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Research Article

On the S-Invariance Property for S-Flows

Amin Saif and Adem Kılıçman

Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

Correspondence should be addressed to Adem Kılıçman, akilicman@putra.upm.edu.my

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We define an equivalence relation on a topological space which is acted by topological monoid S as a transformation semigroup. Then, we give some results about the S-invariant classes for this relation. We also provide a condition for the existence of relative S-invariant classes.

1. Introduction

The invariance theory is one of the principal concepts in the topological dynamics system, see [1, 2]. In [3], Colonius and Kliemann introduced the concept of a control set which is relatively invariant with respect to a subset of the phase space of the control system. From a more general point of view, the theory of control sets for semigroup actions was developed by San Martin and Tonelli in [4].

In this paper, we define an equivalence relation on a topological space which is acted by topological monoid S as a transformation semigroup. Then, we provide the necessary and sufficient conditions for the equivalence classes to be S-invariant classes which correspond with the control sets for control systems. Then, we study the S-invariant classes for this relation in X, in particular, and we provide the conditions for the existence and uniqueness of S-invariant classes.

Throughout this paper, $\mathrm{cl}(A)$ will denote the closure set of a set A, and $\mathrm{int}(A)$ will denote the interior set of A and all topological spaces involved Hausdorff.

Definition 1.1 (see [2]). Let S be a monoid with the identity element e and also a topological space. Then, S will be called a topological monoid if the multiplication operation of: $(s,t) \to st$ is continuous mapping from $S \times S$ to S.

Definition 1.2 (see [4]). Let S be a topological monoid and X a topological space. We say that *S acts on X as a transformation semigroup* if there is a continuous map $a : S \times X \to X$ between the product space $S \times X$ and X satisfying

$$a(st, x) = a(s, a(t, x)), \quad \forall s, t \in S, x \in X, \tag{1.1}$$

we further require that $a(e, x) = x$ for all $x \in X$. The triple (S, X, a) is called an S-flow; $s\bar{a}x$ will denote $a(s, x)$. In particular, an S-flow (S, X, a) is called *S-phase flow* if S is a compact space.

The *orbit* of $x \in X$ under S is the set $O_a(x) = \{s\bar{a}x : s \in S\}$. For a subset M of X, $S(M)$ denotes the set $\{s\bar{a}m : s \in S, m \in M\}$. And a subset M is called an *S-invariant* set if $M \neq \emptyset$ and $S(M) \subset M$. A *control* set for S on X is a subset C of X which satisfies

(1) $\mathrm{int}(C) \neq \emptyset$,

(2) for all $x \in C$, $C \subset \mathrm{cl}(O_a(x))$,

(3) C is a maximal with these properties.

Then, we say that a subset $M \subset X$, satisfies the *no-return condition* if $y \in \mathrm{cl}(O_a(x))$ for some $x \in M$ and $\mathrm{cl}(O_a(y)) \cap M \neq \emptyset$, then $y \in M$.

Lemma 1.3 (see [5, Zorn's Lemma]). *If each chain in a partially ordered set has an upper bound, then there is a maximal element of the set.*

2. S-Invariant Classes

Let (S, X, a) be an S-flow. From the action on X, we can define the relation $\sim$ on X by

$$x \sim y \quad \text{if } x \in O_a(y),\ y \in O_a(x),\ x, y \in X. \tag{2.1}$$

It is clear that the relation $\sim$ is an equivalence relation, and $[X]$ will denote the set of all equivalence classes induced by $\sim$ on X. We observe that $[x] \subset O_a(x)$ for all $x \in X$, and if $y \in O_a(x)$, then $O_a(y) \subset O_a(x)$ for all $x, y \in X$.

The following theorem shows that an equivalence class with nonempty interior set is a control set for S on X.

Theorem 2.1. *Let (S, X, a) be an S-phase flow. A class $[x] \in [X]$ with $\mathrm{int}_X([x]) \neq \emptyset$ is a control set for S on X.*

Proof. It is clear that $[x] \subset O_a(x) \subset O_a(y) \subset \mathrm{cl}(O_a(y))$ for all $y \in [x]$. Let C be a subset of X satisfying the property

$$C \subset \mathrm{cl}(O_a(z)), \quad \forall z \in C,\ [x] \subset C. \tag{2.2}$$

Now if $\omega \in C$ then $\omega \in \mathrm{cl}(O_a(z))$ for all $z \in C$. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a closed subset of X for all

$x \in X$ (i.e., $\mathrm{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Then, $\omega \in O_a(z)$ for all $z \in C$. Since $x \in C$, then $\omega \in O_a(x)$. On the other hand, since $x \in [x] \subset C \subset O_a(\omega)$, then $\omega \in [x]$. Hence, $C = [x]$. □

In the following lemma, we give necessary and sufficient conditions for the equivalence classes to be S-invariant classes.

Lemma 2.2. *Let (S, X, a) be an S-flow. A class $[x] \in [X]$ is an S-invariant class if and only if $[x] = O_a(x)$.*

Proof. Suppose that $[x] \in [X]$ is an S-invariant and let $y \in O_a(x)$, then $y = s\overline{a}x$ for some $s \in S$. Since $x \in [x]$, then $y \in S([x]) \subset [x]$. Hence, $O_a(x) \subset [x]$, and we have $[x] \subset O_a(x)$. Therefore, $[x] = O_a(x)$.

Conversely, let $[x] = O_a(x)$ and $y \in S([x])$, then $y = s\overline{a}z$ for some $s \in S$, $z \in [x]$. Hence, $z \in O_a(x)$. Take $z = s'\overline{a}x$ for some $s' \in S$. Hence

$$y = s\overline{a}z = s\overline{a}\left(s'\overline{a}x\right) = ss'\overline{a}x \in O_a(x) = [x]. \tag{2.3}$$

Therefore, $[x]$ is an S-invariant class. □

Theorem 2.3. *Let (S, X, a) be an S-phase flow. Then, for all $x \in X$, there exists an S-invariant class $[y] \subset O_a(x)$.*

Proof. For $x \in X$, consider the family of subsets

$$E_x = \{z : O_a(z) \subset O_a(x)\}. \tag{2.4}$$

We can define the relation $\preceq$ on E_x by

$$x_1 \preceq x_2, \quad \text{if } O_a(x_2) \subset O_a(x_1) \quad \text{for } x_1, x_2 \in E_x. \tag{2.5}$$

Then, it is clear that the family E_x with $\preceq$ is a partially order set. Let $\{z_i : i \in \wedge\}$ be a linearly ordered subset of E_x, where $\wedge$ is an index set. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$. Hence we have a chain $\{O_a(z_i) : i \in \wedge\}$ of closed subsets of a compact $O_a(x)$. Hence the intersection

$$\bigcap_{i \in \wedge} O_a(z_i) \neq \emptyset. \tag{2.6}$$

Take $r \in O_a(z_i)$ for all $i \in \wedge$. Then, $O_a(r) \subset O_a(z_i)$ for all $i \in \wedge$, implies that $O_a(r)$ is a lower bound of the chain $\{O_a(z_i) : i \in \wedge\}$ (i.e., r is an upper bound of the linearly order subset $\{z_i : i \in \wedge\}$ of E_x). Hence, Zorn's lemma implies that the family E_x has a maximal element, say y. Then, $[y] \subset O_a(y) \subset O_a(x)$.

Now, we show that $[y]$ is an S-invariant. Let $z \in O_a(y)$, then $z \in O_a(z) \subset O_a(x)$ and $y \preceq z$, but by the maximality of y, we get that $z \preceq y$, this implies $y \in O_a(z)$. Hence, $z \in [y]$ (i.e., $O_a(y) \subset [y]$) and we have that $[y] \subset O_a(y)$. Then, by Lemma 2.2, $[y]$ is an S-invariant class. □

Now, we propose an open problem that whether S-invariant class is unique?

Theorem 2.4. *Let (S, X, a) be an S-phase flow. Every $[x] \in [X]$ satisfies the no-return condition for all $x \in X$.*

Proof. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$ (i.e., $\mathrm{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Let $z \in O_a(y)$ for some $y \in [x]$ and $O_a(z) \cap [x] \neq \emptyset$. Take $\omega \in O_a(z)$ and $\omega \in [x]$. Hence,

$$x \in O_a(x) \subset O_a(\omega) \subset O_a(z). \tag{2.7}$$

On the other hand, $z \in O_a(y)$ for some $y \in [x]$, we have

$$z \in O_a(z) \subset O_a(y) \subset O_a(x). \tag{2.8}$$

Hence, $z \in [x]$. □

The next theorem states that if M has the no-return condition, then any class $[x]$ is entirely contained in M or M^c. Further M is also an S-invariant if $[x]$ an S-invariant class for all $x \in M$.

Theorem 2.5. *Let (S, X, a) be S-phase flow and M be a subset of X has no-return condition. Then, M is an S-invariant set if $[x]$ is an S-invariant class for all $x \in M$.*

Proof. It is clear that $M \subset \bigcup_{x\in M}[x]$ because $x \in [x]$. Since S is a compact space, X is a Hausdorff space and by the continuity of the action a, then the orbit $O_a(x)$ is a compact closed subset of X for all $x \in X$ (i.e., $\mathrm{cl}(O_a(x)) = O_a(x)$ for all $x \in X$). Let $y \in \bigcup_{x\in M}[x]$, then $y \in [x]$ for some $x \in M$. Hence, $[x] = [y]$ (i.e., $x \in O_a(y)$ and $y \in O_a(x)$). Since $x \in M$, then $O_a(y) \cap M \neq \emptyset$. By the no-return condition, we have that $y \in M$. Hence,

$$M = \bigcup_{x\in M}[x]. \tag{2.9}$$

Now, we show that M is an S-invariant set. Let $y \in S(M)$. Then, $y = s\bar{a}x$ for some $x \in M$. Hence, $y \in O_a(x)$. Since $[x]$ is an S-invariant class then by Lemma 2.2, $[x] = O_a(x)$ and by (2.9), we get that $y \in [x] \subset M$. Hence, M is an S-invariant. □

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