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Research Article

Some Remarks on Spaces of Morrey Type

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We deepen the study of some Morrey type spaces, denoted by $M^{p,\lambda}(\Omega)$, defined on an unbounded open subset Ω of $\mathbb{R}^n$. In particular, we construct decompositions for functions belonging to two different subspaces of $M^{p,\lambda}(\Omega)$, which allow us to prove a compactness result for an operator in Sobolev spaces. We also introduce a weighted Morrey type space, settled between the above-mentioned subspaces.

1. Introduction

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. For $p \in [1, +\infty[$ and $\lambda \in [0, n[$, we consider the space $M^{p,\lambda}(\Omega)$ of the functions g in $L^p_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|^p_{M^{p,\lambda}(\Omega)} = \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda} \int_{\Omega \cap B(x,\tau)} |g(y)|^p dy < +\infty, \tag{1.1}$$

where $B(x, \tau)$ is the open ball with center x and radius τ.

This space of Morrey type, defined by Transirico et al. in [1], is a generalization of the classical Morrey space $L^{p,\lambda}$ and strictly contains $L^{p,\lambda}(\mathbb{R}^n)$ when $\Omega = \mathbb{R}^n$. Its introduction is related to the solvability of certain elliptic problems with discontinuous coefficients in the case of unbounded domains (see e.g., [1–3]).

In the first part of this work, we deepen the study of two subspaces of $M^{p,\lambda}(\Omega)$, denoted by $\widetilde{M}^{p,\lambda}(\Omega)$ and $M^{p,\lambda}_o(\Omega)$, that can be seen, respectively, as the closure of $L^\infty(\Omega)$ and $C^\infty_o(\Omega)$ in $M^{p,\lambda}(\Omega)$. We start proving some characterization lemmas that allow us to construct suitable decompositions of functions in $\widetilde{M}^{p,\lambda}(\Omega)$ and $M^{p,\lambda}_o(\Omega)$. This is done in the spirit of the classical decomposition (L^1, L^∞), proved in [4] by Calderón and Zygmund for L^1, where a given function in L^1 is decomposed, for any $t > 0$, in the sum of a part $f_t \in L^\infty$ (whose norm can be controlled by $\|f_t\|_{L^\infty(\Omega)} < c(n) \cdot t$) and a remaining one $f - f_t \in L^1$. Analogous decompositions can be found also for different functional spaces (see e.g., [5, 6] for decompositions $(L^1, L^{1,\lambda})$, $(L^p, \text{ Sobolev })$, and (L^p, BMO)).

The idea of our decomposition, both for a g in $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$, is the following: for any $h \in \mathbb{R}_+$, the function g can be written as the sum of a "good" part g_h, which is more regular, and of a "bad" part $g - g_h$, whose norm can be controlled by means of a continuity modulus of the function g itself.

Decompositions are useful in different contexts as the proof of interpolation results, norm inequalities and a priori estimates for solutions of boundary value problems.

For instance, in the study of several elliptic problems with solutions in Sobolev spaces, it is sometimes necessary to establish regularity results and a priori estimates for a fixed operator L. These results often rely on the boundedness and possibly on the compactness of the multiplication operator

$$u \in W^{k,q}(\Omega) \longrightarrow gu \in L^q(\Omega), \tag{1.2}$$

which entails the estimate

$$\|g\, u\|_{L^q(\Omega)} \le c \cdot \|g\|_V \cdot \|u\|_{W^{k,q}(\Omega)}, \tag{1.3}$$

where $c \in \mathbb{R}_+$ depends on the regularity properties of Ω and on the summability exponents, and g is a given function in a normed space V satisfying suitable conditions. In some particular cases, this cannot be done for the operator L itself, but there is the need to introduce a suitable class of operators L_h, whose coefficients, more regular, approximate the ones of L. This "deviation" of the coefficients of L_h from the ones of L needs to be done controlling the norms of the approximating coefficients with the norms of the given ones. Hence, it is necessary to obtain estimates where the dependence on the coefficients is expressed just in terms of their norms. Decomposition results play an important role in this approximation process, providing estimates where the constants involved depend just on the norm of the given coefficients and on their moduli of continuity and do not depend on the considered decomposition.

In the framework of Morrey type spaces, in [1], the authors studied, for $k = 1$, the operator defined in (1.2), generalizing a well-known result proved by Fefferman in [7] (cf. also [8]). They established conditions for the boundedness and compactness of this operator. In [2], the boundedness result and the straightforward estimates have been extended to any $k \in \mathbb{N}$.

In view of the above considerations, the second part of this work is devoted to a further analysis of the multiplication operator defined in (1.2), for functions g in $M^{p,\lambda}(\Omega)$. By means of our decomposition results, we are allowed to deduce a compactness result for the operator given in (1.2). The obtained estimates can be used in the study of elliptic problems to prove that the considered operators have closed range or are semi-Fredholm.

The deeper examination of the structure of $M^{p,\lambda}(\Omega)$ and of its subspaces leads us to the definition of a new functional space, that is a weighted Morrey type space, denoted by $M_\rho^{p,\lambda}(\Omega)$.

In literature, several authors have considered different kinds of weighted spaces of Morrey type and their applications to the study of elliptic equations, both in the degenerate case and in the nondegenerate one (see e.g., [9–11]).

In this paper, given a weight ρ in a class of measurable functions $G(\Omega)$ (see § 6 for its definition), we prove that the corresponding weighted space $M_\rho^{p,\lambda}(\Omega)$ is a space settled between $M_o^{p,\lambda}(\Omega)$ and $\widetilde{M}^{p,\lambda}(\Omega)$. In particular, we provide some conditions on ρ that entail $M_o^{p,\lambda}(\Omega) = M_\rho^{p,\lambda}(\Omega)$.

Taking into account the results of this paper, we are now in position to approach the study of some classes of elliptic problems with discontinuous coefficients belonging to the weighted Morrey type space $M_\rho^{p,\lambda}(\Omega)$.

2. Notation and Preliminary Results

Let G be a Lebesgue measurable subset of $\mathbb{R}^n$ and $\Sigma(G)$ be the σ-algebra of all Lebesgue measurable subsets of G. Given $F \in \Sigma(G)$, we denote by $|F|$ its Lebesgue measure and by χ_F its characteristic function. For every $x \in F$ and every $t \in \mathbb{R}_+$, we set $F(x,t) = F \cap B(x,t)$, where $B(x,t)$ is the open ball with center x and radius t, and in particular, we put $F(x) = F(x,1)$.

The class of restrictions to F of functions $\zeta \in C_o^\infty(\mathbb{R}^n)$ with $\overline{F} \cap \operatorname{supp} \ \zeta \subseteq F$ is denoted by $\mathfrak{D}(F)$ and, for $p \in [1,+\infty[$, $L^p_{\mathrm{loc}}(F)$ is the class of all functions $g : F \longrightarrow \mathbb{R}$ such that $\zeta \ g \in L^p(F)$ for any $\zeta \in \mathfrak{D}(F)$.

Let us recall the definition of the classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$.

For $n \geq 2$, $\lambda \in [0,n[$ and $p \in [1,+\infty[$, $L^{p,\lambda}(\mathbb{R}^n)$ is the set of the functions $g \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|g\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{\substack{\tau>0 \\ x\in\mathbb{R}^n}} \tau^{-\lambda/p} \|g\|_{L^p(B(x,\tau))} < +\infty, \tag{2.1}$$

equipped with the norm defined by (2.1).

If Ω is an unbounded open subset of $\mathbb{R}^n$ and t is fixed in $\mathbb{R}_+$, we can consider the space $M^{p,\lambda}(\Omega,t)$, which is larger than $L^{p,\lambda}(\mathbb{R}^n)$ when $\Omega = \mathbb{R}^n$. More precisely, $M^{p,\lambda}(\Omega,t)$ is the set of all functions g in $L^p_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{\tau\in]0,t] \\ x\in\Omega}} \tau^{-\lambda/p} \|g\|_{L^p(\Omega(x,\tau))} < +\infty, \tag{2.2}$$

endowed with the norm defined in (2.2).

We explicitly observe that a diadic decomposition gives for every $t_1, t_2 \in \mathbb{R}_+$ the existence of $c_1, c_2 \in \mathbb{R}_+$, depending only on t_1, t_2, and n, such that

$$c_1 \|g\|_{M^{p,\lambda}(\Omega,t_1)} \leq \|g\|_{M^{p,\lambda}(\Omega,t_2)} \leq c_2 \|g\|_{M^{p,\lambda}(\Omega,t_1)}, \quad \forall g \in M^{p,\lambda}(\Omega,t_1). \tag{2.3}$$

All the norms being equivalent, from now on, we consider the space

$$M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega,1). \tag{2.4}$$

For the reader's convenience, we briefly recall some properties of functions in $L^{p,\lambda}(\mathbb{R}^n)$ and $M^{p,\lambda}(\Omega)$ needed in the sequel.

The first lemma is a particular case of a more general result proved in [12, Proposition 3].

Lemma 2.1. *Let $(J_h)_{h\in\mathbb{N}}$ be a sequence of mollifiers in $\mathbb{R}^n$. If $g \in L^{p,\lambda}(\mathbb{R}^n)$ and*

$$\lim_{y\to 0}\|g(x-y)-g(x)\|_{L^{p,\lambda}(\mathbb{R}^n)} = 0, \tag{2.5}$$

then

$$\lim_{h\to +\infty}\|g - J_h * g\|_{L^{p,\lambda}(\mathbb{R}^n)} = 0. \tag{2.6}$$

The second results concerns the zero extensions of functions in $M^{p,\lambda}(\Omega)$ (see also [1, Remark 2.4]).

Remark 2.2. Let $g \in M^{p,\lambda}(\Omega)$. If we denote by g_0 the zero extension of g outside Ω, then $g_0 \in M^{p,\lambda}(\mathbb{R}^n)$ and for every τ in $]0,1]$

$$\|g_0\|_{M^{p,\lambda}(\mathbb{R}^n,\tau)} \le c_1\|g\|_{M^{p,\lambda}(\Omega,\tau)}, \tag{2.7}$$

where $c_1 \in \mathbb{R}_+$ is a constant independent of g, Ω and τ.

Furthermore, if $\operatorname{diam}(\Omega) < +\infty$, then $g_0 \in L^{p,\lambda}(\mathbb{R}^n)$ and

$$\|g_0\|_{L^{p,\lambda}(\mathbb{R}^n)} \le c_2\|g\|_{M^{p,\lambda}(\Omega)}, \tag{2.8}$$

where $c_2 \in \mathbb{R}_+$ is a constant independent of g and Ω.

For a general survey on Morrey and Morrey type spaces, we refer to [1, 2, 13, 14].

3. The Spaces $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$

This section is devoted to the study of two subspaces of $M^{p,\lambda}(\Omega)$, denoted by $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$. Here, we point out the peculiar characteristics of functions belonging to these sets by means of two characterization lemmas.

Let us put, for $h \in \mathbb{R}_+$ and $g \in M^{p,\lambda}(\Omega)$,

$$F[g](h) = \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|E(x)|\le 1/h}} \|g\chi_E\|_{M^{p,\lambda}(\Omega)}. \tag{3.1}$$

Lemma 3.1. *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$, and $g \in M^{p,\lambda}(\Omega)$. The following properties are equivalent:*

$$g \text{ is in the closure of } L^\infty(\Omega) \quad \text{in } M^{p,\lambda}(\Omega), \tag{3.2}$$

$$\lim_{h \to +\infty} F[g](h) = 0, \tag{3.3}$$

$$\lim_{h \to +\infty} \left(\sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{\substack{x \in \Omega \\ \tau \in]0,1]}} \tau^{-\lambda}|E(x,\tau)| \le 1/h}} \| g \chi_E \|_{M^{p,\lambda}(\Omega)} \right) = 0. \tag{3.4}$$

We denote by $\widetilde{M}^{p,\lambda}(\Omega)$ the subspace of $M^{p,\lambda}(\Omega)$ made up of functions verifying one of the above properties.

Proof of Lemma 3.1. The equivalence between (3.2) and (3.3) is proved in of [1, Lemma 1.3]. Let us show that (3.2) entails (3.4) and vice versa.

Fix g in the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$, then for each $\varepsilon > 0$, there exists a function $g_\varepsilon \in L^\infty(\Omega)$ such that

$$\| g - g_\varepsilon \|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2}. \tag{3.5}$$

Fixed $E \in \Sigma(\Omega)$, from (3.5), it easily follows that

$$\| g \chi_E \|_{M^{p,\lambda}(\Omega)} \le \| (g - g_\varepsilon) \chi_E \|_{M^{p,\lambda}(\Omega)} + \| g_\varepsilon \chi_E \|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2} + \| g_\varepsilon \chi_E \|_{M^{p,\lambda}(\Omega)}. \tag{3.6}$$

On the other hand

$$\| g_\varepsilon \chi_E \|_{M^{p,\lambda}(\Omega)} = \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda/p} \| g_\varepsilon \chi_E \|_{L^p(\Omega(x,\tau))} \le \| g_\varepsilon \|_{L^\infty(\Omega)} \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \left(\tau^{-\lambda} |E(x,\tau)| \right)^{1/p}. \tag{3.7}$$

Therefore, if we set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{2 \| g_\varepsilon \|_{L^\infty(\Omega)}} \right)^p, \tag{3.8}$$

from(3.7), we deduce that, if $\sup_{\tau \in]0,1],\, x \in \Omega} \tau^{-\lambda} |E(x,\tau)| \le 1/h_\varepsilon$, then

$$\| g_\varepsilon \chi_E \|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2}. \tag{3.9}$$

Putting together (3.6) and (3.9), we get (3.4).

Conversely, if we take a function $g \in M^{p,\lambda}(\Omega)$ satisfying (3.4), for any $\varepsilon > 0$, there exists $h_\varepsilon \in \mathbb{R}_+$ such that if $E \in \Sigma(\Omega)$ with $\sup_{\tau\in]0,1],\, x\in\Omega} \tau^{-\lambda}|E(x,\tau)| \le 1/h_\varepsilon$, then $\|g\chi_E\|_{M^{p,\lambda}(\Omega)} < \varepsilon$.

For each $k \in \mathbb{R}_+$, we set

$$E_k = \{x \in \Omega \mid |g(x)| \ge k\}. \tag{3.10}$$

Observe that

$$\|g\|_{M^{p,\lambda}(\Omega)} \ge \sup_{\substack{\tau\in]0,1] \\ x\in\Omega}} \tau^{-\lambda/p} \|g\|_{L^p(E_k(x,\tau))} \ge k \sup_{\substack{\tau\in]0,1] \\ x\in\Omega}} \left(\tau^{-\lambda}|E_k(x,\tau)|\right)^{1/p}. \tag{3.11}$$

Therefore, if we put

$$k_\varepsilon = \|g\|_{M^{p,\lambda}(\Omega)} h_\varepsilon^{1/p}, \tag{3.12}$$

from (3.11), it follows that

$$\sup_{\substack{\tau\in]0,1] \\ x\in\Omega}} \tau^{-\lambda}|E_{k_\varepsilon}(x,\tau)| \le \frac{1}{h_\varepsilon}, \tag{3.13}$$

and then

$$\|g\chi_{E_{k_\varepsilon}}\|_{M^{p,\lambda}(\Omega)} < \varepsilon. \tag{3.14}$$

To end the proof, we define the function $g_\varepsilon = g - g\chi_{E_{k_\varepsilon}}$. Indeed, by construction $g_\varepsilon \in L^\infty(\Omega)$ and by (3.14), one gets that $\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon$. □

Remark 3.2. It is easily seen (see also [1]) that if $g \in \widetilde{M}^{p,\lambda}(\Omega)$, then

$$\lim_{t\to 0} \|g\|_{M^{p,\lambda}(\Omega,t)} = 0. \tag{3.15}$$

Now, we introduce two classes of applications needed in the sequel.

For $h \in \mathbb{R}_+$, we denote by ζ_h a function of class $C_o^\infty(\mathbb{R}^n)$ such that

$$0 \le \zeta_h \le 1, \qquad \zeta_h|_{\overline{B(0,h)}} = 1, \quad \operatorname{supp}\zeta_h \subset B(0,2h). \tag{3.16}$$

To define the second class, we first fix f in $\mathfrak{D}(\overline{\mathbb{R}_+})$ satisfying

$$0 \le f \le 1, \qquad f(t) = 1 \quad \text{if } t \le \frac{1}{2}, \qquad f(t) = 0 \quad \text{if } t \ge 1, \tag{3.17}$$

and $\alpha \in C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ equivalent to dist $(\cdot, \partial\Omega)$ (for more details on the existence of such an α, see for instance [15]). Hence, for $h \in \mathbb{R}_+$, we put

$$\psi_h : x \in \overline{\Omega} \longrightarrow (1 - f(h\ \alpha(x))) f\left(\frac{|x|}{2h}\right). \tag{3.18}$$

It is easy to prove that ψ_h belongs to $C_o^\infty(\Omega)$, for any $h \in \mathbb{R}_+$. Moreover,

$$0 \le \psi_h \le 1, \qquad \psi_{h_{|\overline{\Omega}_h}} = 1, \qquad \operatorname{supp}\ \psi_h \subset \overline{\Omega}_{2h}, \tag{3.19}$$

where

$$\Omega_h = \left\{ x \in \Omega \mid |x| < h,\ \alpha(x) > \frac{1}{h} \right\}. \tag{3.20}$$

Lemma 3.3. *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$, and $g \in M^{p,\lambda}(\Omega)$. The following properties are equivalent:*

$$g \textit{ is in the closure of } \ C_o^\infty(\Omega) \quad \textit{in } M^{p,\lambda}(\Omega), \tag{3.21}$$

$$\lim_{h \to +\infty} \left(\| (1 - \zeta_h)\ g \|_{M^{p,\lambda}(\Omega)} + F[g](h) \right) = 0, \tag{3.22}$$

$$\lim_{h \to +\infty} \left(\| (1 - \psi_h)\ g \|_{M^{p,\lambda}(\Omega)} + F[g](h) \right) = 0, \tag{3.23}$$

$$\lim_{t \to 0} \|g\|_{M^{p,\lambda}(\Omega,\ t)} + \lim_{|x| \to +\infty} \left(\sup_{\tau \in]0,1]} \tau^{-\lambda/p} \|g\|_{L^p(\Omega(x,\tau))} \right) = 0, \tag{3.24}$$

$$g \in \widetilde{M}^{p,\lambda}(\Omega), \qquad \lim_{|x| \to +\infty} \left(\sup_{\tau \in]0,1]} \tau^{-\lambda/p} \|g\|_{L^p(\Omega(x,\tau))} \right) = 0. \tag{3.25}$$

The subspace of $M^{p,\lambda}(\Omega)$ of the functions satisfying one of the above properties will be denoted by $M_o^{p,\lambda}(\Omega)$.

Proof of Lemma 3.3. The equivalence between (3.21) and (3.22) is a consequence of (3.3) and of [1, Lemmas 2.1 and 2.5]. The one between (3.21) and (3.24) follows from of [1, Remark 2.2]. Always in [1], see Lemma 2.1 and Remark 2.2, it is proved that (3.21) entails (3.25) and vice versa. Let us show that (3.21) and (3.23) are equivalent too.

Let us firstly assume that g belongs to the closure of $C_o^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$.

Clearly, this entails that g is in the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$, thus by Lemma 3.1, one has that

$$\lim_{h \to +\infty} F[g](h) = 0. \tag{3.26}$$

It remains to show that

$$\lim_{h \to +\infty} \| (1 - \psi_h) g \|_{M^{p,\lambda}(\Omega)} = 0. \tag{3.27}$$

To this aim, observe that fixed $\varepsilon > 0$, there exists $g_\varepsilon \in C_o^\infty(\Omega)$ such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon. \tag{3.28}$$

On the other hand, if we consider the sets Ω_h defined in (3.20), one has

$$\Omega/\Omega_h = \{x \in \Omega \mid |x| \geq h\} \cup \left\{x \in \Omega \mid \alpha(x) \leq \frac{1}{h}\right\}. \tag{3.29}$$

Therefore, since g_ε has a compact support, there exists $h_\varepsilon \in \mathbb{R}_+$

$$(\Omega/\Omega_h) \cap \operatorname{supp} g_\varepsilon = \emptyset, \quad \forall h \geq h_\varepsilon. \tag{3.30}$$

Then, since $\psi_{h|\Omega_h} = 1$, one has that $\operatorname{supp}(1-\psi_h) \subset \Omega \setminus \Omega_h$, hence $(1-\psi_h)g_\varepsilon = 0$ for all $h \geq h_\varepsilon$.

The above considerations together with (3.28) give, for any $h \geq h_\varepsilon$,

$$\|(1-\psi_h)g\|_{M^{p,\lambda}(\Omega)} = \|(1-\psi_h)(g-g_\varepsilon)\|_{M^{p,\lambda}(\Omega)} \leq \|g-g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon, \tag{3.31}$$

that is, (3.27).

Conversely, assume that $g \in M^{p,\lambda}(\Omega)$ and that (3.23) holds.

First of all, we observe that denoted by g_o the zero extension of g to $\mathbb{R}^n$, by (2.7) of Remark 2.2, there exists a positive constant c_1, independent of g, ψ_h and of Ω, such that

$$\|(1-\psi_h)g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} \leq c_1\|(1-\psi_h)g\|_{M^{p,\lambda}(\Omega)}. \tag{3.32}$$

Furthermore, by (3.23), we get that fixed $\varepsilon > 0$, there exists h_ε such that

$$\|(1-\psi_{h_\varepsilon})g\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2c_1}. \tag{3.33}$$

Therefore,

$$\|(1-\psi_{h_\varepsilon})g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} < \frac{\varepsilon}{2}. \tag{3.34}$$

Set

$$\Phi_\varepsilon = \psi_{h_\varepsilon} g_o, \tag{3.35}$$

by construction

$$\operatorname{supp} \Phi_\varepsilon \subset \operatorname{supp} \psi_{h_\varepsilon} \subset \overline{\Omega}_{2h_\varepsilon}. \tag{3.36}$$

Hence, taking into account (2.8) of Remark 2.2, one has that

$$\Phi_\varepsilon \in L^{p,\lambda}(\mathbb{R}^n). \tag{3.37}$$

On the other hand, (3.23) together with Lemma 3.1 give that $g \in \widetilde{M}^{p,\lambda}(\Omega)$, then from Remark 3.2, we get

$$\lim_{t\to 0}\|g\|_{M^{p,\lambda}(\Omega,t)} = 0. \tag{3.38}$$

So, using (2.7) of Remark 2.2, we have that $\Phi_\varepsilon \in M^{p,\lambda}(\mathbb{R}^n)$ and

$$\lim_{t\to 0}\|\Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n,\, t)} = 0. \tag{3.39}$$

Arguing as in [16, Lemma 1.2], from (3.36)–(3.39), we conclude that

$$\lim_{y\to 0}\|\Phi_\varepsilon(x-y) - \Phi_\varepsilon(x)\|_{L^{p,\lambda}(\mathbb{R}^n)} = 0. \tag{3.40}$$

We are now in the hypotheses of Lemma 2.1. Hence, denoted by $(J_k)_{k\in\mathbb{N}}$ a sequence of mollifiers in $\mathbb{R}^n$, we can find a positive integer $k_\varepsilon > h_\varepsilon$ such that

$$\|\Phi_\varepsilon - J_{k_\varepsilon} * \Phi_\varepsilon\|_{L^{p,\lambda}(\mathbb{R}^n)} < \frac{\varepsilon}{2}. \tag{3.41}$$

Set $g_\varepsilon = J_{k_\varepsilon} * \Phi_\varepsilon$, one has $g_\varepsilon \in C_o^\infty(\Omega)$. Furthermore, using (3.34) and (3.41), we get

$$\begin{aligned}
\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} &\le \|g_o - J_{k_\varepsilon} * \Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n)} \\
&\le \|g_o - \Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n)} + \|\Phi_\varepsilon - J_{k_\varepsilon} * \Phi_\varepsilon\|_{M^{p,\lambda}(\mathbb{R}^n)} \\
&\le \|g_o - \psi_{h_\varepsilon} g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} + \|\Phi_\varepsilon - J_{k_\varepsilon} * \Phi_\varepsilon\|_{L^{p,\lambda}(\mathbb{R}^n)} \\
&\le \|(1-\psi_{h_\varepsilon}) g_o\|_{M^{p,\lambda}(\mathbb{R}^n)} + \frac{\varepsilon}{2} < \varepsilon,
\end{aligned} \tag{3.42}$$

this concludes the proof. □

4. Decompositions of Functions in $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$

The characterizations of the spaces $\widetilde{M}^{p,\lambda}(\Omega)$ and $M_o^{p,\lambda}(\Omega)$ naturally lead us to the introduction of the following *moduli of continuity*.

Let g be a function in $\widetilde{M}^{p,\lambda}(\Omega)$. A modulus of continuity of g in $\widetilde{M}^{p,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \leq \widetilde{\sigma}^{p,\lambda}[g](h),$$
$$\lim_{h \to +\infty} \widetilde{\sigma}^{p,\lambda}[g](h) = 0. \tag{4.1}$$

If g belongs to $M_o^{p,\lambda}(\Omega)$, a modulus of continuity of g in $M_o^{p,\lambda}(\Omega)$ is an application $\sigma_o{}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|(1-\zeta_h)g\|_{M^{p,\lambda}(\Omega)} + F[g](h) \leq \sigma_o{}^{p,\lambda}[g](h),$$
$$\lim_{h \to +\infty} \sigma_o{}^{p,\lambda}[g](h) = 0. \tag{4.2}$$

Let us show now the decomposition results.

Lemma 4.1. *Let $\lambda \in [0,n[$, $p \in [1,+\infty[$, and $g \in \widetilde{M}^{p,\lambda}(\Omega)$. For any $h \in \mathbb{R}_+$, one has*

$$g = g'_h + g''_h, \tag{4.3}$$

with $g''_h \in L^\infty(\Omega)$ and

$$\|g'_h\|_{M^{p,\lambda}(\Omega)} \leq \widetilde{\sigma}^{p,\lambda}[g](h), \qquad \|g''_h\|_{L^\infty(\Omega)} \leq h^{1/p}\|g\|_{M^{p,\lambda}(\Omega)}. \tag{4.4}$$

Proof. Given $g \in \widetilde{M}^{p,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$, we introduce the set

$$E_h = \left\{ x \in \Omega \mid |g(x)| \geq \ h^{1/p}\|g\|_{M^{p,\lambda}(\Omega)} \right\}. \tag{4.5}$$

Observe that

$$\begin{aligned} |E_h(x)| &\leq \int_{\Omega(x)\cap E_h} \frac{|g(y)|^p}{\|g\|^p_{M^{p,\lambda}(\Omega)} h} dy \\ &\leq \frac{1}{\|g\|^p_{M^{p,\lambda}(\Omega)} h} \int_{\Omega(x)} |g(y)|^p dy \leq \frac{1}{\|g\|^p_{M^{p,\lambda}(\Omega)} h} \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda} \|g\|^p_{L^p(\Omega(x,\tau))} = \frac{1}{h}. \end{aligned} \tag{4.6}$$

Set

$$g'_h = g\chi_{E_h} = \begin{cases} g & \text{if } x \in E_h, \\ 0 & \text{if } x \in \Omega/E_h, \end{cases} \qquad g''_h = g - g\chi_{E_h} = \begin{cases} 0 & \text{if } x \in E_h, \\ g & \text{if } x \in \Omega/E_h. \end{cases} \tag{4.7}$$

In view of (4.6),

$$\|g_h'\|_{M^{p,\lambda}(\Omega)} = \|g\chi_{E_h}\|_{M^{p,\lambda}(\Omega)} \le F[g](h) \le \widetilde{\sigma}^{p,\lambda}[g](h), \tag{4.8}$$

this gives the first inequality in (4.4), the second one easily follows from (4.5). □

Lemma 4.2. *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in M_o^{p,\lambda}(\Omega)$. For any $h \in \mathbb{R}_+$, one has*

$$g = \phi_h' + \phi_h'', \tag{4.9}$$

with

$$\|\phi_h'\|_{M^{p,\lambda}(\Omega)} \le \sigma_o{}^{p,\lambda}[g](h), \qquad |\phi_h''| \le \zeta_h h^{1/p} \|g\|_{M^{p,\lambda}(\Omega)}. \tag{4.10}$$

Proof. To prove this second decomposition result, we exploit again the definition of the set E_h introduced in (4.5) and inequality (4.6).

In this case, for any $h \in \mathbb{R}_+$, we define

$$\begin{aligned} \phi_h' &= g(1-\zeta_h) + \zeta_h g\chi_{E_h} = \begin{cases} g & \text{if } x \in E_h, \\ g(1-\zeta_h) & \text{if } x \in \Omega/E_h, \end{cases} \\ \phi_h'' &= \zeta_h(g - g\chi_{E_h}) = \begin{cases} 0 & \text{if } x \in E_h, \\ g\zeta_h & \text{if } x \in \Omega/E_h. \end{cases} \end{aligned} \tag{4.11}$$

To obtain the first inequality in (4.10), we observe that (4.6) gives

$$\begin{aligned} \|\phi_h'\|_{M^{p,\lambda}(\Omega)} &\le \|g(1-\zeta_h)\|_{M^{p,\lambda}(\Omega)} + \|\zeta_h g\chi_{E_h}\|_{M^{p,\lambda}(\Omega)} \\ &\le \|g(1-\zeta_h)\|_{M^{p,\lambda}(\Omega)} + \|g\chi_{E_h}\|_{M^{p,\lambda}(\Omega)} \\ &\le \|g(1-\zeta_h)\|_{M^{p,\lambda}(\Omega)} + F[g](h) \le \sigma_o{}^{p,\lambda}[g](h). \end{aligned} \tag{4.12}$$

The second one is a consequence of (4.5). □

5. A Compactness Result

In this section, as application, we use the previous results to prove the compactness of a multiplication operator on Sobolev spaces.

To this aim, let us recall an imbedding theorem proved in [2, Theorem 3.2].

Let us specify the assumptions:

(h_1) Ω is an open subset of $\mathbb{R}^n$ having the cone property with cone C, the parameters k, r, p, q, λ satisfy one of the following conditions:

(h_2) $k \in \mathbb{N}$, $1 \le p \le q \le r < +\infty$, $0 \le \lambda < n$, $\gamma = 1/q - 1/p + k/n > 0$, with $r > q$ when $p = n/k > 1$ and $\lambda = 0$, and with $\lambda > n(1 - r\gamma)$ when $r\gamma < 1$,

(h_3) $k = 1$, $1 < p = q < r \le n$, $\lambda = n - r$.

Theorem 5.1. *Under hypothesis (h_1) and if (h_2) or (h_3) holds, for any $u \in W^{k,p}(\Omega)$ and for any $g \in M^{r,\lambda}(\Omega)$, one has $gu \in L^q(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$, depending on n, k, p, q, r, λ, and C, such that*

$$\|gu\|_{L^q(\Omega)} \le c\|g\|_{M^{r,\lambda}(\Omega)}\|u\|_{W^{k,p}(\Omega)}. \tag{5.1}$$

Putting together Lemma 4.1 and Theorem 5.1, we easily have the following result.

Corollary 5.2. *Under hypothesis (h_1) and if (h_2) or (h_3) holds, for any $g \in \widetilde{M}^{r,\lambda}(\Omega)$ and for any $h \in \mathbb{R}_+$, one has*

$$\|gu\|_{L^q(\Omega)} \le c \cdot \tilde{\sigma}^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + h^{1/r} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{L^q(\Omega)}, \tag{5.2}$$

for each $u \in W^{k,p}(\Omega)$, where $c \in \mathbb{R}_+$ is the constant of (5.1).

If g is in $M_o^{r,\lambda}(\Omega)$, the previous estimate can be improved as showed in the corollary below.

Corollary 5.3. *Under hypothesis (h_1) and if (h_2) or (h_3) holds, for any $g \in M_o^{r,\lambda}(\Omega)$ and for any $h \in \mathbb{R}_+$, there exists an open set $A_h \subset\subset \Omega$ with the cone property, such that*

$$\|gu\|_{L^q(\Omega)} \le c \cdot {\sigma_o}^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + h^{1/r} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{L^q(A_h)}, \tag{5.3}$$

for each $u \in W^{k,p}(\Omega)$, where $c \in \mathbb{R}_+$ is the constant of (5.1).

Proof. Fix $g \in M_o^{r,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$. In view of Lemma 4.2 and Theorem 5.1, for any $u \in W^{k,p}(\Omega)$, we have

$$\begin{aligned}
\|gu\|_{L^q(\Omega)} &\le \|\phi'_h u\|_{L^q(\Omega)} + \|\phi''_h u\|_{L^q(\Omega)} \\
&\le c\|\phi'_h\|_{M^{r,\lambda}(\Omega)} \cdot \|u\|_{W^{k,p}(\Omega)} + \|\phi''_h u\|_{L^q(\Omega)} \\
&\le c \cdot {\sigma_o}^{r,\lambda}[g](h) \cdot \|u\|_{W^{k,p}(\Omega)} + \|\phi''_h u\|_{L^q(\Omega)}.
\end{aligned} \tag{5.4}$$

Using again Lemma 4.2, we obtain

$$\|\phi''_h u\|_{L^q(\Omega)} \le \|g\|_{M^{r,\lambda}(\Omega)} h^{1/r} \left(\int_\Omega |\zeta_h u|^q dx \right)^{1/q} \le \|g\|_{M^{r,\lambda}(\Omega)} h^{1/r} \left(\int_{\operatorname{supp} \zeta_h} |u|^q dx \right)^{1/q}. \tag{5.5}$$

Putting together (5.4) and (5.5), we get (5.3), with A_h obtained as follows: fixed $d_h \in]0, \operatorname{dist}(\operatorname{supp} \zeta_h, \partial\Omega)/2[$ and $\theta \in]0, \pi/2[$, the set A_h is union of the open cones $\mathcal{C} \subset\subset \Omega$ with opening θ, height d_h and such that $\mathcal{C} \cap \operatorname{supp} \ \zeta_h \neq \emptyset$. □

We are now in position to prove the compactness result.

Corollary 5.4. *Suppose that condition (h_1) is satisfied, that (h_2) or (h_3) holds, and fix $g \in M_o^{r,\lambda}(\Omega)$. Then, the operator*

$$u \in W^{k,p}(\Omega) \longrightarrow gu \in L^q(\Omega) \tag{5.6}$$

is compact.

Proof. Observe that if $\Omega' \subset\subset \Omega$ is a bounded open set with the cone property, the operator

$$u \in W^{k,p}(\Omega) \longrightarrow u \in L^q(\Omega') \tag{5.7}$$

is compact.

Indeed, if $\Omega' \subset\subset \Omega$ is a bounded open set, the operator

$$u \in W^{k,p}(\Omega) \longrightarrow u_{|_{\Omega'}} \in W^{k,p}(\Omega') \tag{5.8}$$

is linear and bounded. Moreover, since Ω' has the cone property, the Rellich-Kondrachov Theorem (see e.g., [17]) applies and gives that the operator

$$w \in W^{k,p}(\Omega') \longrightarrow w \in L^q(\Omega') \tag{5.9}$$

is compact.

Let us consider now a sequence $(u_n)_{n\in\mathbb{N}}$ bounded in $W^{k,p}(\Omega)$, and let $M \in \mathbb{R}_+$ be such that $\|u_n\|_{W^{k,p}(\Omega)} \leq M$ for all $n \in \mathbb{N}$. According to the above considerations, fixed $\varepsilon > 0$, there exist a subsequence $(u_{n_m})_{m\in\mathbb{N}}$ and $\nu \in \mathbb{N}$ such that

$$\|u_{n_m} - u_{n_l}\|_{L^q(\Omega')} \leq \varepsilon, \quad \forall m, l > \nu. \tag{5.10}$$

On the other hand, given $g \in M_o^{r,\lambda}(\Omega)$ and $h \in \mathbb{R}_+$, in view of Corollary 5.3, there exists a constant $c \in \mathbb{R}_+$ and an open set $A_h \subset\subset \Omega$ with the cone property, independent of u_n, such that

$$\|gu_n\|_{L^q(\Omega)} \leq c \cdot {\sigma_o}^{r,\lambda}[g](h) \cdot \|u_n\|_{W^{k,p}(\Omega)} + h^{1/r} \cdot \|g\|_{M^{r,\lambda}(\Omega)} \cdot \|u_n\|_{L^q(A_h)}. \tag{5.11}$$

From (5.11) and (5.10) written for $\varepsilon = (c \cdot {\sigma_o}^{r,\lambda}[g](h))/(h^{1/r} \cdot \ \|g\|_{M^{r,\lambda}(\Omega)})$ and $\Omega' = A_h$, for $m, l > \nu$, one has

$$\|gu_{n_m} - gu_{n_l}\|_{L^q(\Omega)} \leq c \cdot \sigma_o^{r,\lambda}[g](h) \cdot (2M + 1). \tag{5.12}$$

By (5.12) and (4.2), we conclude that $(gu_{n_m})_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega)$, which gives the compactness of the operator defined in (5.6). □

6. The Space $M_\rho^{p,\lambda}(\Omega)$

In this section, we introduce some weighted spaces of Morrey type settled between $M_o^{p,\lambda}(\Omega)$ and $\widetilde{M}^{p,\lambda}(\Omega)$. To this aim, given $d \in \mathbb{R}_+$, we consider the set $G(\Omega, d)$ defined in [18] as the class of measurable weight functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega\\ |x-y|<d}} \frac{\rho(x)}{\rho(y)} < +\infty. \tag{6.1}$$

It is easy to show that $\rho \in G(\Omega, d)$ if and only if there exists $\gamma \in \mathbb{R}_+$, independent on x and y, such that

$$\gamma^{-1}\rho(y) \le \rho(x) \le \gamma\rho(y), \quad \forall y \in \Omega,\ \forall x \in \Omega(y, d). \tag{6.2}$$

Furthermore,

$$\rho, \rho^{-1} \in L^\infty_{\text{loc}}\left(\overline{\Omega}\right). \tag{6.3}$$

We put

$$G(\Omega) = \bigcup_{d>0} G(\Omega, d). \tag{6.4}$$

For $p \in [1, +\infty[$, $s \in \mathbb{R}$, and $\rho \in G(\Omega)$, we denote by $L^p_s(\Omega)$ the Banach space made up of measurable functions $g : \Omega \to \mathbb{R}$ such that $\rho^s g \in L^p(\Omega)$ equipped with the norm

$$\|g\|_{L^p_s(\Omega)} = \|\rho^s g\|_{L^p(\Omega)}. \tag{6.5}$$

It can be proved that the space $C^\infty_o(\Omega)$ is dense in $L^p_s(\Omega)$ (see e.g., [18, 19]).

From now on, we consider $\rho \in G(\Omega) \cap L^\infty(\Omega)$, and we denote by d the positive real number such that $\rho \in G(\Omega, d)$.

Lemma 6.1. *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in M^{p,\lambda}(\Omega)$. The following properties are equivalent:*

$$g \text{ is in the closure of } L^{\infty}_{-1/p}(\Omega) \quad \text{in } M^{p,\lambda}(\Omega), \tag{6.6}$$

$$\lim_{h\to+\infty}\left(\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{\substack{x\in\Omega\\ \tau\in]0,d]}} \tau^{-\lambda}\rho(x)|E(x,\tau)|\le 1/h}} \|g\chi_E\|_{M^{p,\lambda}(\Omega)}\right) = 0, \tag{6.7}$$

$$\lim_{h\to+\infty}\left(\sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}\rho(x)|E(x,d)|\le 1/h}} \|g\chi_E\|_{M^{p,\lambda}(\Omega)}\right) = 0. \tag{6.8}$$

We denote by $M^{p,\lambda}_{\rho}(\Omega)$ the set of functions satisfying one of the above properties.

Proof of Lemma 6.1. We start proving the equivalence between (6.6) and (6.7). This proof is in the spirit of the one of Lemma 3.1. For the reader's convenience, we write down just few lines pointing out the main differences.

If (6.6) holds, fixed $\varepsilon > 0$, there exists a function $g_\varepsilon \in L^{\infty}_{-1/p}(\Omega)$ such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2}. \tag{6.9}$$

From (6.9), we get that for any $E \in \Sigma(\Omega)$,

$$\|g\chi_E\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2} + \|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega)}. \tag{6.10}$$

Furthermore, in view of the equivalence of the spaces $M^{p,\lambda}(\Omega, d)$ and $M^{p,\lambda}(\Omega)$ given by (2.3) and taking into account (6.2),

$$\begin{aligned}\|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega)} &\le c_1\|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega,d)} = c_1 \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda/p}\|g_\varepsilon\chi_E\|_{L^p(\Omega(x,\tau))}\\ &\le c_1\gamma^{1/p}\|g_\varepsilon\|_{L^{\infty}_{-1/p}(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\tau^{-\lambda}\rho(x)|E(x,\tau)|\right)^{1/p},\end{aligned} \tag{6.11}$$

where $c_1 \in \mathbb{R}_+$ depends only on n and d. Hence, set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{2\, c_1\gamma^{1/p}\|g_\varepsilon\|_{L^{\infty}_{-1/p}(\Omega)}}\right)^{p}, \tag{6.12}$$

from (6.11) we deduce that if $\sup_{\tau\in]0,d],\, x\in\Omega} \tau^{-\lambda}\rho(x)|E(x,\tau)| \leq 1/h_\varepsilon$, then

$$\left\|g_\varepsilon \chi_E\right\|_{M^{p,\lambda}(\Omega)} \leq \frac{\varepsilon}{2} . \tag{6.13}$$

Putting together (6.10) and (6.13), we obtain (6.7).

Now, assume that g is a function in $M^{p,\lambda}(\Omega)$ and that (6.7) holds. Then, for any $\varepsilon > 0$, there exists $h_\varepsilon \in \mathbb{R}_+$ such that if $E \in \Sigma(\Omega)$ with $\sup_{\tau\in]0,d],\, x\in\Omega} \tau^{-\lambda}\rho(x)|E(x,\tau)| \leq 1/h_\varepsilon$, then

$$\left\|g \chi_E\right\|_{M^{p,\lambda}(\Omega)} < \varepsilon. \tag{6.14}$$

For each $k \in \mathbb{R}_+$, we define the set

$$G_k = \left\{x \in \Omega \mid \rho^{-1/p}(x)\left|g(x)\right| \geq k\right\}. \tag{6.15}$$

Using again (2.3), there exists $c_2 \in \mathbb{R}_+$ depending on the same parameters as c_1 such that

$$\begin{aligned}\left\|g\right\|_{M^{p,\lambda}(\Omega)} &\geq c_2\left\|g\right\|_{M^{p,\lambda}(\Omega,d)} \geq c_2 \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda/p}\left\|g\right\|_{L^p(G_k(x,\tau))} \\ &\geq c_2\gamma^{-1/p}k \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\tau^{-\lambda}\rho(x)|G_k(x,\tau)|\right)^{1/p}.\end{aligned} \tag{6.16}$$

Therefore, if we put

$$k_\varepsilon = \frac{\gamma^{1/p}h_\varepsilon^{1/p}\left\|g\right\|_{M^{p,\lambda}(\Omega)}}{c_2}, \tag{6.17}$$

from (6.16), we obtain

$$\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda}\rho(x)|G_{k_\varepsilon}(x,\tau)| \leq \frac{1}{h_\varepsilon}, \tag{6.18}$$

and then

$$\left\|g\chi_{G_{k_\varepsilon}}\right\|_{M^{p,\lambda}(\Omega)} < \varepsilon. \tag{6.19}$$

We conclude setting $g_\varepsilon = g - g\chi_{G_{k_\varepsilon}}$. Indeed, by (6.15), $g_\varepsilon \in L^\infty_{-1/p}(\Omega)$ and (6.19) gives that $\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon$.

Arguing similarly, we prove also that (6.6) entails (6.8) and vice versa. Indeed, if $g \in M^{p,\lambda}(\Omega)$ and (6.6) holds, we can obtain as before (6.10) and (6.11).

On the other hand, there exists a constant $c_3 = c_3(n)$ such that

$$\begin{aligned}
\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\tau^{-\lambda}\cdot\rho(x)\cdot|E(x,\tau)|\right)^{1/p} &\le \|\rho\|^{\lambda/np}_{L^\infty(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda/p}\cdot\rho^{(n-\lambda)/np}(x)\cdot|E(x,\tau)|^{\lambda/np} \\
&\quad \cdot|E(x,\tau)|^{(n-\lambda)/np} \\
&\le c_3\cdot\|\rho\|^{\lambda/np}_{L^\infty(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\rho(x)\cdot|E(x,\tau)|\right)^{(n-\lambda)/np}.
\end{aligned} \tag{6.20}$$

Putting together (6.11) and (6.20), we obtain

$$\|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega)} \le c_4\gamma^{1/p}\|g_\varepsilon\|_{L^\infty_{-1/p}(\Omega)}\|\rho\|^{\lambda/np}_{L^\infty(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\rho(x)|E(x,\tau)|\right)^{(n-\lambda)/np}, \tag{6.21}$$

where $c_4 = c_1\cdot c_3$. Now, set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{2c_4\gamma^{1/p}\|g_\varepsilon\|_{L^\infty_{-1/p}(\Omega)}\|\rho\|^{\lambda/np}_{L^\infty(\Omega)}}\right)^{np/(n-\lambda)}, \tag{6.22}$$

from (6.21), we deduce that if $\sup_{\tau\in]0,d],\, x\in\Omega}\rho(x)|E(x,\tau)| \le 1/h_\varepsilon$, then

$$\|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2}. \tag{6.23}$$

From (6.10) and (6.23), we obtain (6.8).

Conversely, assume that (6.8) holds. We consider again the sets G_k introduced in (6.15). From (6.16), we get

$$\|g\|_{M^{p,\lambda}(\Omega)} \ge c_2\|g\|_{M^{p,\lambda}(\Omega,d)} \ge c_2 d^{-\lambda/p}\gamma^{-1/p}k\sup_{x\in\Omega}\left(\rho(x)|G_k(x,d)|\right)^{1/p}. \tag{6.24}$$

Therefore, if we put

$$k_\varepsilon = \frac{d^{\lambda/p}\gamma^{1/p}h_\varepsilon^{1/p}\|g\|_{M^{p,\lambda}(\Omega)}}{c_2}, \tag{6.25}$$

from (6.24), we obtain

$$\sup_{x\in\Omega}\rho(x)|G_{k_\varepsilon}(x,d)| \le \frac{1}{h_\varepsilon}, \tag{6.26}$$

and then, (6.8) being verified,

$$\|g\chi_{G_{k_\varepsilon}}\|_{M^{p,\lambda}(\Omega)} < \varepsilon. \tag{6.27}$$

We conclude the proof setting $g_\varepsilon = g - g\chi_{G_{k_\varepsilon}}$. Indeed, clearly $g_\varepsilon \in L^\infty_{-1/p}(\Omega)$ and (6.27) gives $\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \varepsilon$. □

Arguing in the spirit of Section 4, we want to obtain a decomposition result also for functions in $M^{p,\lambda}_\rho(\Omega)$. To this aim, we put for $h \in \mathbb{R}_+$ and $g \in M^{p,\lambda}(\Omega)$

$$D[g](h) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega} \rho(x)|E(x,d)|\le 1/h}} \|g\chi_E\|_{M^{p,\lambda}(\Omega)} \tag{6.28}$$

In view of the previous lemma, we can define a modulus of continuity of a function g in $M^{p,\lambda}_\rho(\Omega)$ as a map $\sigma_\rho{}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{gathered} D[g](h) \le \sigma^{p,\lambda}_\rho[g](h), \\ \lim_{h\to+\infty} \sigma^{p,\lambda}_\rho[g](h) = 0. \end{gathered} \tag{6.29}$$

Lemma 6.2. *Let $\lambda \in [0, n[$, $p \in [1, +\infty[$ and $g \in M^{p,\lambda}_\rho(\Omega)$. For any $h \in \mathbb{R}_+$, one has*

$$g = \varphi'_h + \varphi''_h, \tag{6.30}$$

with $\varphi''_h \in L^\infty_{-1/p}(\Omega)$ and

$$\|\varphi'_h\|_{M^{p,\lambda}(\Omega)} \le \sigma_\rho{}^{p,\lambda}[g](h), \qquad \|\varphi''_h\|_{L^\infty_{-1/p}(\Omega)} \le c\gamma^{1/p}h^{1/p}\|g\|_{M^{p,\lambda}(\Omega)}, \tag{6.31}$$

where c is a positive constant only depending on $n, d, p,$ and λ and where γ is that of (6.2).

Proof. Fix $g \in M^{p,\lambda}_\rho(\Omega)$, for any $h \in \mathbb{R}_+$, we set

$$\varphi'_h = g\chi_{G_h} = \begin{cases} g & \text{if } x \in G_h, \\ 0 & \text{if } x \in \Omega/G_h, \end{cases} \qquad \varphi''_h = g - g\,\chi_{G_h} = \begin{cases} 0 & \text{if } x \in G_h, \\ g & \text{if } x \in \Omega/G_h, \end{cases} \tag{6.32}$$

where

$$G_h = \left\{ x \in \Omega \mid \rho^{-1/p}(x)|g(x)| \ge d^{\lambda/p}\gamma^{1/p}h^{1/p}\|g\|_{M^{p,\lambda}(\Omega,d)} \right\}. \tag{6.33}$$

The thesis followed by (6.2) and (2.3) arguing as in the proof of Lemma 4.1. □

Let us show the following inclusion.

Lemma 6.3. *Let $\lambda \in [0, n[$ and $p \in [1, +\infty[$. Then, $L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega) \subset M^{p,\lambda}_{\rho}(\Omega)$, for all $\alpha \in \mathbb{R}_+$.*

Proof. For $\alpha \geq 1/p$, clearly $L^{\infty}_{-\alpha}(\Omega) \subset L^{\infty}_{-1/p}(\Omega)$ and then (6.6) holds. On the other hand, for $\alpha < 1/p$, we can show that if $g \in L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$, then (6.7) holds. Indeed, observe that by (2.3), there exists a constant $c_1 = c_1(n, d)$ such that for any $E \in \Sigma(\Omega)$

$$\begin{aligned}\|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leq c_1 \|g\chi_E\|_{M^{p,\lambda}(\Omega,d)} &= c_1 \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda/p} \|g\chi_E\|_{L^p(\Omega(x,\tau))} \\ &\leq c_1\gamma^{\alpha}\|g\|_{L^{\infty}_{-\alpha}(\Omega)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda/p}\rho^{\alpha}(x)|E(x,\tau)|^{1/p}.\end{aligned} \tag{6.34}$$

Moreover, there exists a constant $c_2 = c_2(n)$ such that

$$\begin{aligned}\sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \tau^{-\lambda/p}\rho^{\alpha}(x)|E(x,\tau)|^{1/p} &= \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\tau^{-\lambda}\ \rho(x)|E(x,\tau)|\right)^{\alpha}\left(\tau^{-\lambda}|E(x,\tau)|\right)^{1/p-\alpha} \\ &\leq c_2 d^{(n-\lambda)(1/p-\alpha)} \sup_{\substack{\tau\in]0,d]\\ x\in\Omega}} \left(\tau^{-\lambda}\rho(x)|E(x,\tau)|\right)^{\alpha}.\end{aligned} \tag{6.35}$$

Hence, fixed $\varepsilon > 0$ and set

$$\frac{1}{h_\varepsilon} = \left(\frac{\varepsilon}{c_1 \cdot c_2\gamma^{\alpha}\|g\|_{L^{\infty}_{-\alpha}(\Omega)} d^{(n-\lambda)(1/p-\alpha)}}\right)^{1/\alpha}, \tag{6.36}$$

we deduce that, if $\sup_{\tau\in]0,d],x\in\Omega}\tau^{-\lambda}\rho(x)|E(x,\tau)| \leq 1/h_\varepsilon$, then $\|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \varepsilon$. □

Now, we can prove a further characterization of $M^{p,\lambda}_{\rho}(\Omega)$.

Lemma 6.4. *Let $\lambda \in [0, n[, p \in [1, +\infty[$. Then, $M^{p,\lambda}_{\rho}(\Omega)$ is the closure of $\bigcup_{\alpha\in\mathbb{R}_+} L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ in $M^{p,\lambda}(\Omega)$.*

Proof. Clearly, if $g \in M^{p,\lambda}_{\rho}(\Omega)$ by (6.6), one has also that g is in the closure of $\bigcup_{\alpha\in\mathbb{R}_+} L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ in $M^{p,\lambda}(\Omega)$.

Conversely, let us prove that if g belongs to the closure of $\bigcup_{\alpha\in\mathbb{R}_+} L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ in $M^{p,\lambda}(\Omega)$, then (6.8) holds. Indeed, given $\varepsilon > 0$, there exists a function $g_\varepsilon \in L^{\infty}_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$, for an $\alpha \in \mathbb{R}_+$, such that

$$\|g - g_\varepsilon\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2}. \tag{6.37}$$

Hence, given $E \in \Sigma(\Omega)$

$$\|g\chi_E\|_{M^{p,\lambda}(\Omega)} \leq \|(g - g_\varepsilon)\chi_E\|_{M^{p,\lambda}(\Omega)} + \|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega)} < \frac{\varepsilon}{2} + \|g_\varepsilon\chi_E\|_{M^{p,\lambda}(\Omega)}. \tag{6.38}$$

Now, observe that since $g_\varepsilon \in L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$ by Lemma 6.3, we get $g_\varepsilon \in M^{p,\lambda}_\rho(\Omega)$, and therefore, using (6.8) of Lemma 6.1, we obtain that if $\sup_{x\in\Omega}\rho(x)|E(x,d)| \le 1/h$, then

$$\|g_\varepsilon \chi_E\|_{M^{p,\lambda}(\Omega)} \le \frac{\varepsilon}{2}. \tag{6.39}$$

This, together with (6.38), ends the proof. □

A straightforward consequence of the definitions (3.21) of Lemma 3.3, (6.6) of Lemma 6.1, and (3.2) of Lemma 3.1 is given by the following result.

Lemma 6.5. *Let $\lambda \in [0,n[$ and $p \in [1,+\infty[$. Then, $M^{p,\lambda}_o(\Omega) \subset M^{p,\lambda}_\rho(\Omega) \subset \widetilde{M}^{p,\lambda}(\Omega)$.*

Let us show that if ρ vanishes at infinity, the first inclusion stated in the lemma above becomes an identity.

Lemma 6.6. *Let $\lambda \in [0,n[$ and $p \in [1,+\infty[$. If ρ is such that*

$$\lim_{|x|\to+\infty} \rho(x) = 0, \tag{6.40}$$

then $M^{p,\lambda}_o(\Omega) = M^{p,\lambda}_\rho(\Omega)$.

Proof. We show the inclusion $M^{p,\lambda}_\rho(\Omega) \subset M^{p,\lambda}_o(\Omega)$, the converse being stated in Lemma 6.5. In view of Lemma 6.4, it is enough to verify that if (6.40) holds, then $L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega) \subset M^{p,\lambda}_o(\Omega)$, for any $\alpha \in \mathbb{R}_+$.

To this aim, given $\alpha \in \mathbb{R}_+$, we fix $g \in L^\infty_{-\alpha}(\Omega) \cap M^{p,\lambda}(\Omega)$, and we prove that (3.25) is satisfied. Observe that by Lemmas 6.3 and 6.5 $g \in \widetilde{M}^{p,\lambda}(\Omega)$. Moreover, for any $x \in \Omega$ and if $1 \le d$ there exists a constant $c = c(n)$ such that

$$\begin{aligned}\sup_{\tau\in]0,1]} \tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))} &\le \gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \sup_{\tau\in]0,1]} \tau^{-\lambda/p}\rho^\alpha(x)|\Omega(x,\tau)|^{1/p} \\ &\le c\gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)} \sup_{\tau\in]0,1]} \tau^{(n-\lambda)/p}\rho^\alpha(x) = c\gamma^\alpha\|g\|_{L^\infty_{-\alpha}(\Omega)}\rho^\alpha(x).\end{aligned} \tag{6.41}$$

On the other hand, if $d < 1$, clearly one has

$$\sup_{\tau\in]0,1]} \tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))} = \max\left\{\sup_{\tau\in]0,d]} \tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))}, \sup_{\tau\in]d,1]} \tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))}\right\}. \tag{6.42}$$

We can treat the first term on the right-hand side of this last equality as done in (6.41) obtaining

$$\sup_{\tau\in]0,d]} \tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))} \le d^{(n-\lambda)/p} c\gamma^\alpha \|g\|_{L^\infty_{-\alpha}(\Omega)}\rho^\alpha(x), \tag{6.43}$$

the constant $c = c(n)$ being the one of (6.41).

Concerning the second one, observe that for any $x \in \Omega$ and $\tau \in]d, 1]$, we have the inclusion $\Omega(x,\tau) \subset Q(x,\tau)$, where $Q(x,\tau)$ denotes an n-dimensional cube of center x and edge 2τ. Now, there exists a positive integer k such that we can decompose the cube $Q(x,1)$ in k cubes of edge less than $d/2$ and center x_i, with $x_i \in \Omega$ for $i = 1,\dots,k$. Therefore, $Q(x,1) \subset \bigcup_{i=1}^{k} B(x_i, d/2)$. Hence, for any $x \in \Omega$ and $\tau \in]d,1]$, we have, arguing as before with opportune modifications,

$$\tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))} \leq d^{-\lambda/p}\sum_{i=1}^{k}\|g\|_{L^p(\Omega(x_i,d/2))} \leq kd^{(n-\lambda)/p}c\gamma^{\alpha}\|g\|_{L^{\infty}_{-\alpha}(\Omega)}\rho^{\alpha}(x), \tag{6.44}$$

the constant $c = c(n)$ being the same of (6.41).

The thesis follows then from (6.41), (6.42), (6.43), and (6.44) passing to the limit as $|x| \to +\infty$, as a consequence of hypothesis (6.40). □

From the latter result, we easily obtain the following lemma.

Lemma 6.7. *Let $\lambda \in [0, n[$ and $p \in [1, +\infty[$. If $\rho,\ \sigma \in G(\Omega) \cap L^{\infty}(\Omega)$ and*

$$\lim_{|x|\to+\infty}\rho(x) = \lim_{|x|\to+\infty}\sigma(x) = 0, \tag{6.45}$$

then $M^{p,\lambda}_{\rho}(\Omega) = M^{p,\lambda}_{\sigma}(\Omega)$.

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