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Research Article

A Bäcklund Transformation for the Burgers Hierarchy

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We give a Bäcklund transformation in a unified form for each member in the Burgers hierarchy. By applying the Bäcklund transformation to the trivial solutions, we generate some solutions of the Burgers hierarchy.

1. Introduction

Let

$$P_{-1} = 1, \qquad P_0(u) = u, \tag{1.1}$$

and for $j \geq 1$, define the differential expressions $P_j(u, \ldots, \partial_x^j u)$ recursively as follows:

$$P_j\left(u, \ldots, \partial_x^j u\right) = (u + \partial_x) P_{j-1}\left(u, \ldots, \partial_x^{j-1} u\right). \tag{1.2}$$

Then the Burgers hierarchy is defined by

$$u_t = \partial_x P_j\left(u, \ldots, \partial_x^j u\right), \quad j \geq 1. \tag{1.3}$$

The first few members of the hierarchy (1.3) are

$$u_t = 2uu_x + u_{xx}, \tag{1.4}$$

$$u_t = 3u^2\ u_x + 3u_x^2 + 3uu_{xx} + u_{xxx}, \tag{1.5}$$

$$u_t = 4u^3u_x + 12uu_x^2 + 6u^2u_{xx} + 10u_xu_{xx} + 4uu_{xxx} + u_{xxxx}, \tag{1.6}$$

with (1.4) being just the Burgers equation.

There is much literature on the Burgers hierarchy. Olver [1] derived the hierarchy (1.3) from the point of view of infinitely many symmetries. The work in [2] showed that the Cole-Hopf transformation

$$w \longmapsto u = \frac{w_x}{w} \tag{1.7}$$

transforms solutions of the linear equation

$$w_t = \partial_x^{j+1} w \tag{1.8}$$

to that of (1.3). With the help of the Cole-Hopf transformation (1.9), Taflin [3] and Tasso [4] showed, respectively, that the Burgers equation (1.4) and the second member (1.5) of the hierarchy (1.3) can be written in the Hamiltonian form. More recently, Talukdar et al. [5] constructed an appropriate Lagrangian by solving the inverse problem of variational calculus and then Hamiltonized (1.5) to get the relevant Poisson structure. Furthermore, they pointed out that their method is applicable to each member of (1.3). Pickering [6] proved explicitly that each member of (1.3) passes the Weiss-Tabor-Carnevale Painlevé test.

This paper is devoted to the study of Bäcklund transformation for the Burgers hierarchy. Bäcklund transformation was named after the Swedish mathematical physicist and geometer Albert Victor Bäcklund(1845-1922), who found in 1883 [7], when studying the surfaces of constant negative curvature, that the sine-Gordon equation

$$u_{xt} = \sin u \tag{1.9}$$

has the following property: if u solves (1.9), then for an arbitrary non-zero constant λ, the system on v

$$\begin{aligned} v_x &= u_x - 2\lambda \sin\frac{u+v}{2}, \\ v_t &= -u_t + \frac{2}{\lambda}\sin\frac{u-v}{2} \end{aligned} \tag{1.10}$$

is integrable; moreover, v also solves (1.9). So (1.10) gives a transformation $u \mapsto v$, now called Bäcklund transformation, which takes one solution of (1.9) into another. For example, substituting the trivial solution $u(x,t) \equiv 0$ into (1.10) yields one-soliton solution:

$$v(x,t) = 4\arctan\exp\left(\alpha - \lambda x - \frac{1}{\lambda}t\right), \tag{1.11}$$

where α is an arbitrary constant. By repeating this procedure one can get multiple-soliton solutions. Some other nonlinear partial differential equations (PDEs), such as KdV equation [8]

$$u_t = 6uu_x + u_{xxx}, \tag{1.12}$$

modified KdV equation [9]

$$u_t = u^2u_x + u_{xxx}, \tag{1.13}$$

Burgers equation (1.4) [10], and a generalized Burgers equation [11]

$$u_t + b(t)uu_x + a(t)u_{xx} = 0, \tag{1.14}$$

also possess Bäcklund transformations. Now Bäcklund transformation has become a useful tool for generating solutions to certain nonlinear PDEs. Much literature is devoted to searching Bäcklund transformations for some nonlinear PDEs (see, e.g., [12–15]). In this paper, we give a Bäcklund transformation for each member in the Burgers hierarchy. As an application, by applying our Bäcklund transformation to the trivial solutions, we generate some new solutions of (1.3).

2. Bäcklund Transformation

First, the differential expressions P_j have the following property.

Theorem 2.1. *For an arbitrary constant λ, let*

$$u = v + \frac{v_x}{\lambda + v}. \tag{2.1}$$

Then

$$P_j\left(u,\ldots,\partial_x^j u\right) = \frac{\lambda P_j\left(v,\ldots,\partial_x^j v\right) + P_{j+1}\left(v,\ldots,\partial_x^{j+1} v\right)}{\lambda + v}, \quad j \geq 1. \tag{2.2}$$

Proof. We use induction to prove (2.2).

First, for $j = 1$,

$$\begin{aligned} P_1(u, u_x) &= u^2 + u_x \\ &= \left(v + \frac{v_x}{\lambda + v}\right)^2 + v_x - \frac{{v_x}^2}{(\lambda + v)^2} + \frac{v_{xx}}{\lambda + v} \\ &= \frac{\lambda(v^2 + v_x) + v^3 + 3vv_x + v_{xx}}{\lambda + v} \\ &= \frac{\lambda P_1(v, v_x) + P_2(v, v_x, v_{xx})}{\lambda + v}. \end{aligned} \tag{2.3}$$

So (2.2) is true for $j = 1$.

Next, fix a $k > 1$, and assume that (2.2) is true for $j = k - 1$. Then

$$\begin{aligned} P_k\left(u, \ldots, \partial_x^k u\right) &= \left(v + \frac{v_x}{\lambda + v} + \partial_x\right) P_{k-1}\left(u, \ldots, \partial_x^{k-1} u\right) \\ &= \left(v + \frac{v_x}{\lambda + v} + \partial_x\right) \frac{\lambda P_{k-1}(v, \ldots, \partial_x^{k-1} v) + P_k(v, \ldots, \partial_x^k v)}{\lambda + v} \\ &= \frac{(v + \partial_x)\left(\lambda P_{k-1}(v, \ldots, \partial_x^{k-1} v) + P_k(v, \ldots, \partial_x^k v)\right)}{\lambda + v} \\ &= \frac{\lambda P_k(v, \ldots, \partial_x^k v) + P_{k+1}(v, \ldots, \partial_x^{k+1} v)}{\lambda + v}; \end{aligned} \tag{2.4}$$

that is, (2.2) is valid for $j = k$.

Therefore, (2.2) is always true for $j \geq 1$. □

Now we state our main result.

Theorem 2.2. *If u is a solution of* (1.3), *then the system on v*

$$\begin{aligned} v_x &= (\lambda + v)(u - v), \\ v_t &= (\lambda + v) \sum_{k=0}^{j} (-\lambda)^{j-k} \left(P_k\left(u, \ldots, \partial_x^k u\right) - v P_{k-1}\left(u, \ldots, \partial_x^{k-1} u\right)\right) \end{aligned} \tag{2.5}$$

is integrable; moreover, v also satisfies (1.3). *Therefore,* (2.5) *defines a Bäcklund transformation $u \mapsto v$, in a unified form, for each member of the Burgers hierarchy* (1.3).

Proof. By (1.3) and (2.5) we have

$$\begin{aligned} v_{xt} &= (\lambda+v)(u-v)\sum_{k=0}^{j}(-\lambda)^{j-k}\left(P_k\left(u,\dots,\partial_x^k u\right)-vP_{k-1}\left(u,\dots,\partial_x^{k-1}u\right)\right) \\ &\quad+(\lambda+v)\partial_x P_j\left(v,\dots,\partial_x^j v\right) \\ &\quad-(\lambda+v)^2\sum_{k=0}^{j}(-\lambda)^{j-k}\left(P_k\left(u,\dots,\partial_x^k u\right)-vP_{k-1}\left(u,\dots,\partial_x^{k-1}u\right)\right), \end{aligned} \tag{2.6}$$

$$\begin{aligned} v_{tx} &= (\lambda+v)(u-v)\sum_{k=0}^{j}(-\lambda)^{j-k}\left(P_k\left(u,\dots,\partial_x^k u\right)-vP_{k-1}\left(u,\dots,\partial_x^{k-1}u\right)\right) \\ &\quad+(\lambda+v)\sum_{k=0}^{j}(-\lambda)^{j-k}\partial_x P_k\left(v,\dots,\partial_x^j v\right) \\ &\quad-v(\lambda+v)\sum_{k=0}^{j-1}(-\lambda)^{j-1-k}\partial_x P_k\left(v,\dots,\partial_x^j v\right) \\ &\quad-(\lambda+v)^2(u-v)\sum_{k=0}^{j}(-\lambda)^{j-k}P_{k-1}\left(u,\dots,\partial_x^k u\right) \\ &= (\lambda+v)(u-v)\sum_{k=0}^{j}(-\lambda)^{j-k}\left(P_k\left(u,\dots,\partial_x^k u\right)-vP_{k-1}\left(u,\dots,\partial_x^{k-1}u\right)\right) \\ &\quad+(\lambda+v)\partial_x P_j\left(v,\dots,\partial_x^j v\right)-(\lambda+v)^2\sum_{k=0}^{j-1}(-\lambda)^{j-1-k}\partial_x P_k\left(v,\dots,\partial_x^j v\right) \\ &\quad-(\lambda+v)^2(u-v)\sum_{k=0}^{j}(-\lambda)^{j-k}P_{k-1}\left(u,\dots,\partial_x^k u\right) \\ &= (\lambda+v)(u-v)\sum_{k=0}^{j}(-\lambda)^{j-k}\left(P_k\left(u,\dots,\partial_x^k u\right)-vP_{k-1}\left(u,\dots,\partial_x^{k-1}u\right)\right) \\ &\quad+(\lambda+v)\partial_x P_j\left(v,\dots,\partial_x^j v\right) \\ &\quad-(\lambda+v)^2\sum_{k=0}^{j}(-\lambda)^{j-k}\left(P_k\left(u,\dots,\partial_x^k u\right)-vP_{k-1}\left(u,\dots,\partial_x^{k-1}u\right)\right); \end{aligned} \tag{2.7}$$

therefore $v_{xt}=v_{tx}$; that is, (2.5) is an integrable system associated with (1.3).

From the first equation of (2.5) we have

$$u=v+\frac{v_x}{\lambda+v}. \tag{2.8}$$

So

$$P_0(u) - v = \frac{v_x}{\lambda + v} = \frac{\partial_x P_0(v)}{\lambda + v}. \tag{2.9}$$

On the other hand, by (2.2)

$$P_k\left(u, \ldots, \partial_x^k u\right) - vP_{k-1}\left(u, \ldots, \partial_x^{k-1} u\right) = \frac{\lambda \partial_x P_{k-1}\left(v, \ldots, \partial_x^{k-1} v\right) + \partial_x P_k\left(v, \ldots, \partial_x^k v\right)}{\lambda + v}, \quad k \geq 1. \tag{2.10}$$

Substituting (2.9) and (2.10) into the second equation of (2.5) yields

$$v_t = \partial_x P_j\left(v, \ldots, \partial_x^j v\right); \tag{2.11}$$

that is, v also satisfies the Burgers hierarchy (1.3). □

3. Exact Solutions

In this section we always assume that λ is an arbitrary nonzero constant.

From a known solution u of (1.3), the first equation of (2.5) gives

$$v(x,t) = \frac{e^{\int(\lambda+u)\mathrm{d}x} - \lambda \int e^{\int(\lambda+u)\mathrm{d}x}\mathrm{d}x - \lambda c(t)}{\int e^{\int(\lambda+u)\mathrm{d}x}\mathrm{d}x + c(t)}, \tag{3.1}$$

with the "integration constant" $c(t)$ satisfying a first-order ordinary differential equation determined by the second equation of (2.5).

Example 3.1. Take the trivial solution $u(x,t) \equiv 1$ of (1.3). Then from (1.2) we have

$$P_j\left(u, \ldots, \partial_x^j u\right) \equiv 1 \quad \text{for } j \geq 1. \tag{3.2}$$

So (2.5) becomes

$$\begin{gathered} v_x = (\lambda + v)(1 - v), \\ v_t = \frac{(\lambda + v)(1 - v)\left(1 - (-\lambda)^{j+1}\right)}{1 + \lambda}. \end{gathered} \tag{3.3}$$

Solving (3.3) gives the following solution of (1.3):

$$v(x,t) = \frac{e^{(1+\lambda)x + (1+(-1)^j \lambda^{j+1})t} + \lambda e^c}{e^{(1+\lambda)x + (1+(-1)^j \lambda^{j+1})t} - e^c}, \tag{3.4}$$

where c is an arbitrary constant.

Note that (3.4) is a traveling wave solution.

Example 3.2. By the Cole-Hopf transformation (1.7),

$$u(x,t) = \frac{1}{x} \tag{3.5}$$

is a solution of (1.3). Then from (1.2) we have

$$P_j\left(u,\ldots,\partial_x^j u\right) \equiv 0 \quad \text{for } j \geq 1. \tag{3.6}$$

So (2.5) becomes

$$\begin{gathered} v_x = (\lambda + v)\left(\frac{1}{x} - v\right), \\ v_t = (\lambda + v)\left((-\lambda)^j\left(\frac{1}{x} - v\right) - \frac{(-\lambda)^{j-1}v}{x}\right). \end{gathered} \tag{3.7}$$

Solving (3.7) gives the following solution of (1.3):

$$v(x,t) = \frac{\lambda e^{\lambda(x+(-\lambda)^j t)} + \lambda e^c}{(-1+\lambda x)e^{\lambda(x+(-\lambda)^j t)} - e^c}. \tag{3.8}$$

Note that (3.8) is not a traveling wave solution.

Example 3.3. By the Cole-Hopf transformation (1.7),

$$u(x,t) = \frac{2}{x} \tag{3.9}$$

is a solution of (1.3) for $j \geq 2$. Then from (1.2) we have

$$P_1(u,u_x) = \frac{2}{x^2}, \qquad P_j\left(u,\ldots,\partial_x^j u\right) \equiv 0 \quad \text{for } j \geq 2. \tag{3.10}$$

So (2.5) becomes

$$\begin{gathered} v_x = (\lambda + v)\left(\frac{2}{x} - v\right), \\ v_t = (\lambda + v)\left((-\lambda)^j\left(\frac{2}{x} - v\right) + 2(-\lambda)^{j-1}\left(\frac{1}{x^2} - \frac{v}{x}\right) - \frac{2(-\lambda)^{j-2}v}{x^2}\right). \end{gathered} \tag{3.11}$$

Solving (3.11) gives the following solution of (1.3) for $j \geq 2$:

$$v(x,t) = \frac{2\lambda(-1+\lambda x)e^{\lambda(x+(-\lambda)^j t)} + \lambda e^c}{(2-2\lambda x+\lambda^2 x^2)e^{\lambda(x+(-\lambda)^j t)} - e^c}. \tag{3.12}$$

Note that (3.12) is not a traveling wave solution.

Example 3.4. By the Cole-Hopf transformation (1.7),

$$u(x,t) = \frac{3}{x} \tag{3.13}$$

is a solution of (1.3) for $j \geq 3$. Then from (1.2) we have

$$P_1(u,u_x) = \frac{6}{x^2}, \qquad P_2(u,u_x,u_{xx}) = \frac{6}{x^3}, \qquad P_j\left(u,\ldots,\partial_x^j u\right) \equiv 0 \quad \text{for } j \geq 3. \tag{3.14}$$

So (2.5) becomes

$$\begin{gathered} v_x = (\lambda+v)\left(\frac{3}{x}-v\right), \\ v_t = (\lambda+v)\left((-\lambda)^j\left(\frac{3}{x}-v\right) + 3(-\lambda)^{j-1}\left(\frac{2}{x^2}-\frac{v}{x}\right) + 6(-\lambda)^{j-2}\left(\frac{1}{x^3}-\frac{v}{x^2}\right) - \frac{6(-\lambda)^{j-3}v}{x^3}\right). \end{gathered} \tag{3.15}$$

Solving (3.15) gives the following solution of (1.3) for $j \geq 3$:

$$v(x,t) = \frac{3\lambda\left(2-2\lambda x+\lambda^2 x^2\right)e^{\lambda(x+(-\lambda)^j t)} + \lambda e^c}{(-6+6\lambda x-3\lambda^2 x^2+\lambda^3 x^3)e^{\lambda(x+(-\lambda)^j t)} - e^c}. \tag{3.16}$$

Note that (3.16) is not a traveling wave solution.

Remark 3.5. In general, for an arbitrary positive integer k,

$$u(x,t) = \frac{k}{x} \tag{3.17}$$

is a solution of (1.3) for $j \geq k$. Substituting (3.17) into (2.5) gives the following solution of (1.3) for $j \geq k$:

$$v(x,t) = \frac{\left(\partial f\left(x,x^2,\ldots,x^k\right)/\partial x\right)e^{\lambda(x+(-\lambda)^j t)} + \lambda e^c}{f\left(x,x^2,\ldots,x^k\right)e^{\lambda(x+(-\lambda)^j t)} - e^c}, \tag{3.18}$$

where

$$f\left(x, x^2, \ldots, x^k\right) = (-1)^k k! + (-1)^{k-1} k! \lambda x + (-1)^{k-2} \frac{k!}{2!} \lambda^2 x^2 + \cdots - k\lambda^{k-1} x^{k-1} + \lambda^k x^k. \quad (3.19)$$

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