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Research Article

A Class of Fan-Browder Type Fixed-Point Theorem and Its Applications in Topological Space

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A fixed-point theorem is proved under noncompact setting of general topological spaces. By applying the fixed-point theorem, several new existence theorems of solutions for equilibrium problems are proved under noncompact setting of topological spaces. These theorems improve and generalize the corresponding results in related literature.

1. Introduction

Let X and Y be nonempty sets, let $T : X \to Y$ be a single-valued mapping, let $A : X \to 2^X$ be a set-valued mapping, let $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ and $\phi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ be functions. The quasi-equilibrium problem QEP(T, A, f) is to find $\hat{x} \in X$ such that

$$\hat{x} \in A(\hat{x}), \quad f(\hat{x}, T(\hat{x})) \leq f(y, T(\hat{x})), \quad \forall y \in A(\hat{x}). \tag{1.1}$$

The QEP(T, A, f) was introduced and studied by Noor and Oettli [1]. Cubiotti [2] and Ding [3] proved some existence theorems of solutions for the QEP(T, A, f) in finite-dimensional space $\mathbb{R}^n$ and topological vector spaces, respectively.

The quasi-equilibrium problem QEP(A, ϕ) is to find $\hat{x} \in X$ such that

$$\hat{x} \in A(\hat{x}), \quad \phi(y, \hat{x}) \geq 0, \quad \forall y \in A(\hat{x}). \tag{1.2}$$

The QEP(A, ϕ) has been studied by many authors; for example, see [4–8] and others.

The QEP(T, A, f) and QEP(A, ϕ) include many optimization problems, Nash-type equilibrium problems, quasivariational inequality problems, quasi-complementary problems, and others as special cases; see [1–8] and the references therein.

In this paper, we first prove a new Fan-Browder-type fixed-point theorem under noncompact setting of general topological spaces. Next, by applying the fixed-point theorem, some new existence theorems of solutions for the QEP(T, A, f) and QEP(A, ϕ) are proved in noncompact setting of general topological spaces. These results include a number of important known results in the fields as special cases.

2. Preliminaries

For a set X, we will denote by 2^X and $\langle X \rangle$ the family of all subsets of X and the family of all nonempty finite subsets of X. For any $A \in \langle X \rangle$, let $|A|$ denote the cardinality of A. Let Δ_n denote the standard n-dimensional simplex with vertices $\{e_0, e_1, \ldots, e_n\}$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$. A set-valued mapping $F : X \to 2^Y$ is a function from a set X into the power set 2^Y of Y, that is, a function with the value $F(x) \subset Y$ for each $x \in X$ and the fiber $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for each $y \in Y$. For $A \subset X$, let $F(A) = \bigcup\{F(x) : x \in A\}$.

A subset A of X is said to be compactly open (resp., compactly closed) in X if, for any nonempty compact subset K of X, $A \cap K$ is open (resp., closed) in K. The following notions were introduced by Ding [9]. For any given nonempty subset A of X, we define the compact closure and the compact interior of A, denoted by $\mathrm{ccl}(A)$ and $\mathrm{cint}(A)$, as

$$\begin{aligned} \mathrm{ccl}(A) &= \bigcap\{B \subset X : A \subset B \text{ and } B \text{ is compactly closed in } X\}, \\ \mathrm{cint}(A) &= \bigcap\{B \subset X : B \subset A \text{ and } B \text{ is compactly open in } X\}. \end{aligned} \tag{2.1}$$

If A is a subset of a vector space, then $\mathrm{co}(A)$ denotes the convex hull of A.

If X and Y are two topological spaces and $G : X \to 2^Y$ is a set-valued mapping, then G is said to be transfer compactly open-valued (resp., transfer compactly closed-valued) on X if, for each $x \in X$ and for each compact subset K of Y with $G(x) \cap K \neq \emptyset$, $y \in G(x) \cap K$ (resp., $y \notin G(x) \cap K$) implies that there exists a point $x' \in X$ such that $y \in \mathrm{int}_K(G(x') \cap K)$ (resp., $y \notin \mathrm{cl}_K(G(x') \cap K)$). A set-valued mapping $G : X \to 2^Y$ is said to have the compactly local intersection property on X if, for each nonempty compact subset K of X and for each $x \in X$ with $G(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x in X such that $\bigcap_{z \in N(x) \cap K} G(z) \neq \emptyset$; see [10]. A multivalued map $G(x)$ is said to be transfer open-valued [11], if for any $x \in X$, $y = g(x)$ there exists a $Z \subset X$ such that $y \in \mathrm{int}_Y(Z)$.

In [12], Deng and Xia introduced the following concept which is crucial to the study of KKM theory in general topological spaces.

Let X be a nonempty set and let Y be a topological space. $W : X \to 2^Y$ is said to be a generalized relatively KKM (R-KKM) mapping if, for each $A \in \langle X \rangle$ with $|A| = n + 1$, there exists a continuous mapping $\varphi_A : \Delta_n \to Y$ such that, for each $J \in \langle A \rangle$,

$$\varphi_A(\Delta_J) \subset W(J), \tag{2.2}$$

where Δ_J is the standard subsimplex of Δ_n corresponding to J.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

In order to prove our main theorems, we need the following results.

Lemma 2.1. *Let X and Y be two topological spaces and $G : X \to 2^Y$ a set-valued mapping with nonempty values. Then the following conditions are equivalent:*

(I) *G has the compactly local intersection property,*

(II) *for each nonempty compact subset K of X and for each $y \in Y$, there exists an open subset O_y of X (which may be empty) such that $O_y \cap K \subset G^{-1}(y)$ and $K = \bigcup_{y\in Y}(O_y \cap K)$,*

(III) *for each nonempty compact subset K of X, there exists a set-valued mapping $F : X \to 2^Y$ such that for each $y \in Y$, $F^{-1}(y)$ is open or empty in X and $F^{-1}(y) \cap K \subset G^{-1}(y)$ for each $y \in Y$, and $K = \bigcup_{y\in Y}(F^{-1}(y) \cap K)$,*

(IV) *for each nonempty compact subset K of X and for each $x \in K$, there exists $y \in Y$ such that $x \in \operatorname{cint} G^{-1}(y) \cap K$ and $K = \bigcup_{y\in Y}(\operatorname{cint} G^{-1}(y) \cap K) = \bigcup_{y\in Y}(G^{-1}(y) \cap K)$,*

(V) *$G^{-1} : Y \to 2^X$ is transfer compactly open-valued on X,*

(VI) *$X = \bigcup_{y\in Y} \operatorname{cint} G^{-1}(y)$,*

(VII) *for each $y \in Y$, $G^{-1}(y) = \{x \in X; y \in G(x)\}$ contains a relatively open subset O_y of Y (O_y could be empty set for some y) such that $\bigcup_{y\in Y} O_y = Y$,*

(VIII) *let $S, T : k \to 2^k$ be two multivalued maps, $\operatorname{co}(G(x)) \subset T(x)$ and $G(x)$ is nonempty, and G^{-1} is open in X.*

Proof. By Lemma 1.1 of Ding in [10], (I), (II), (III), (IV), and (V) are equivalent. By Lemma 2.2 of Lin and Ansari in [13], (V) and (VI) are equivalent, and by Ansari in [14], (VI), (VII), and (VIII) are equivalent. This completes our proof. □

Remark 2.2. Lemma 2.1 includes Lemma 1.1 of Ding in [10] and Lemma 2.2 of Ansari in [13] as special cases.

Lemma 2.3 (see [15]). *Let X and Y be topological spaces, let D be a nonempty closed subset of X, and let $\Phi, \Psi : X \to 2^Y$ be two set-valued mappings such that $\Phi(x) \subset \Psi(x)$ for each $x \in X$. Suppose that $\Phi^{-1}, \Psi^{-1} : Y \to 2^X$ are both transfer compactly open-valued on Y. Then the mapping $G : X \to 2^Y$ defined by*

$$G(x) = \begin{cases} \Phi(x) & \text{if } x \in D, \\ \Psi(x) & \text{if } x \in X \setminus D \end{cases} \tag{2.3}$$

is such that $G^{-1} : Y \to 2^X$ is also transfer compactly open-valued on Y.

3. Fan-Browder Type Fixed-Point Theorem

Theorem 3.1. *Let X be a topological space, let K be a nonempty compact subset of X, and let $G : X \to 2^X$ be such that*

(i) *G has nonempty values and satisfies one of the conditions (I)–(VIII) in Lemma 2.1,*

(ii) *either,*

(a) *for each* $N \in \mathcal{F}(X)$, *there is a nonempty compact subset* L_N *of* X *containing* N *such that there is a generalized R-KKM mapping* $W : L_N \to 2^{L_N}$ *satisfying, for each* $x \in X$, *the fact that* $M \in \langle G(x) \cap L_N \rangle$ *implies that* $W(M) \subset G(x)$, *having, moreover, that* $L_N \setminus K \subset \bigcup_{y \in L_N} \operatorname{cint} G^{-1}(y)$,

or

(b) *if* X *is a nonempty convex set of a topological vector space, for each* $N \in F(X)$, *there is a nonempty compact subset* L_N *of* X *containing* N *such that there is a mapping* $W : L_N \to 2^L$ *satisfying, for each* $D \subset N$, *the fact that* $W(D)$ *is convex and, for each* $x \in X$, *the fact that* $M \in \langle G(x) \cap L_N \rangle$ *implies that* $W(M) \subset G(x)$.

Then there exists a point $x_0 \in X$ *such that* $x_0 \in G(x_0)$.

Proof. By (i) and Lemma 2.1, we have $X = \bigcup_{y \in X} \operatorname{cint} G^{-1}(y)$. Since K is a nonempty compact subset of X, there exists a finite set $N \in \mathcal{F}(X)$ such that

$$K = \bigcup_{y \in N} \left(\operatorname{cint} G^{-1}(y) \cap K \right). \tag{3.1}$$

For the N, consider the compact subset $L_N \subset X$ in condition (ii) satisfying

$$L_N \setminus K \subset \bigcup_{y \in L_N} \operatorname{cint} G^{-1}(y). \tag{3.2}$$

By (3.1), we have

$$L_N \cap K \subset \bigcup_{y \in N} \left(\operatorname{cint} G^{-1}(y) \cap L_N \right). \tag{3.3}$$

Noting that $N \subset L_N$, it follows from (3.2) and (3.3) that $L_N = \bigcup_{y \in L_N} (\operatorname{cint} G^{-1}(y) \cap L_N)$. Since L_N is compact, there exists a finite set $\{y_0, \ldots, y_m\} \subset L_N$ such that

$$L_N = \bigcup_{i=0}^{m} \left(\operatorname{cint} G^{-1}(y_i) \cap L_N \right). \tag{3.4}$$

Since $W : L_N \to 2^{L_N}$ is a generalized R-KKM mapping, there exists a continuous mapping $f : \Delta_m \to L_N$ such that $f(\Delta_J) \subset W(J)$ for each $\emptyset \neq J \subset \{y_0, \ldots, y_m\}$, where Δ_J is the face of Δ_m corresponding to $J \subset \{y_0, \ldots, y_m\}$. Let $\{\varphi_i\}_{i=0}^m$ be a continuous partition of unity subordinated to the open covering $\{\operatorname{cint} G^{-1}(y_i) \cap L_N\}_{i=0}^m$, that is, for each $i \in \{0, 1, \ldots, m\}$, $\varphi_i : L_N \to [0,1]$ is continuous, and

$$\{x \in L_N : \varphi_i(x) \neq 0\} \subset \operatorname{cint} G^{-1}(y_i) \cap L_N \subset G^{-1}(y_i) \tag{3.5}$$

such that $\sum_{i=0}^m \varphi_i(x) = 1$ for all $x \in L_N$. Define $\varphi : L_N \to \Delta_m$ by

$$\varphi(x) = (\varphi_0(x), \varphi_1(x), \ldots, \varphi_m(x)), \quad \forall x \in L_N. \tag{3.6}$$

Then $\varphi(x) \in \Delta_{J(x)}$ for all $x \in L_N$, where $J(x) = \{y_j = y_0, y_1, \ldots, y_m : \varphi_j(x) \neq 0\}$. Therefore, we have

$$f(\varphi(x)) \in f(\Delta_{J(x)}) \subset W(J(x)) \subset G(x), \quad \forall x \in L_N. \tag{3.7}$$

It is easy to see that $\varphi \circ f : \Delta_m \to \Delta_m$ is continuous. By Browder's fixed-point theorem, there exists $z \in \Delta_m$ such that $z = (\varphi \circ f)(z)$. Let $\hat{x} = f(z)$, then we have $\hat{x} = f(z) = f(\varphi(z)) \subset G(\hat{x})$. This completes our proof. □

Remark 3.2. Theorem 3.1 generalized Theorem 2.1 of Ding in [15] by dropping all the contractible conditions. Theorem 3.1 is also a noncompact variant of [8, Theorem 2] in general topological spaces.

4. Equilibrium Existence of $QEP(T, A, f)$ and $QEP(A, f)$

First, we prove the following equilibrium existence theorems of QEP(T, A, f).

Theorem 4.1. *Let X be a topological space, let K be a nonempty compact subset of X, and let Y be a nonempty set. Let $T : X \to Y$, $A : X \to 2^X$, and $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be such that*

(i) *A has nonempty values on X and satisfies one of the conditions (I)–(VIII) in Lemma 2.1,*

(ii) *the set $D = \{x \in X : x \in A(x)\}$ is nonempty closed in X,*

(iii) *the mapping $B^{-1} : X \to 2^X$ is transfer compactly open-valued, where $B : X \to 2^X$ is defined by*

$$B(x) = \{y \in A(x) : f(x, T(x)) - f(y, T(x)) > 0\}, \tag{4.1}$$

(iv) *for each $N \in \mathcal{F}(X)$, there is a nonempty compact subset L_N of X containing N such that there is a generalized R-KKM mapping $W : L_N \to 2^{L_N}$ satisfying, for each $x \in D$, the fact that $M \in \langle B(x) \cap L_N \rangle$ implies that $W(M) \subset B(x)$ and, for each $x \in X \setminus D$, the fact that $M \in \langle A(x) \cap L_N \rangle$ implies that $W(M) \subset A(x)$. Moreover, for each $x \in L_N \setminus K$, if $x \in X \setminus D$, then there exists $y \in L_N$ such that $x \in \operatorname{cint} A^{-1}(y)$; if $x \in D$, then there exists $y \in L_N$ such that $x \in \operatorname{cint} B^{-1}(y)$.*

Then there exists $\hat{x} \in X$ such that

$$\hat{x} \in A(\hat{x}), \quad f(\hat{x}, T(\hat{x})) \leq f(y, T(\hat{x})), \quad \forall y \in A(\hat{x}), \tag{4.2}$$

that is, $\hat{x}$ is a solution of the QEP(T, A, f).

Proof. Define a mapping $G : X \to 2^X$ by

$$G(x) = \begin{cases} B(x) & \text{if } x \in D, \\ A(x) & \text{if } x \in X \setminus D. \end{cases} \tag{4.3}$$

From conditions (i), (iii) and Lemma 2.1, it follows that the mappings $B^{-1}, A^{-1} : X \to 2^X$ are both transfer compactly open-valued on X. Note that $B(x) \subset A(x)$ for each $x \in X$ and D is nonempty closed by (ii); by Lemma 2.3, $G^{-1} : X \to 2^X$ is also transfer compactly open-valued on X. From condition (iv), it follows that, for each $N \in \mathcal{F}(X)$ and for each $x \in X$, $M \in \langle G(x) \cap L_N \rangle$ implies that $W(M) \subset G(x)$. Moreover, $L_N \setminus K \subset \bigcup_{y \in L_N} \operatorname{cint} G^{-1}(y)$. Hence, condition (ii) of Theorem 3.1 holds. Now assume that, for each $x \in D$, $B(x) \neq \emptyset$. Then for each $x \in X$, $G(x) \neq \emptyset$ by (i). It is easy to see that all conditions of Theorem 3.1 are satisfied. By Theorem 3.1, there exists $\hat{x} \in X$ such that $\hat{x} \in G(\hat{x})$. By the definition of D and G, we must have $\{x \in X : x \in G(x)\} \subset D$. It follows that $\hat{x} \in B(\hat{x}) \cap D$. In particular, we obtain $f(\hat{x}, T(\hat{x})) - f(\hat{x}, T(\hat{x})) > 0$ which is impossible. Therefore, there exists $\hat{x} \in D$ such that $B(\hat{x}) = \emptyset$, that is, $\hat{x} \in A(\hat{x})$ and $f(\hat{x}, T(\hat{x})) \leq f(y, T(\hat{x}))$ for all $y \in A(\hat{x})$. This completes the proof. □

Remark 4.2. Theorem 4.1 generalized Theorem 3.1 of Ding in [15] by dropping all the contractible conditions. Theorem 4.1 also improved and generalized [3, Theorem 2.1] and [2, Theorem 4.2] from topological vector spaces to general noncompact topological spaces without linear structure under much weaker assumptions.

Theorem 4.3. *Let X and Y be two topological spaces and let K be a nonempty compact subset of X. Let $T : X \to Y$, $A : X \to 2^X$, and $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be such that*

(i) *A has nonempty values on X such that $A^{-1} : X \to 2^X$ is compactly open-valued,*

(ii) *the set $D = \{x \in X : x \in A(x)\}$ is nonempty closed in X,*

(iii) *T and f are continuous,*

(iv) *for each $N \in \mathcal{F}(X)$, there is a nonempty compact subset L_N of X containing N such that there is a generalized R-KKM mapping $W : L_N \to 2^{L_N}$ satisfying, for each $x \in D$, the fact that $M \in \langle A(x) \cap P(x) \cap L_N \rangle$ implies that $W(M) \subset A(x) \cap P(x)$, where $P : X \to 2^X$ is defined by $P(x) = \{y \in X : f(x, T(x)) - f(y, T(x)) > 0\}$, and, for each $x \in X \setminus D$, the fact that $M \in \langle A(x) \cap L_N \rangle$ implies that $W(M) \subset A(x)$. Moreover, for each $x \in L_N \setminus K$, if $x \in X \setminus D$, then $A(x) \cap L_N \neq \emptyset$; if $x \in D$, then there exists $y \in A(x) \cap L_N$ satisfying $f(y, T(x)) < f(x, T(x))$.*

Then there exists $\hat{x} \in X$ such that

$$\hat{x} \in A(\hat{x}), \quad f(\hat{x}, T(\hat{x})) \leq f(y, T(\hat{x})), \quad \forall y \in A(\hat{x}), \tag{4.4}$$

that is, $\hat{x}$ is a solution of the QEP(T, A, f).

Proof. Since T and f are both continuous, we have that, for each $y \in X$, $P^{-1}(y) = \{x \in X : f(x, T(x)) - f(y, T(x)) > 0\}$ is open in X, and hence, $P^{-1} : X \to 2^X$ has compactly open values. It follows that the mapping $B : X \to 2^X$ defined by $B(x) = A(x) \cap P(x)$ is such that the mapping $B^{-1} = (A \cap P)^{-1} = A^{-1} \cap P^{-1}$ also has compactly open values on X, and hence, it is also transfer compactly open-valued on X. By condition (iv), for each $x \in L_N \setminus K$, if $x \in X \setminus D$, we have $A(x) \cap L_N \neq \emptyset$, and hence, there exists $y \in L_N$ such that $x \in A^{-1}(y) = \operatorname{cint} A^{-1}(y)$; if $x \in D$, we have $y \in L_N$ and $x \in A^{-1}(y) \cap P^{-1}(y) = \operatorname{cint}(A^{-1}(y) \cap P^{-1}(y))$. Hence, condition (iv) implies that condition (iv) of Theorem 4.1 is satisfied. It is easy to see that all conditions of Theorem 4.1 are satisfied. By Theorem 4.1, the conclusion of Theorem 4.3 holds. This completes the proof. □

Second, we prove the following equilibrium existence theorem of QEP(A, f).

Theorem 4.4. *Let X be a topological space, let K be a nonempty compact subset of X. Let $A : X \to 2^X$ and $f : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$ be such that*

(i) *A has nonempty values on X such that $A^{-1} : X \to 2^X$ has compactly open values and the mapping $\operatorname{cl} A : X \to 2^X$ defined by $(\operatorname{cl} A)(x) = \operatorname{cl} A(x)$ (the closure of $A(x)$) is upper semicontinuous,*

(ii) *f is a continuous function,*

(iii) *for each $N \in \mathcal{F}(X)$, there is a nonempty compact subset L_N of X containing N such that there is a generalized R-KKM mapping $W : L_N \to 2^{L_N}$ satisfying, for each $x \in \overline{D}$, the fact that $M \in \langle A(x) \cap P(x) \cap L_N \rangle$ implies that $W(M) \subset A(x) \cap P(x)$, where $\overline{D} = \{x \in X : x \in \operatorname{cl} A(x)\}$ and $P(x) = \{y \in X : f(x,x) - f(y,x) > 0\}$, and for each $x \in X \setminus \overline{D}$, the fact that $M \in \langle A(x) \cap L_N \rangle$ implies that $W(M) \subset A(x)$. Moreover, for each $x \in L_N \setminus K$, if $x \in X \setminus \overline{D}$, then $A(x) \cap L_N \neq \emptyset$; if $x \in \overline{D}$, then there exists $y \in A(x) \cap L_N$ satisfying $f(y,x) < f(x,x)$.*

Then there exists $\hat{x} \in X$ such that

$$\hat{x} \in A(\hat{x}), \quad f(\hat{x},\hat{x}) \leq f(y,\hat{x}), \quad \forall y \in A(\hat{x}). \tag{4.5}$$

If one we further assumes that $f(x,x) \geq 0$ for all $x \in X$, then one has that

$$\hat{x} \in A(\hat{x}), \quad f(y,\hat{x}) \geq 0, \quad \forall y \in A(\hat{x}), \tag{4.6}$$

that is, $\hat{x}$ is a solution of the QEP(A, f).

Proof. Since $\operatorname{cl} A : X \to 2^X$ is upper semicontinuous with closed values, the set $\overline{D} = \{x \in X : x \in \operatorname{cl} A(x)\}$ must be closed in X. By letting $Y = X$, with T being the identity mapping and $\overline{D}$ being in place of D, it is easy to see that all conditions of Theorem 4.3 are satisfied. By Theorem 4.3, there exists $\hat{x} \in X$ such that $\hat{x} \in A(\hat{x})$ and $f(\hat{x},\hat{x}) \leq f(y,\hat{x})$, for all $y \in A(\hat{x})$. If $f(x,x) \geq 0$ for all $x \in X$, then we must have $\hat{x} \in A(\hat{x})$ and $f(y,\hat{x}) \geq 0$, for all $y \in A(\hat{x})$, that is, $\hat{x}$ is a solution of the QEP(A, f). □

Remark 4.5. Theorem 4.4 generalized Theorem 3.3 of Ding in [15] by dropping all the contractible conditions. Theorem 4.4 is also a noncompact variant of [4, Theorem 6.4.21], [5, Theorem 3.1], Theorem 2.1 of [6, 7], and [8, Theorem 4] in general topological spaces.

From Theorem 4.4, we obtain the following existence result for generalized quasi-equilibrium problems.

Theorem 4.6. *Let X and Y be two topological spaces and let K be a nonempty compact subset of X. Let $T : X \to 2^Y$ have a continuous selection $g : X \to Y$. Let $A : X \to 2^X$ and $\phi : X \times Y \times X \to \mathbb{R} \cup \{\pm\infty\}$ be such that*

(i) *A satisfies condition (i) of Theorem 4.4,*

(ii) *ϕ is a continuous function,*

(iii) *for each $N \in \mathcal{F}(X)$, there is a nonempty compact subset L_N of X containing N such that there is a generalized R-KKM mapping $W : L_N \to 2^{L_N}$ satisfying, for each $x \in \overline{D}$,*

the fact that $M \in \langle A(x) \cap P(x) \cap L_N \rangle$ *implies that* $W(M) \subset A(x) \cap P(x)$, *where* $\overline{D} = \{x \in X : x \in \operatorname{cl} A(x)\}$ *and* $P(x) = \{z \in X : \phi(x, g(x), x) - \phi(x, g(x), z) > 0\}$, *and, for each* $x \in X \setminus \overline{D}$, *the fact that* $M \in \langle A(x) \cap L_N \rangle$ *implies that* $W(M) \subset A(x)$. *Moreover, for each* $x \in L_N \setminus K$, *if* $x \in X \setminus \overline{D}$, *then* $A(x) \cap L_N \neq \emptyset$; *if* $x \in \overline{D}$, *then there exists* $y \in A(x) \cap L_N$ *satisfying* $\phi(x, g(x), y) < \phi(x, g(x), x)$.

Then there exists $\widehat{x} \in X$ *and* $\widehat{y} = g(\widehat{x}) \in T(\widehat{x})$ *such that*

$$\widehat{x} \in A(\widehat{x}), \quad \phi(\widehat{x}, \widehat{y}, \widehat{x}) \leq \phi(\widehat{x}, \widehat{y}, z), \quad \forall z \in A(\widehat{x}). \tag{4.7}$$

If one further assumes that $\phi(x, g(x), x) \geq 0$ *for all* $x \in X$, *then one has that*

$$\widehat{x} \in A(\widehat{x}), \quad \phi(\widehat{x}, \widehat{y}, z) \geq 0, \quad \forall z \in A(\widehat{x}). \tag{4.8}$$

Proof. Define $f : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ by

$$f(z, x) = \phi(x, g(x), z), \quad \forall (z, x) \in X \times X. \tag{4.9}$$

Then the conclusion of Theorem 4.6 holds from Theorem 4.4. □

Remark 4.7. Theorem 4.6 generalized Theorem 3.4 of Ding in [15] by dropping all the contractible conditions. Theorem 4.6 is a noncompact variant of [8, Corollary 5] and [16, Theorem 3.1] in general topological spaces.

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