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Research Article

Weighted Differentiation Composition Operators from the Mixed-Norm Space to the *n*th Weigthed-Type Space on the Unit Disk

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The boundedness and compactness of the weighted differentiation composition operator from the mixed-norm space to the nth weighted-type space on the unit disk are characterized.

1. Introduction

Throughout this paper $\mathbb{D}$ will denote the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $H^\infty = H^\infty(\mathbb{D})$ the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

The mixed norm space $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^q_{H_{p,q,\gamma}} = \int_0^1 M_p^q(f,r)(1-r)^\gamma dr < \infty, \tag{1.1}$$

where

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p}. \tag{1.2}$$

A positive continuous function on $\mathbb{D}$ is called *weight*. Let $\mu(z)$ be a weight and $n \in \mathbb{N}_0$. The nth *weighted-type space* on $\mathbb{D}$, denoted by $\mathcal{W}_\mu^{(n)}(\mathbb{D})$, consists of all $f \in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f) := \sup_{z\in\mathbb{D}} \mu(z)\left|f^{(n)}(z)\right| < \infty. \tag{1.3}$$

The space was recently introduced by this author in [1] as an extension of several weighted-type spaces which attracted a lot of attention in last few decades. For instance, when $n = 0$, the space becomes the weighted-type space $H_\mu^\infty(\mathbb{D})$ (see, e.g., [2–4]), when $n = 1$, the Bloch-type space $\mathcal{B}_\mu(\mathbb{D})$ (see, e.g., [5–7]), and for $n = 2$, the Zygmund-type space $\mathcal{Z}_\mu(\mathbb{D})$. Some information on Zygmund-type spaces on $\mathbb{D}$ and some operators on them can be found, for example, in [8–10] and on the unit ball, for example, in [11, 12].

The quantity $b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f)$ is a seminorm on the nth weighted-type space $\mathcal{W}_\mu^{(n)}(\mathbb{D})$ and a norm on $\mathcal{W}_\mu^{(n)}(\mathbb{D})/\mathbb{P}_{n-1}$, where $\mathbb{P}_{n-1}$ is the set of all polynomials whose degrees are less than or equal to $n-1$. A natural norm on the nth weighted-type space is introduced as follows:

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} = \sum_{j=0}^{n-1}\left|f^{(j)}(0)\right| + b_{\mathcal{W}_\mu^{(n)}(\mathbb{D})}(f). \tag{1.4}$$

With this norm the nth weighted-type space becomes a Banach space.

The little nth weighted-type space, denoted by $\mathcal{W}_{\mu,0}^{(n)}(\mathbb{D})$, is a closed subspace of $\mathcal{W}_\mu^{(n)}(\mathbb{D})$ consisting of those f for which

$$\lim_{|z|\to 1} \mu(z)\left|f^{(n)}(z)\right| = 0. \tag{1.5}$$

An analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$ induces the composition operator C_φ on $H(\mathbb{D})$, defined by $C_\varphi(f)(z) = f(\varphi(z))$ for $f \in H(\mathbb{D})$ (see, e.g., [8, 13–16]).

Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, and $m \in \mathbb{N}$. Then the weighted differentiation composition operator, denoted by $D_{\varphi,u}^m$, is defined on $H(\mathbb{D})$ by

$$D_{\varphi,u}^m f(z) = u(z) f^{(m)}(\varphi(z)), \quad f \in H(\mathbb{D}). \tag{1.6}$$

Recently there has been some interest in studying some particular cases of operator $D_{\varphi,u}^m$ (see, e.g., [17–25]). For some other products of linear operators on spaces of holomorphic functions see also recent papers [11, 26–32].

Here we study the boundedness and compactness of the operator $D_{\varphi,u}^m$ from $H_{p,q,\gamma}$ to nth weighted-type spaces, where $n \in \mathbb{N}$.

Throughout this paper, constants are denoted by C; they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Auxiliary Results

Here we quote some auxiliary results which will be used in the proofs of the main results. The first lemma can be proved in a standard way (see, e.g., in [13, Proposition 3.11] or in [15, Lemma 3]).

Lemma 2.1. *Assume that $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, $p, q > 0$, $\gamma > -1$, φ is an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is compact if and only if $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $D^m_{\varphi,u} f_k \to 0$ in $\mathcal{W}^{(n)}_\mu$ as $k \to \infty$.*

The next lemma is known, but we give a proof of it for the benefit of the reader.

Lemma 2.2. *Assume that $n \in \mathbb{N}_0$, $0 < p, q < \infty$, $-1 < \gamma < \infty$ and $f \in H_{p,q,\gamma}$. Then there is a positive constant C independent of f such that*

$$\left|f^{(n)}(z)\right| \le C \frac{\|f\|_{H_{p,q,\gamma}}}{\left(1-|z|^2\right)^{(\gamma+1)/q+1/p+n}}. \tag{2.1}$$

Proof. By the monotonicity of the integral means, using the well-known asymptotic formula

$$\int_0^1 M_p^q(f,r)(1-r)^\gamma dr \asymp |f(0)|^q + \int_0^1 M_p^q\left(f^{(n)},r\right)(1-r)^{\gamma+nq} dr, \tag{2.2}$$

and Theorem 7.2.5 in [33], we have that

$$\begin{aligned}
\|f\|^q_{H_{p,q,\gamma}} &\ge \int_{(1+|z|)/2}^1 M_p^q\left(f^{(n)},r\right)(1-r)^{\gamma+nq} dr \\
&\ge C M_p^q\left(f^{(n)}, \frac{1+|z|}{2}\right)\left(1-|z|^2\right)^{\gamma+1+nq} \\
&\ge C\left(1-|z|^2\right)^{\gamma+1+nq+q/p}\left|f^{(n)}(z)\right|^q,
\end{aligned} \tag{2.3}$$

from which the result follows. □

The following lemma can be found in [34].

Lemma 2.3. *For $\beta > -1$ and $m > 1 + \beta$ one has*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \le C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1. \tag{2.4}$$

A proof of the next lemma can be found in [35, Lemma 2.3].

Lemma 2.4. *Assume $a > 0$ and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n-1 \\ a(a+1) & (a+1)(a+2) & \cdots & (a+n-1)(a+n) \\ & & \cdots & \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1\ ) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix}. \tag{2.5}$$

Then $D_n(a) = \prod_{j=1}^{n-1} j!$.

The following formula

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=1}^{n} f^{(k)}(\varphi(z)) \sum_{k_1,\ldots,k_n} \frac{n!}{k_1! \cdots k_n!} \prod_{j=1}^{n} \left(\frac{\varphi^{(j)}(z)}{j!} \right)^{k_j}, \tag{2.6}$$

where the second sum is over all nonnegative integers $k_1, k_2, \ldots, k_n$ satisfying $k = k_1 + k_2 + \cdots + k_n$ and $k_1 + 2k_2 + \cdots + nk_n = n$, is attributed to Faà di Bruno [36]. By using Bell polynomials $B_{n,k}(x_1, \ldots, x_{n-k+1})$ it can be written as follows:

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}\Big(\varphi'(z), \varphi''(z), \ldots, \varphi^{(n-k+1)}(z)\Big). \tag{2.7}$$

For $n \in \mathbb{N}$ the last sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), \varphi''(z), \ldots, \varphi^{(n+1)}(z)) = 0$; however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1 and we will use it.

The Leibnitz formula along with (2.6) yields

$$\big(u(z) g(\varphi(z))\big)^{(n)} = \sum_{l=0}^{n} C_l^n u^{(n-l)}(z) \sum_{k=0}^{l} g^{(k)}(\varphi(z)) B_{l,k}\Big(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\Big). \tag{2.8}$$

Hence we have the next result.

Lemma 2.5. *Assume that $g, u \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$. Then*

$$\big(u(z) g(\varphi(z))\big)^{(n)} = \sum_{k=0}^{n} g^{(k)}(\varphi(z)) \sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\Big(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\Big). \tag{2.9}$$

3. The Boundedness and Compactness of $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}$

This section characterizes the boundedness and compactness of the operator $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}$.

Theorem 3.1. *Suppose that $m, n \in \mathbb{N}$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded if and only if for each $k \in \{0, 1, \ldots, n\}$*

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+m+k}} < \infty. \tag{3.1}$$

Moreover if $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded, then the following asymptotic relation holds

$$\left\| D^m_{\varphi,u} \right\|_{H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu / \mathbb{P}_{n-1}} \asymp \sum_{k=0}^{n} I_k. \tag{3.2}$$

Proof. First assume that $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded; then there exists a constant C such that

$$\left\| D^m_{\varphi,u} f \right\|_{\mathcal{W}^{(n)}_\mu} \le C \|f\|_{H_{p,q,\gamma}} \tag{3.3}$$

for all $f \in H_{p,q,\gamma}$.

For a fixed $w \in \mathbb{D}$, $t \ge (\gamma+1)/q$, and constants $c_1, \ldots, c_{n+1}$, set

$$g_w(z) = \sum_{j=1}^{n+1} \frac{c_j}{\prod_{l=0}^{m-1}(j+t+1/p+l)} \widehat{g}_{w,j}(z), \tag{3.4}$$

where

$$\widehat{g}_{w,j}(z) = \frac{\left(1-|w|^2\right)^{j+t-(\gamma+1)/q}}{(1-\overline{w}z)^{1/p+j+t}}, \quad j = 1, \ldots, n+1. \tag{3.5}$$

By [33, Theorem 1.4.10], we get

$$M_p\left(\widehat{g}_{w,j}, r\right) \le C \frac{\left(1-|w|^2\right)^{j+t-(\gamma+1)/q}}{(1-r|w|)^{j+t}}, \quad j = 1, \ldots, n+1. \tag{3.6}$$

Applying Lemma 2.3, we have that

$$\begin{aligned} \|\widehat{g}_{w,j}\|_{H_{p,q,\gamma}}^q &= \int_0^1 M_p^q(\widehat{g}_{w,j}, r)(1-r)^\gamma dr \\ &\le C\int_0^1 \frac{\left(1-|w|^2\right)^{q(j+t)-(\gamma+1)}}{(1-r|w|)^{q(j+t)}}(1-r)^\gamma dr \\ &\le C. \end{aligned} \tag{3.7}$$

Therefore $g_w \in H_{p,q,\gamma}$, and moreover $\sup_{w\in\mathbb{D}}\|g_w\|_{H_{p,q,\gamma}} < \infty$.

Now we show that for each $s \in \{m, m+1, \ldots, m+n\}$, there are constants $c_1, c_2, \ldots, c_{n+1}$, such that

$$g_w^{(s)}(w) = \frac{\overline{w}^s}{\left(1-|w|^2\right)^{s+(\gamma+1)/q+1/p}}, \quad g_w^{(t)}(w) = 0, \qquad t \in \{m, \ldots, m+n\} \setminus \{s\}. \tag{3.8}$$

By differentiating function g_w, for each $s \in \{m, \ldots, m+n\}$, (3.8) becomes

$$\begin{gathered} c_1 + c_2 + \cdots + c_{n+1} = 0, \\ \left(t+p^{-1}+m+1\right)c_1 + \left(t+p^{-1}+m+2\right)c_2 + \cdots + \left(t+p^{-1}+m+n+1\right)c_{n+1} = 0, \\ \vdots \\ \prod_{j=1}^{s-m}\left(t+p^{-1}+m+j\right)c_1 + \cdots + \prod_{j=1}^{s-m}\left(t+p^{-1}+m+n+j\right)c_{n+1} = 1, \\ \vdots \\ \prod_{j=1}^{n}\left(t+p^{-1}+m+j\right)c_1 + \cdots + \prod_{j=1}^{n}\left(t+p^{-1}+m+n+j\right)c_{n+1} = 0. \end{gathered} \tag{3.9}$$

Applying Lemma 2.4 with $a = t + 1/p + m + 1 > 0$ and where $n \to n+1$, we see that the determinant of system (3.9) is different from zero, as claimed.

By $g_{w,k}$, $k \in \{0, 1, \ldots, n\}$, denote the corresponding family of functions which satisfy (3.8) with $s = m + k$. Then, for each fixed $k \in \{0, 1, \ldots, n\}$, inequality (3.3) along with (2.9) and (3.8) implies that for each $\varphi(w) \ne 0$

$$\begin{aligned} &\frac{\mu(w)\left|\varphi(w)\right|^{k+m}\left|\sum_{l=k}^n C_l^n u^{(n-l)}(w) B_{l,k}\left(\varphi'(w), \ldots, \varphi^{(l-k+1)}(w)\right)\right|}{\left(1-\left|\varphi(w)\right|^2\right)^{(\gamma+1)/q+1/p+k+m}} \\ &\quad \le C\sup_{w\in\mathbb{D}}\left\|D_{\varphi,u}^m\left(g_{\varphi(w),k}\right)\right\|_{\mathcal{W}_\mu^{(n)}} \le C\left\|D_{\varphi,u}^m\right\|_{H_{p,q,\gamma}\to\mathcal{W}_\mu^{(n)}}. \end{aligned} \tag{3.10}$$

From (3.10) it follows that for each $k \in \{0, 1, \ldots, n\}$,

$$\sup_{|\varphi(z)|>1/2} \frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} \leq C\|D_{\varphi,u}^m\|_{H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}}. \tag{3.11}$$

Let

$$h_k(z) = z^k, \quad k = m, \ldots, n+m. \tag{3.12}$$

Then clearly

$$\|h_k\|_{H_{p,q,\gamma}} \leq 1, \quad \text{for each } k \in \mathbb{N}. \tag{3.13}$$

By formula (2.9) applied to the function $f(z) = h_m(z)$ we get

$$\begin{aligned} \left(D_{\varphi,u}^m h_m\right)^{(n)}(z) &= h_m^{(m)}(\varphi(z)) \sum_{l=0}^{n} C_l^n u^{(n-l)}(z) B_{l,0}\left(\varphi'(z), \ldots, \varphi^{(l+1)}(z)\right) \\ &= m! \sum_{l=0}^{n} C_l^n u^{(n-l)}(z) B_{l,0}\left(\varphi'(z), \ldots, \varphi^{(l+1)}(z)\right), \end{aligned} \tag{3.14}$$

which along with the boundedness of the operator $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}$ and (3.13) implies that

$$m! \sup_{z\in\mathbb{D}} \mu(z) \left| \sum_{l=0}^{n} C_l^n u^{(n-l)}(z) B_{l,0}\left(\varphi'(z), \ldots, \varphi^{(l+1)}(z)\right) \right| \leq \left\| D_{\varphi,u}^m(z^m) \right\|_{\mathcal{W}_\mu^{(n)}} \leq \left\| D_{\varphi,u}^m \right\|_{H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}}. \tag{3.15}$$

Now assume that we have proved that for $j \in \{0, 1, \ldots, k-1\}$ and a $k \leq n$

$$\sup_{z\in\mathbb{D}} \mu(z) \left| \sum_{l=j}^{n} C_l^n u^{(n-l)}(z) B_{l,j}\left(\varphi'(z), \ldots, \varphi^{(l-j+1)}(z)\right) \right| \leq C \left\| D_{\varphi,u}^m \right\|_{H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}}. \tag{3.16}$$

Applying (2.9) to the function $f(z) = h_{m+k}(z)$, $k \in \{0, 1, \ldots, n\}$, and noticing that $h_{m+k}^{(s)}(z) \equiv 0$ for $s > m+k$, we get

$$\begin{aligned} \left(D_{\varphi,u}^m h_{m+k}\right)^{(n)}(z) &= \sum_{j=0}^{k} h_{m+k}^{(m+j)}(\varphi(z)) \sum_{l=j}^{n} C_l^n u^{(n-l)}(z) B_{l,j}\left(\varphi'(z), \ldots, \varphi^{(l-j+1)}(z)\right) \\ &= \sum_{j=0}^{k} (m+k) \cdots (k-j+1) (\varphi(z))^{k-j} \sum_{l=j}^{n} C_l^n u^{(n-l)}(z) B_{l,j}\left(\varphi'(z), \ldots, \varphi^{(l-j+1)}(z)\right). \end{aligned} \tag{3.17}$$

From (3.17), the boundedness of the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$, the fact that $\|\varphi\|_\infty \le 1$, the triangle inequality, noticing that $(m+k)!$ is the coefficient at $\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}(\varphi'(z),\dots,\varphi^{(l-k+1)}(z))$, and finally using hypothesis (3.16) we get

$$\sup_{z\in\mathbb{B}} \mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \le C\left\|D^m_{\varphi,u}\right\|_{H_{p,q,\gamma}\to\mathcal{W}^{(n)}_\mu}. \tag{3.18}$$

Hence by induction, (3.18) holds for each $k \in \{0,1,\dots,n\}$.

From (3.18), for each fixed $k \in \{0,1,\dots,n\}$

$$\begin{aligned} &\sup_{|\varphi(z)|\le 1/2} \frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}(\varphi'(z),\dots,\varphi^{(l-k+1)}(z))\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} \\ &\quad\le C\sup_{z\in\mathbb{B}} \mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \le C\left\|D^m_{\varphi,u}\right\|_{H_{p,q,\gamma}\to\mathcal{W}^{(n)}_\mu}. \end{aligned} \tag{3.19}$$

Inequalities (3.11) and (3.19) imply

$$\sum_{k=0}^n I_k \le C\left\|D^m_{\varphi,u}\right\|_{H_{p,q,\gamma}\to\mathcal{W}^{(n)}_\mu}. \tag{3.20}$$

Now assume that (3.1) holds. Then for any $f \in H_{p,q,\gamma}$, by (2.9) and Lemma 2.2 we have

$$\begin{aligned} \mu(z)\left|\left(D^m_{\varphi,u} f\right)^{(n)}(z)\right| &= \mu(z)\left|\sum_{k=0}^n f^{(m+k)}(\varphi(z))\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \\ &\le \mu(z)\sum_{k=0}^n \left|f^{(m+k)}(\varphi(z))\right|\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \end{aligned} \tag{3.21}$$

$$\le C\|f\|_{H_{p,q,\gamma}} \sum_{k=0}^n \frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}(\varphi'(z),\dots,\varphi^{(l-k+1)}(z))\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} \tag{3.22}$$

$$\le C\|f\|_{H_{p,q,\gamma}} \sum_{k=0}^n \sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}(\varphi'(z),\dots,\varphi^{(l-k+1)}(z))\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} \tag{3.23}$$

We also have that for each $s \in \{1, \ldots, n-1\}$

$$\left|\left(D^m_{\varphi,u} f\right)^{(s)}(0)\right| = \left|\sum_{k=0}^{s} f^{(m+k)}(\varphi(0)) \sum_{l=k}^{s} C_l^s u^{(s-l)}(0) B_{l,k}\left(\varphi'(0), \ldots, \varphi^{(l-k+1)}(0)\right)\right|$$
$$\leq C\|f\|_{H_{p,q,\gamma}} \sum_{k=0}^{s} \frac{\left|\sum_{l=k}^{s} C_l^s u^{(s-l)}(0) B_{l,k}(\varphi'(0), \ldots, \varphi^{(l-k+1)}(0))\right|}{\left(1-|\varphi(0)|^2\right)^{(\gamma+1)/q+1/p+m+k}}, \tag{3.24}$$
$$\left|\left(D^m_{\varphi,u} f\right)(0)\right| = |u(0)|\left|f^{(m)}(\varphi(0))\right| \leq C|u(0)| \frac{\|f\|_{H_{p,q,\gamma}}}{\left(1-|\varphi(0)|^2\right)^{(\gamma+1)/q+1/p+m}}.$$

Using (3.23), (3.24), and (3.1) it follows that the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu}$ is bounded.

From (3.23) and (3.20) the asymptotic relation (3.2) follows. □

Theorem 3.2. *Suppose that $m, n \in \mathbb{N}$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$ is bounded if and only if $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu}$ is bounded and for each $k \in \{0, 1, \ldots, n\}$*

$$\lim_{|z|\to 1} \mu(z) \left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right| = 0. \tag{3.25}$$

Proof. The boundedness of $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$ clearly implies that $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu}$ is bounded. Applying (2.9) to the function $f(z) = h_m(z)$ and using the assumption $D^m_{\varphi,u}(h_m) \in \mathcal{W}^{(n)}_{\mu,0}$ it follows that

$$\mu(z)\left|\left(D^m_{\varphi,u} h_m\right)^{(n)}(z)\right| = m!\mu(z)\left|\sum_{l=0}^{n} C_l^n u^{(n-l)}(z) B_{l,0}\left(\varphi'(z), \ldots, \varphi^{(l+1)}(z)\right)\right| \longrightarrow 0, \tag{3.26}$$

as $|z| \to 1$, which is (3.25) for $k = 0$.

Assume that we have proved the following inequalities:

$$\lim_{|z|\to 1} \mu(z) \left|\sum_{l=j}^{n} C_l^n u^{(n-l)}(z) B_{l,j}\left(\varphi'(z), \ldots, \varphi^{(l-j+1)}(z)\right)\right| = 0, \tag{3.27}$$

for $j \in \{0, 1, \ldots, k-1\}$ and a $k \leq n$.

Applying formula (2.9) to the function $f(z) = h_{m+k}(z), k \in \{0, 1, \ldots, n\}$, we get (3.17). From (3.17), by using the boundedness of function φ, the triangle inequality, noticing that the coefficient at $\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z))$ is independent of z, and finally using hypothesis (3.27), we easily obtain

$$\lim_{|z|\to 1}\mu(z)\left|\sum_{l=k}^{n}C_l^n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| = 0. \tag{3.28}$$

Hence by induction we get that (3.25) holds for each $k \in \{0,1,\dots,n\}$.

Now assume that $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded and (3.25) holds for each $k \in \{0,1,\dots,n\}$. For each polynomial p we have

$$\begin{aligned}\mu(z)\left|\left(D^m_{\varphi,u}p\right)^{(n)}(z)\right| &= \mu(z)\left|\sum_{k=0}^{n}p^{(k)}(\varphi(z))\sum_{l=k}^{n}C_l^n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \\ &\le \sum_{k=0}^{n}\left\|p^{(k)}\right\|_\infty \mu(z)\left|\sum_{l=k}^{n}C_l^n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \longrightarrow 0,\end{aligned} \tag{3.29}$$

as $|z| \to 1$.

From (3.29) we have that, for each polynomial p, $D^m_{\varphi,u}p \in \mathcal{W}^{(n)}_{\mu,0}$. The set of all polynomials is dense in $H_{p,q,\gamma}$, so we have that for each $f \in H_{p,q,\gamma}$, there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{H_{p,q,\gamma}} \to 0$ as $k \to \infty$. Thus the boundedness of $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ implies

$$\left\|D^m_{\varphi,u}f - D^m_{\varphi,u}p_k\right\|_{\mathcal{W}^{(n)}_\mu} \le \left\|D^m_{\varphi,u}\right\|_{H_{p,q,\gamma}\to\mathcal{W}^{(n)}_\mu}\|f - p_k\|_{H_{p,q,\gamma}} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.30}$$

Hence $D^m_{\varphi,u}(H_{p,q,\gamma}) \subseteq \mathcal{W}^{(n)}_{\mu,0}$, from which the boundedness of $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$ follows, completing the proof of the theorem. □

Theorem 3.3. *Suppose that $m, n \in \mathbb{N}$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$, and μ is a weight. Then the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is compact if and only if $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded and for each $k \in \{0,1,\dots,n\}$*

$$\lim_{|\varphi(z)|\to 1}\frac{\mu(z)\left|\sum_{l=k}^{n}C_l^n u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} = 0. \tag{3.31}$$

Proof. First assume that $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_\mu$ is bounded and (3.31) holds. By Theorem 3.1 we have that for each $k \in \{0,1,\dots,n\}$, (3.1) holds.

Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in $H_{p,q,\gamma}$ such that $\sup_{i\in\mathbb{N}}\|f_i\|_{H_{p,q,\gamma}} \le L$ and f_i converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that for each $k \in \{0,1,\dots,n\}$ and $\delta < |\varphi(z)| < 1$

$$\frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} < \varepsilon. \tag{3.32}$$

We have

$$\begin{aligned}
&\left\|D_{\varphi,u}^m f_i\right\|_{\mathcal{W}_\mu^{(n)}} \\
&\quad= \sup_{z\in\mathbb{D}}\mu(z)\left|\left(D_{\varphi,u}^m f_i\right)^{(n)}(z)\right| + \sum_{j=0}^{n-1}\left|\left(D_{\varphi,u}^m f_i\right)^{(j)}(0)\right| \\
&\quad= \sup_{z\in\mathbb{D}}\mu(z)\left|\sum_{k=0}^{n} f_i^{(m+k)}(\varphi(z))\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \\
&\qquad+ \sum_{j=0}^{n-1}\left|\sum_{k=0}^{j} f_i^{(m+k)}(\varphi(0))\sum_{l=k}^{j} C_l^j u^{(j-l)}(0) B_{l,k}\left(\varphi'(0),\dots,\varphi^{(l-k+1)}(0)\right)\right| \\
&\quad\le \left(\sup_{|\varphi(z)|\le\delta} + \sup_{|\varphi(z)|>\delta}\right)\mu(z)\sum_{k=0}^{n}\left|f_i^{(m+k)}(\varphi(z))\right|\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \\
&\qquad+ \sum_{j=0}^{n-1}\left|\sum_{k=0}^{j} f_i^{(m+k)}(\varphi(0))\sum_{l=k}^{j} C_l^j u^{(j-l)}(0) B_{l,k}\left(\varphi'(0),\dots,\varphi^{(l-k+1)}(0)\right)\right| = J_1 + J_2 + J_3.
\end{aligned} \tag{3.33}$$

Now we estimate J_1, J_2, and J_3:

$$\begin{aligned}
J_1 &= \sup_{|\varphi(z)|\le\delta}\mu(z)\sum_{k=0}^{n}\left|f_i^{(m+k)}(\varphi(z))\right|\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \\
&\le \sum_{k=0}^{n}\sup_{|w|\le\delta}\left|f_i^{(m+k)}(w)\right|\sup_{|\varphi(z)|\le\delta}\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right| \\
&\le \sum_{k=0}^{n}\sup_{|w|\le\delta}\left|f_i^{(m+k)}(w)\right|\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\dots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+m+k}} \\
&= \sum_{k=0}^{n}\sup_{|w|\le\delta}\left|f_i^{(m+k)}(w)\right| I_k \longrightarrow 0, \quad \text{as } i \longrightarrow \infty,
\end{aligned} \tag{3.34}$$

where in (3.34) we have used the fact that from $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ it follows that for each $s \in \mathbb{N}$, $f_i^{(s)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$.

The fact that

$$J_3 = \sum_{j=0}^{n-1}\left|\sum_{k=0}^{j} f_i^{(m+k)}(\varphi(0))\sum_{l=k}^{j} C_l^j u^{(j-l)}(0) B_{l,k}\left(\varphi'(0),\ldots,\varphi^{(l-k+1)}(0)\right)\right| \longrightarrow 0, \tag{3.35}$$

as $i \to \infty$, is proved similarly; so we omit it.

By Lemma 2.2 and (3.32) we have that

$$J_2 \le C\|f_i\|_{H_{p,q,\gamma}} \sum_{k=0}^{n} \sup_{|\varphi(z)|>\delta} \frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} < C\varepsilon(n+1)L. \tag{3.36}$$

From (3.34), (3.35), and (3.36) we obtain

$$\lim_{i\to\infty}\left\|D_{\varphi,u}^m f_i\right\|_{\mathcal{W}_\mu^{(n)}} = 0. \tag{3.37}$$

From this and applying Lemma 2.1 the implication follows.

Now assume that $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}$ is compact; then clearly $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}$ is bounded. Let $(z_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. If such a sequence does not exist, then the conditions in (3.31) automatically hold.

Let $g_{w,k}$, $k \in \{0,1,\ldots,n\}$ be as in Theorem 3.1. Then the sequences $(g_{\varphi(z_i),k})_{i\in\mathbb{N}}$ are bounded and $g_{\varphi(z_i),k} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. Since $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_\mu^{(n)}$ is compact, we have that for each $k \in \{0,1,\ldots,n\}$

$$\lim_{i\to\infty}\left\|D_{\varphi,u}^m g_{\varphi(z_i),k}\right\|_{\mathcal{W}_\mu^{(n)}} = 0. \tag{3.38}$$

On the other hand, from (3.10) we obtain

$$\left\|D_{\varphi,u}^m g_{\varphi(z_i),k}\right\|_{\mathcal{W}_\mu^{(n)}} \ge \frac{C\mu(z_i)|\varphi(z_i)|^{k+m}\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z_i) B_{l,k}\left(\varphi'(z_i),\ldots,\varphi^{(l-k+1)}(z_i)\right)\right|}{\left(1-|\varphi(z_i)|^2\right)^{(\gamma+1)/q+1/p+k+m}}, \tag{3.39}$$

which along with $|\varphi(z_i)| \to 1$ as $i \to \infty$ and (3.38) implies that

$$\lim_{i\to\infty}\frac{\mu(z_i)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z_i) B_{l,k}\left(\varphi'(z_i),\ldots,\varphi^{(l-k+1)}(z_i)\right)\right|}{\left(1-|\varphi(z_i)|^2\right)^{(\gamma+1)/q+1/p+k+m}}, \tag{3.40}$$

for each $k \in \{0,1,\ldots,n\}$, from which (3.31) holds in this case. □

4. The Compactness of the Operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$

The compactness of $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$ is characterized here. The proof of the next lemma is similar to the proof of the corresponding result in [14].

Lemma 4.1. *Suppose that $n \in \mathbb{N}_0$ and μ is a radial weight such that $\lim_{|z|\to 1}\mu(z) = 0$. A closed set K in $\mathcal{W}^{(n)}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)\left|f^{(n)}(z)\right| = 0. \tag{4.1}$$

Theorem 4.2. *Suppose that $m, n \in \mathbb{N}$, $0 < p, q < \infty$, $-1 < \gamma < \infty$, φ is an analytic self-map of the unit disk, $u \in H(\mathbb{D})$ and μ is a radial weight such that $\lim_{|z|\to 1}\mu(z) = 0$. Then the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$ is compact if and only if for each $k \in \{0, 1, \ldots, n\}$*

$$\lim_{|z|\to 1} \frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1 - |\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} = 0. \tag{4.2}$$

Proof. First assume that $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu,0}$ is compact. Then it is bounded and since the test functions in (3.12) belong to $H_{p,q,\gamma}(\mathbb{D})$, we have that (3.25) holds. Beside this the operator $D^m_{\varphi,u} : H_{p,q,\gamma} \to \mathcal{W}^{(n)}_{\mu}$ is compact too, so that (3.31) holds. Hence, if $\|\varphi\|_\infty < 1$, from (3.25) for each $k \in \{0, 1, \ldots, n\}$ we get

$$\begin{aligned} &\frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1 - |\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} \\ &\quad\le \frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1 - \|\varphi\|_\infty^2\right)^{(\gamma+1)/q+1/p+k+m}} \longrightarrow 0, \end{aligned} \tag{4.3}$$

as $|z| \to 1$, hence we obtain (4.2) in this case.

Now assume $\|\varphi\|_\infty = 1$. Let $(\varphi(z_i))_{i\in\mathbb{N}}$ be a sequence such that $|\varphi(z_i)| \to 1$ as $i \to \infty$. Then from (3.31) we have that for every $\varepsilon > 0$, there is an $r \in (0, 1)$ such that for each $k \in \{0, 1, \ldots, n\}$

$$\frac{\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right|}{\left(1 - |\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} < \varepsilon \tag{4.4}$$

when $r < |\varphi(z)| < 1$, and from (3.25) there exists a $\sigma \in (0, 1)$ such that for $\sigma < |z| < 1$

$$\mu(z)\left|\sum_{l=k}^n C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z), \ldots, \varphi^{(l-k+1)}(z)\right)\right| < \varepsilon\left(1 - r^2\right)^{(\gamma+1)/q+1/p+k+m}. \tag{4.5}$$

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} < \varepsilon. \tag{4.6}$$

On the other hand, if $|\varphi(z)| \le r$ and $\sigma < |z| < 1$, from (4.5) we obtain

$$\begin{aligned}
&\frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{\left(1-|\varphi(z)|^2\right)^{(\gamma+1)/q+1/p+k+m}} \\
&\quad < \frac{\mu(z)\left|\sum_{l=k}^{n} C_l^n u^{(n-l)}(z) B_{l,k}\left(\varphi'(z),\ldots,\varphi^{(l-k+1)}(z)\right)\right|}{(1-r^2)^{(\gamma+1)/q+1/p+k+m}} < \varepsilon.
\end{aligned} \tag{4.7}$$

Combining the last two inequalities we obtain (4.2), as desired.

Now assume that (4.2) holds. Taking the supremum in (3.22) over f in the unit ball of $H_{p,q,\gamma}$, then letting $|z| \to 1$ is such obtained inequality and using (4.2) we get

$$\lim_{|z|\to 1} \sup_{\|f\|_{H_{p,q,\gamma}} \le 1} \mu(z)\left|\left(D_{\varphi,u}^m f\right)^{(n)}(z)\right| = 0. \tag{4.8}$$

Hence by Lemma 4.1 the compactness of the operator $D_{\varphi,u}^m : H_{p,q,\gamma} \to \mathcal{W}_{\mu,0}^{(n)}$ follows. □

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