4] for more details).

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.6}$$

for all $x, y \in E$. In a Hilbert space H, we have $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$.

Let C be a closed and convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T is said to be *relatively nonexpansive* [6–8] if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [6, 7]. T is said to be *relatively quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in F(T)$ and $x \in C$. It is obvious that the class of relatively quasi-nonexpansive mappings is more general than the class of relatively nonexpansive mappings. The class of relatively quasi-nonexpansive mappings was studied by many authors (see, for example, [9–12]). Recall that T is closed if

$$x_n \longrightarrow x, \quad Tx_n \longrightarrow y \text{ imply } Tx = y. \tag{2.7}$$

The aim of this paper is to introduce a new hybrid projection algorithm for finding a common element in the solutions set of a system of equilibrium problems and the common fixed points set of an infinitely countable family of closed and relatively quasi-nonexpansive mappings in the frameworks of Banach spaces.

We will need the following lemmas.

Lemma 2.1 (Kamimura and Takahashi [8]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$, $\{y_n\}$be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Let E be a reflexive, strictly convex, and smooth Banach space, and let C be a nonempty, closed, and convex subset of E. The *generalized projection mapping*, introduced by Alber [13], is a mapping $\Pi_C : E \to C$ that assigns to an arbitrary point $x \in E$ the minimum point of the function $\phi(y, x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution of the minimization problem

$$\phi(\overline{x}, x) = \min\{\phi(y, x) : y \in C\}. \tag{2.8}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional ϕ and strict monotonicity of the duality mapping J (see, for instance, [4, 8, 13–15]). In a Hilbert space, Π_C is coincident with the metric projection.

Lemma 2.2 (Alber [13], Kamimura and Takahashi [8]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E, let $x \in E$, and let $z \in C$. Then $z = \Pi_C x$ if and only if*

$$\langle y - z, Jx - Jz \rangle \leq 0, \quad \forall y \in C. \tag{2.9}$$

Lemma 2.3 (Alber [13], Kamimura and Takahashi [8]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y) \quad \forall x \in C,\ y \in E. \tag{2.10}$$

Lemma 2.4 (Qin et al. [16]). *Let E be a uniformly convex, smooth Banach space, and let C be a closed and convex subset of E. Let T be a closed and relatively quasi-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone; that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

Lemma 2.5 (Blum and Oettli [17]). *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.11}$$

Lemma 2.6 (Takahashi and Zembayashi [18]). *Let C be a closed and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). For all $r > 0$ and $x \in E$, define the mapping $T_r^f : E \to C$ as follows:*

$$T_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \tag{2.12}$$

Then, the following statements hold:

(1) T_r^f *is single valued;*

(2) T_r^f *is of firmly nonexpansive type [19]; that is, for all $x, y \in E$,*

$$\left\langle T_r^f x - T_r^f y, JT_r^f x - JT_r^f y \right\rangle \leq \left\langle T_r^f x - T_r^f y, Jx - Jy \right\rangle; \tag{2.13}$$

(3) $F(T_r^f) = EP(f)$;

(4) $EP(f)$ *is closed and convex.*

Lemma 2.7 (Takahashi and Zembayashi [18]). *Let C be a closed and convex subset of a smooth, strictly, and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $r > 0$. Then, for all $x \in E$ and $q \in F(T_r^f)$,*

$$\phi\left(q, T_r^f x\right) + \phi\left(T_r^f x, x\right) \leq \phi(q, x). \tag{2.14}$$

3. Strong Convergence Theorems

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself. Assume that $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} EP(f_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{gathered} y_{n,i} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JT_i x_n), \\ u_{n,i} = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{gathered} \tag{3.1}$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \ldots, M$ are sequences which satisfy the following conditions:

(B1) $\limsup_{n \to \infty} \alpha_n < 1$;

(B2) $\liminf_{n \to \infty} r_{j,n} > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. We divide our proof into six steps as follows.

Step 1. $F \subset C_n$ for all $n \geq 1$.

From Lemma 2.4 we know that $F(T_i)$ is closed, and convex for all $i \geq 1$. From Lemma 2.6(4), we also know that $\mathrm{EP}(f_j)$ is closed and convex for each $j = 1, 2, \ldots, M$. Hence $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} \mathrm{EP}(f_j))$ is a nonempty, closed and convex subset of C. Clearly $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For each $z \in C_k$ and $i \geq 1$, we see that $\phi(z, u_{k,i}) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k \rangle - 2\langle z, Ju_{k,i} \rangle \leq \|x_k\|^2 - \|u_{k,i}\|^2. \tag{3.2}$$

By the construction of the set C_{k+1}, we see that

$$\begin{aligned} C_{k+1} &= \left\{ z \in C_k : \sup_{i \geq 1} \phi(z, u_{k,i}) \leq \phi(z, x_k) \right\} \\ &= \bigcap_{i=1}^{\infty} \{ z \in C_k : \phi(z, u_{k,i}) \leq \phi(z, x_k) \}. \end{aligned} \tag{3.3}$$

Hence C_{k+1} is also closed and convex.

It is obvious that $F \subset C_1 = C$. Now, suppose that $F \subset C_k$ for some $k \in \mathbb{N}$, and let $p \in F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} \mathrm{EP}(f_j))$. Then

$$\begin{aligned} \phi(p, u_{k,i}) &= \phi\left(p, T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{k,i}\right) \\ &\leq \phi\left(p, T_{r_{M-1,n}}^{f_{M-1}} T_{r_{M-2,n}}^{f_{M-2}} \cdots T_{r_{1,n}}^{f_1} y_{k,i}\right) \\ &\;\vdots \\ &\leq \phi\left(p, T_{r_{1,n}}^{f_1} y_{k,i}\right) \\ &\leq \phi(p, y_{k,i}) \\ &= \phi\left(p, J^{-1}(\alpha_k J x_k + (1-\alpha_k) J T_i x_k)\right) \\ &= \|p\|^2 - 2\langle p, \alpha_k J x_k + (1-\alpha_k) J T_i x_k \rangle \\ &\quad + \|\alpha_k J x_k + (1-\alpha_k) J T_i x_k\|^2 \\ &\leq \|p\|^2 - 2\alpha_k \langle p, J x_k \rangle - 2(1-\alpha_k)\langle p, J T_i x_k \rangle \\ &\quad + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_i x_k\|^2 \\ &= \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, T_i x_k) \\ &\leq \phi(p, x_k). \end{aligned} \tag{3.4}$$

Hence $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$.

Step 2. $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

From $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad n \geq 1. \tag{3.5}$$

From Lemma 2.3 we get that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0). \tag{3.6}$$

Combining (3.5) and (3.6), we get that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

Step 3. $\{x_n\}$ is a Cauchy sequence in C.

Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$ for $m > n$, we obtain from Lemma 2.3 that

$$\begin{aligned}\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n}x_0) &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0)\\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \tag{3.7}$$

We see that $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$, which implies with Lemma 2.1 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence. By the completeness of the space E and the closedness of the set C, we can assume that $x_n \to q \in C$ as $n \to \infty$. Moreover, we get that

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{3.8}$$

Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we have for all $i \geq 1$ that

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) \longrightarrow 0. \tag{3.9}$$

Applying Lemma 2.1 to (3.8) and (3.9), we derive

$$\lim_{n\to\infty}\|u_{n,i} - x_n\| = 0, \quad \forall i \geq 1. \tag{3.10}$$

This shows that $u_{n,i} \to q$ as $n \to \infty$ for all $i \geq 1$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain that

$$\lim_{n\to\infty}\|Ju_{n,i} - Jx_n\| = 0, \quad \forall i \geq 1. \tag{3.11}$$

Step 4. $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Denote $\Theta_n^j = T_{r_{j,n}}^{f_j} T_{r_{j-1,n}}^{f_{j-1}} \cdots T_{r_{1,n}}^{f_1}$ for any $j \in \{1, 2, \ldots, M\}$ and $\Theta_n^0 = I$ for all $n \geq 1$. We note that $u_{n,i} = \Theta_n^M y_{n,i}$ for all $i \geq 1$. From (3.4) we observe that

$$\phi\left(p, \Theta_n^{M-1}y_{n,i}\right) \leq \phi\left(p, \Theta_n^{M-2}y_{n,i}\right) \leq \cdots \leq \phi(p, y_{n,i}) \leq \phi(p, x_n), \quad \forall i \geq 1. \tag{3.12}$$

Since $p \in \mathrm{EP}(f_M) = F(T^{f_M}_{r_{M,n}})$ for all $n \geq 1$, it follows from (3.12) and Lemma 2.7 that

$$\begin{aligned}\phi\left(u_{n,i}, \Theta_n^{M-1} y_{n,i}\right) &\leq \phi\left(p, \Theta_n^{M-1} y_{n,i}\right) - \phi(p, u_{n,i}) \\ &\leq \phi(p, x_n) - \phi(p, u_{n,i}).\end{aligned} \tag{3.13}$$

From (3.10) and (3.11), we get that $\lim_{n\to\infty}\phi(u_{n,i}, \Theta_n^{M-1} y_{n,i}) = 0$ for all $i \geq 1$. From Lemma 2.1, we have

$$\lim_{n\to\infty}\left\|u_{n,i} - \Theta_n^{M-1} y_{n,i}\right\| = 0, \quad \forall i \geq 1. \tag{3.14}$$

From (3.10) and (3.14), we have

$$\lim_{n\to\infty}\left\|x_n - \Theta_n^{M-1} y_{n,i}\right\| = 0, \quad \forall i \geq 1, \tag{3.15}$$

and hence,

$$\lim_{n\to\infty}\left\|Jx_n - J\Theta_n^{M-1} y_{n,i}\right\| = 0, \quad \forall i \geq 1. \tag{3.16}$$

Again, since $p \in \mathrm{EP}(f_{M-1}) = F(T^{f_{M-1}}_{r_{M-1,n}})$ for all $n \geq 1$, it follows from (3.12) and Lemma 2.7 that

$$\begin{aligned}\phi\left(\Theta_n^{M-1} y_{n,i}, \Theta_n^{M-2} y_{n,i}\right) &\leq \phi\left(p, \Theta_n^{M-2} y_{n,i}\right) - \phi\left(p, \Theta_n^{M-1} y_{n,i}\right) \\ &\leq \phi(p, x_n) - \phi\left(p, \Theta_n^{M-1} y_{n,i}\right).\end{aligned} \tag{3.17}$$

From (3.15) and (3.16), we also have

$$\lim_{n\to\infty}\left\|\Theta_n^{M-1} y_{n,i} - \Theta_n^{M-2} y_{n,i}\right\| = 0, \quad \forall i \geq 1. \tag{3.18}$$

Hence, from (3.15) and (3.18), we get

$$\lim_{n\to\infty}\left\|x_n - \Theta_n^{M-2} y_{n,i}\right\| = 0, \quad \forall i \geq 1, \tag{3.19}$$

$$\lim_{n\to\infty}\left\|Jx_n - J\Theta_n^{M-2} y_{n,i}\right\| = 0, \quad \forall i \geq 1. \tag{3.20}$$

In a similar way, we can verify that

$$\lim_{n\to\infty}\left\|\Theta_n^{M-2} y_{n,i} - \Theta_n^{M-3} y_{n,i}\right\| = \cdots = \lim_{n\to\infty}\left\|\Theta_n^{1} y_{n,i} - y_{n,i}\right\| = 0 \tag{3.21}$$

for all $i \geq 1$,

$$\lim_{n\to\infty}\left\|x_n - \Theta_n^{M-3} y_{n,i}\right\| = \cdots = \lim_{n\to\infty}\left\|x_n - y_{n,i}\right\| = 0 \tag{3.22}$$

for all $i \geq 1$,

$$\lim_{n\to\infty}\left\|Jx_n - J\Theta_n^{M-3} y_{n,i}\right\| = \cdots = \lim_{n\to\infty}\left\|Jx_n - Jy_{n,i}\right\| = 0 \tag{3.23}$$

for all $i \geq 1$. Hence, we can conclude that

$$\lim_{n\to\infty}\left\|\Theta_n^{j} y_{n,i} - \Theta_n^{j-1} y_{n,i}\right\| = 0 \tag{3.24}$$

for each $j = 1, 2, \ldots, M$ and $i \geq 1$. Observe that

$$\begin{aligned}\left\|Jy_{n,i} - Jx_n\right\| &= \|\alpha_n Jx_n + (1-\alpha_n)JT_i x_n - Jx_n\| \\ &= (1-\alpha_n)\|JT_i x_n - Jx_n\|,\end{aligned} \tag{3.25}$$

then we obtain from (B1) and (3.23) that

$$\lim_{n\to\infty}\|JT_i x_n - Jx_n\| = 0, \quad \forall i \geq 1. \tag{3.26}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets, we get that

$$\lim_{n\to\infty}\|T_i x_n - x_n\| = 0, \quad \forall i \geq 1. \tag{3.27}$$

Since T_i is closed for all $i \geq 1$ and $x_n \to q$, we conclude that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Step 5. $q \in \bigcap_{j=1}^{M} \mathrm{EP}(f_j)$.

From (3.24) and (B2), we have that $\|J\Theta_n^{j} y_{n,i} - J\Theta_n^{j-1} y_{n,i}\|/r_{j,n} \to 0$ as $n \to \infty$. Then, for each $j = 1, 2, \ldots, M$, we obtain that

$$f_j\left(\Theta_n^{j} y_{n,i}, y\right) + \frac{1}{r_{j,n}}\left\langle y - \Theta_n^{j} y_{n,i}, J\Theta_n^{j} y_{n,i} - J\Theta_n^{j-1} y_{n,i}\right\rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

From (A2) we have that

$$\begin{aligned}\left\|y - \Theta_n^{j} y_{n,i}\right\| \frac{\left\|J\Theta_n^{j} y_{n,i} - J\Theta_n^{j-1} y_{n,i}\right\|}{r_{j,n}} &\geq \frac{1}{r_{j,n}}\left\langle y - \Theta_n^{j} y_{n,i}, J\Theta_n^{j} y_{n,i} - J\Theta_n^{j-1} y_{n,i}\right\rangle \\ &\geq -f_j\left(\Theta_n^{j} y_{n,i}, y\right) \geq f_j\left(y, \Theta_n^{j} y_{n,i}\right), \quad \forall y \in C.\end{aligned} \tag{3.29}$$

From (A4) and the fact that $\Theta_n^j y_{n,i} \to q$ for $i \geq 1$, we get $f_j(y,q) \leq 0$ for all $y \in C$. For each $0 < t < 1$ and $y \in C$, denote $y_t = ty + (1-t)q$. Then $y_t \in C$, which implies that $f_j(y_t, q) \leq 0$. From (A1) and (A4), we obtain that $0 = f_j(y_t, y_t) \leq t f_j(y_t, y) + (1-t) f_j(y_t, q) \leq t f_j(y_t, y)$. Thus, $f_j(y_t, y) \geq 0$. From (A3), we have $f_j(q, y) \geq 0$ for all $y \in C$ and $j = 1, 2, \ldots, M$. Hence $q \in \bigcap_{j=1}^{M} \mathrm{EP}(f_j)$.

Step 6. $q = \Pi_F x_0$.

From $x_n = \Pi_{C_n} x_0$, we have

$$\langle Jx_0 - Jx_n, x_n - z \rangle \geq 0 \quad \forall z \in C_n. \tag{3.30}$$

Since $F \subset C_n$, we also have

$$\langle Jx_0 - Jx_n, x_n - p \rangle \geq 0 \quad \forall p \in F. \tag{3.31}$$

Letting $n \to \infty$ in (3.31), we obtain that

$$\langle Jx_0 - Jq, q - p \rangle \geq 0 \quad \forall p \in F. \tag{3.32}$$

From Lemma 2.2 we conclude that $q = \Pi_F x_0$. This completes the proof. □

Remark 3.2. Theorem 3.1 improves and extends Theorem 3.1 of Takahashi and Zembayashi in [3] in the following senses:

(i) from the case of an equilibrium problem to a finite family of equilibrium problems;

(ii) from a single relatively nonexpansive mapping to an infinitely countable family of relatively quasi-nonexpansive mappings;

(iii) if $M = 1$ and $T_i = T$ for all $i \geq 1$, then our restriction on $\{\alpha_n\}$ is weaker than that of Theorem 3.1 of [3].

Remark 3.3. The iteration (3.1) is a modification of (1.4) in the following ways.

(i) We use the composition of mappings $\{T_{r_{j,n}}^{f_j}\}_{j=1}^{M}$ in the second step.

(ii) We construct the set C_{n+1} by using the concept of supremum concerning an infinitely countable family of closed and relatively quasi-nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$. If $M = 1$ and $T_i = T$ for all $i \geq 1$, then the iteration (3.1) reduces to that of (1.4).

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 3.1, then we have the following corollary.

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies* *conditions (A1)–(A4), and let* $\{T_i\}_{i=1}^{\infty}$ *be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from* C *into itself. Assume that* $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} EP(f_j)) \neq \emptyset$. *For any initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$*, define the sequence* $\{x_n\}$ *as follows:*

$$\begin{gathered}
y_{n,i} = T_i x_n, \\
u_{n,i} = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1.
\end{gathered} \tag{3.33}$$

If $\liminf_{n\to\infty} r_{j,n} > 0$ *for each* $j = 1, 2, \ldots, M$*, then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

4. Applications

In this section, we give several applications of Theorem 3.1 in the framework of Banach spaces and Hilbert spaces.

Let $A : C \to E^*$ be a nonlinear mapping. The classical variational inequality problem is to find that $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0 \quad \forall y \in C. \tag{4.1}$$

The solutions set of (4.1) is denoted by $VI(C, A)$. For each $r > 0$ and $x \in E$, define the mapping $T_r^A : E \to C$ as follows:

$$T_r^A(x) = \left\{ z \in C : \langle Az, y - z \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \tag{4.2}$$

Theorem 4.1. *Let* E *be a uniformly convex and uniformly smooth Banach space, and let* C *be a nonempty, closed, and convex subset of* E. *Let* $\{A_j\}_{j=1}^{M}$ *be continuous and monotone operators from* C *to* E^**, and let* $\{T_i\}_{i=1}^{\infty}$ *be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from* C *into itself such that* $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} VI(C, A_j)) \neq \emptyset$. *For any initial point* $x_0 \in E$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$*, define the sequence* $\{x_n\}$ *as follows:*

$$\begin{gathered}
y_{n,i} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_i x_n), \\
u_{n,i} = T_{r_{M,n}}^{A_M} T_{r_{M-1,n}}^{A_{M-1}} \cdots T_{r_{1,n}}^{A_1} y_{n,i}, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1.
\end{gathered} \tag{4.3}$$

Assume that $\{\alpha_n\}$ *and* $\{r_{j,n}\}$ *for* $j = 1, 2, \ldots, M$ *are sequences which satisfy conditions (B1) and (B2) of Theorem 3.1.*

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof. Define $f_j(x, y) = \langle A_j x, y - x\rangle$ for all $x, y \in C$ and $j = 1, 2, \ldots, M$. First, we see that $F(T_{r_j}^{f_j}) = \mathrm{EP}(f_j) = VI(C, A_j) = F(T_{r_j}^{A_j})$ for each $j = 1, 2, \ldots, M$.

Next, we show that $\{f_j\}_{j=1}^{M}$ satisfy conditions (A1)–(A4).

(A1) Consider $f_j(x, x) = \langle A_j x, x - x\rangle = 0$ for all $x \in C$ and $j = 1, 2, \ldots, M$.

(A2) For each $x, y \in C$ and $j = 1, 2, \ldots, M$, we observe that

$$\begin{aligned} f_j(x, y) + f_j(y, x) &= \langle A_j x, y - x\rangle + \langle A_j y, x - y\rangle \\ &= \langle A_j x - A_j y, y - x\rangle. \end{aligned} \tag{4.4}$$

By the monotonicity of A_j, we obtain that f_j is monotone. Thus $\{f_j\}_{j=1}^{M}$ satisfy condition (A2).

(A3) For each $x, y, z \in C$ and $j = 1, 2, \ldots, M$, we have by the continuity of A_j that

$$\begin{aligned} \limsup_{t\downarrow 0} f_j(tz + (1-t)x, y) &= \limsup_{t\downarrow 0} \langle A_j(tz + (1-t)x), y - (tz + (1-t)x)\rangle \\ &= \langle A_j x, y - x\rangle \\ &= f_j(x, y). \end{aligned} \tag{4.5}$$

This shows that $\{f_j\}_{j=1}^{M}$ satisfy condition (A3).

(A4) Let $u, v \in C$ and $s \in (0, 1)$. Then, for each $x \in C$ and $j = 1, 2, \ldots, M$, we have

$$\begin{aligned} f_j(x, su + (1-s)v) &= \langle A_j x, su + (1-s)v - x\rangle \\ &= s\langle A_j x, u - x\rangle + (1-s)\langle A_j x, v - x\rangle \\ &= s f_j(x, u) + (1-s) f_j(x, v). \end{aligned} \tag{4.6}$$

Thus f_j is convex in the second variable. Let $u_n \in C$ and $\lim_{n\to\infty} u_n = u$. Then

$$\begin{aligned} f_j(x, u) &= \langle A_j x, u - x\rangle \\ &= \lim_{n\to\infty} \langle A_j x, u_n - x\rangle \\ &= \lim_{n\to\infty} f_j(x, u_n). \end{aligned} \tag{4.7}$$

This shows that f_j is lower semicontinuous in the second variable. Hence $\{f_j\}_{j=1}^{M}$ satisfy condition (A4). From Theorem 3.1 we obtain the desired result. □

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.1, then we have the following corollary.

Corollary 4.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E. Let $\{A_j\}_{j=1}^{M}$ be continuous and monotone operators from C to E^*, and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} VI(C, A_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
y_{n,i} &= T_i x_n, \\
u_{n,i} &= T_{r_{M,n}}^{A_M} T_{r_{M-1,n}}^{A_{M-1}} \cdots T_{r_{1,n}}^{A_1} y_{n,i}, \\
C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1.
\end{aligned} \tag{4.8}$$

If $\liminf_{n\to\infty} r_{j,n} > 0$ *for each $j = 1, 2, \ldots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

Let $\varphi : C \to \mathbb{R}$ be a real-valued function. The convex minimization problem is to find that $\hat{x} \in C$ such that

$$\varphi(\hat{x}) \leq \varphi(y) \quad \forall y \in C. \tag{4.9}$$

The solutions set of (4.9) *is denoted by $CMP(\varphi)$. For each $r > 0$ and $x \in E$, define the mapping $T_r^{\varphi} : E \to C$ as follows:*

$$T_r^{\varphi}(x) = \left\{ z \in C : \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq \varphi(z), \forall y \in C \right\}. \tag{4.10}$$

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E. Let $\{\varphi_j\}_{j=1}^{M}$ be lower semicontinuous and convex functions from C to $\mathbb{R}$, and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} CMP(\varphi_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
y_{n,i} &= J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_i x_n), \\
u_{n,i} &= T_{r_{M,n}}^{\varphi_M} T_{r_{M-1,n}}^{\varphi_{M-1}} \cdots T_{r_{1,n}}^{\varphi_1} y_{n,i}, \\
C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\
x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1.
\end{aligned} \tag{4.11}$$

Assume that $\{\alpha_n\}$ *and* $\{r_{j,n}\}$ *for* $j = 1, 2, \ldots, M$ *are sequences which satisfy conditions (B1) and (B2) of Theorem 3.1.*

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof. Define $f_j(x, y) = \varphi_j(y) - \varphi_j(x)$ for all $x, y \in C$ and $j = 1, 2, \ldots, M$. Then $F(T_{r_j}^{f_j}) = \mathrm{EP}(f_j) = CMP(\varphi_j) = F(T_{r_j}^{\varphi_j})$ for each $j = 1, 2, \ldots, M$, and therefore $\{f_j\}_{j=1}^M$ satisfy conditions (A1) and (A2).

Next, we show that $\{f_j\}_{j=1}^M$ satisfy conditions (A3) and (A4). For each $x, y, z \in C$, we have by the lower semicontinuity of φ_j that

$$\begin{aligned}\limsup_{t\downarrow 0} f_j(tz + (1-t)x, y) &= \limsup_{t\downarrow 0}\left(\varphi_j(y) - \varphi_j(tz + (1-t)x)\right)\\ &\le \varphi_j(y) - \varphi_j(x)\\ &= f_j(x, y).\end{aligned} \tag{4.12}$$

This implies that $\{f_j\}_{j=1}^M$ satisfy condition (A3).

Let $u, v \in C$ and $s \in (0, 1)$. For each $x \in C$, we have by the convexity of φ_j that

$$\begin{aligned}f_j(x, su + (1-s)v) &= \varphi_j(su + (1-s)v) - \varphi_j(x)\\ &\le s\varphi_j(u) + (1-s)\varphi_j(v) - \varphi_j(x)\\ &= s\left(\varphi_j(u) - \varphi_j(x)\right) + (1-s)\left(\varphi_j(v) - \varphi_j(x)\right)\\ &= sf_j(x, u) + (1-s)f_j(x, v).\end{aligned} \tag{4.13}$$

On the other hand, let $u_n \in C$ and $\lim_{n\to\infty} u_n = u$. By the lower semicontinuity of φ_j we have

$$\begin{aligned}f_j(x, u) &= \varphi_j(u) - \varphi_j(x)\\ &\le \liminf_{n\to\infty}\left(\varphi_j(u_n) - \varphi_j(x)\right)\\ &= \liminf_{n\to\infty} f_j(x, u_n).\end{aligned} \tag{4.14}$$

Thus $\{f_j\}_{j=1}^M$ satisfy condition (A4). From Theorem 3.1 we also obtain the desired result. □

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.3, then we have the following corollary.

Corollary 4.4. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty, closed, and convex subset of E. Let* $\{\varphi_j\}_{j=1}^M$ *be lower semicontinuous and convex* *functions from C to $\mathbb{R}$, and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} CMP(\varphi_j)) \neq \emptyset$. For any initial point $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_{n,i} &= T_i x_n, \\ u_{n,i} &= T_{r_{M,n}}^{\varphi_M} T_{r_{M-1,n}}^{\varphi_{M-1}} \cdots T_{r_{1,n}}^{\varphi_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{aligned} \tag{4.15}$$

If $\liminf_{n\to\infty} r_{j,n} > 0$ *for each $j = 1, 2, \ldots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

As a direct consequence of Theorem 3.1, we obtain the following application in a Hilbert space.

Theorem 4.5. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4), and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and quasi-nonexpansive mappings from C into itself such that $F := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{M} EP(f_j)) \neq \emptyset$. For any initial point $x_0 \in H$ with $x_1 = P_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ as follows:*

$$\begin{aligned} y_{n,i} &= \alpha_n x_n + (1 - \alpha_n) T_i x_n, \\ u_{n,i} &= T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} &= \left\{ z \in C_n : \sup_{i \geq 1} \|z - u_{n,i}\| \leq \|z - x_n\| \right\}, \\ x_{n+1} &= P_{C_{n+1}} x_0, \quad n \geq 1, \end{aligned} \tag{4.16}$$

where P is the metric projection. Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \ldots, M$ are sequences which satisfy conditions (B1) and (B2) of Theorem 3.1.

Then the sequence $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. Taking $E = H$ in Theorem 3.1, the result is obtained immediately. □

Remark 4.6. Theorem 4.5 improves and extends the main results of [20–22] in the following senses:

(i) from the case of an equilibrium problem to a finite family of equilibrium problems;

(ii) from the class of nonexpansive mappings to the class of an infinitely countable family of quasi-nonexpansive mappings.

Acknowledgments

The authors would like to thank Professor Simeon Reich and the referee for valuable suggestions on improving the manuscript and the Thailand Research Fund for financial support. The first author was supported by the Royal Golden Jubilee Grant PHD/0261/2551 and by the Graduate School, Chiang Mai University, Thailand.

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