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Research Article

A New Iterative Method for Finding Common Solutions of a System of Equilibrium Problems, Fixed-Point Problems, and Variational Inequalities

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We introduce a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed point sets of an infinite family of nonexpansive mappings, and the solution set of a variational inequality for a relaxed cocoercive mapping in a Hilbert space. We prove strong convergence theorem. The results in this paper unify and generalize some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H. Let $\{F_k\}_{k\in\Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Combettes and Hirstoaga [1] considered the following system of equilibrium problems:

$$\text{Find } x \in C \text{ such that } (\forall k \in \Gamma),\ (\forall y \in C),\ F_k(x, y) \geq 0. \tag{1.1}$$

If Γ is a singleton, problem (1.1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solutions set of (1.2) is denoted by $\mathrm{EP}(F)$. And clearly the solutions set of problem (1.1) can be written as $\bigcap_{k\in\Gamma} \mathrm{EP}(F_k)$.

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see for instance, [1–4].

Recall that a mapping S of a closed and convex subset C into itself is nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.3}$$

We denote fixed-points set of S by $\mathrm{Fix}(S)$. A mapping $f : C \to C$ is called contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|fx - fy\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

A bounded linear operator B on H is strongly positive, if there is a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$.

Combettes and Hirstoaga [1] introduced an iterative scheme for finding a common element of the solutions set of problem (1.1) in a Hilbert space and obtained a weak convergence theorem. Peng and Yao [2] introduced a new viscosity approximation scheme based on the extragradient method for finding a common element in the solutions set of the problem (1.1), fixed-points set of an infinite family of nonexpansive mappings and the solutions set of the variational inequality for a monotone and Lipschitz continuous mapping in a Hilbert space and obtained a strong convergence theorem. Colao et al. [3] introduced an implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed-points of infinite family of nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Saeidi [4] introduced some iterative algorithms for finding a common element of the solutions set of a system of equilibrium problems and of fixed-points set of a finite family and a left amenable semigroup of nonexpansive mappings in a Hilbert space and obtained some strong convergence theorems.

Several algorithms for problem (1.2) have been proposed (see [5–20]). S. Takahashi and W. Takahashi [5] introduced and studied the following iterative scheme by the viscosity approximation method for finding a common element of the solutions set of problem (1.2) and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$ define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \in N. \end{gathered} \tag{1.5}$$

Shang et al. [6] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the solutions set of problem (1.2) and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S u_n, \quad \forall n \in N. \end{gathered} \tag{1.6}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.6) converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(S) \cap \operatorname{EP}(F), \tag{1.7}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \operatorname{Fix}(S) \cap \operatorname{EP}(F)} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{1.8}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). If $C = H$, the algorithm (1.6) was also studied by Plubtieng and Punpaeng [7].

Let $A : C \to H$ be a monotone mapping. The variational inequality problem is to find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \tag{1.9}$$

for all $y \in C$. The solutions set of the variational inequality problem is denoted by $\operatorname{VI}(C, A)$. Qin et al. [8] introduced the following general iterative scheme for finding a common element of the solutions set of problem (1.2), the solutions set of a variational inequality and fixed-points set of a nonexpansive mapping in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S P_C (I - s_n A) u_n, \quad \forall n \in N. \end{gathered} \tag{1.10}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{s_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.10) converge strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(S) \cap \operatorname{VI}(C, A) \cap \operatorname{EP}(F). \tag{1.11}$$

Qin et al. [9] introduced the following general iterative scheme for finding a common element of the solutions set of problem (1.2) and fixed-points set of a finite family of nonexpansive mappings in a Hilbert space. Let an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$F(u_n, y) + \frac{1}{\beta_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n \gamma f(W_n x_n) + (1 - \alpha_n B) W_n P_C (I - s_n A) u_n, \quad \forall n \in N, \tag{1.12}$$

where W_n is the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$. They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{s_n\}$ and $\{\beta_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.12) converge strongly to the unique solution of the variational inequality

$$\left\langle (B - \gamma f)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) \cap \operatorname{VI}(C, A) \cap \operatorname{EP}(F). \tag{1.13}$$

A typical problem is to minimize a quadratic function over the fixed-points set of a nonexpansive mapping S on a real Hilbert space H, that is,

$$\min_{x \in \operatorname{Fix}(S)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{1.14}$$

where b is a given point in H. In 2003, Xu [21] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial point $x_0 \in H$, chosen arbitrarily:

$$x_{n+1} = (I - \alpha_n B) S x_n + \alpha_n u, \quad n \geq 0, \tag{1.15}$$

converges strongly to the unique solution of the minimization problem (1.15) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Marino and Xu [22] combine the iterative method (1.15) with the viscosity approximation in [23] and consider the following general iterative method: with the initial point $x_0 \in H$, chosen arbitrarily:

$$x_{n+1} = (1 - \alpha_n B) S x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.16}$$

They proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.16) converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in \operatorname{Fix}(S) \tag{1.17}$$

which is the optimality condition for the minimization problem

$$\min_{x \in \operatorname{Fix}(S)} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{1.18}$$

where h is a potential function for γf.

Recently, Qin et al. [24] introduced the following general iterative process: with the initial point $x_1 \in C$, chosen arbitrarily:

$$y_n = P_C(I - s_n A)x_n,$$
$$x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - r_n A)y_n, \quad \forall n \in N, \tag{1.19}$$

where W_n is the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n1}, \lambda_{n2}, \ldots, \lambda_{nN}$. They proved that if the sequences of parameters $\{\alpha_n\}$, $\{r_n\}$ and $\{s_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$, $\{y_n\}$ generated by (1.19) converge strongly to a point x^* which is the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{VI}(C, A). \tag{1.20}$$

Inspired and motivated by above works, we introduce a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed-points set of a family of infinitely nonexpansive mappings and the solutions set of a variational inequality for a relaxed cocoercive mapping in a Hilbert space. We prove strong convergence theorem. The results in this paper unify, generalize and extend some well-known results in [6–9, 21, 22, 24].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty, closed, and convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. It is well known that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C, $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y \rangle \geq 0 \tag{2.2}$$

for all $x, y \in H$.

It is easy to see that (2.2) is equivalent to

$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.3}$$

for all $x, y \in H$. It is also known that P_C has the following firmly nonexpansive property:

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{2.4}$$

for all $x, y \in H$.

Recall also that a mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \tag{2.5}$$

for all $x, y \in C$. A is said to be μ-cocoercive, if for each $x, y \in C$, we have

$$\langle Ax - Ay, x - y \rangle \geq \mu \|Ax - Ay\|^2, \tag{2.6}$$

for a constant $\mu > 0$. A is said to be relaxed (u, v)-cocoercive, if there exist two constants $u, v > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-u)\|Ax - Ay\|^2 + v\|x - y\|^2, \quad \forall x, y \in C. \tag{2.7}$$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.2) implies the following:

$$\begin{gathered} u \in \mathrm{VI}(C, A) \Longrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0, \\ u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0 \Longrightarrow u \in \mathrm{VI}(C, A). \end{gathered} \tag{2.8}$$

It is also known that H satisfies the Opial's condition [25], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \tag{2.9}$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.10}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$ (see [26]).

For solving the problem (1.1), let us assume that the bifunction F satisfies the following condition:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \le F(x, y); \tag{2.11}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex;

(A5) for each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous.

We recall some lemmas needed later.

Lemma 2.1 (see [1, 10]). *Let C be a nonempty, closed, and convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A5). For $\beta > 0$ and $x \in H$, define the mapping $T_\beta^F : H \to C$ as follows:*

$$T_\beta^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{\beta}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C \right\} \tag{2.12}$$

for all $x \in H$. Then, the following statements hold:

(1) $T_\beta^F(x) \neq \emptyset$;

(2) T_β^F *is single-valued;*

(3) T_β^F *is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_\beta^F(x) - T_\beta^F(y) \right\|^2 \le \left\langle T_\beta^F(x) - T_\beta^F(y), x - y \right\rangle; \tag{2.13}$$

(4) $\mathrm{Fix}(T_\beta^F) = \mathrm{EP}(F)$;

(5) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.2 (see [27]). *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n + \delta_n, \quad n \ge 1, \tag{2.14}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences of numbers which satisfy the conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *or equivalently,* $\prod_{i=1}^{\infty}(1 - \alpha_n) = 0$;

(ii) $\limsup_{n \to \infty} \beta_n \le 0$;

(iii) $\delta_n \ge 0 (n \ge 1)$, $\sum_{n=1}^{\infty} \delta_n < \infty$;

Then, $\lim_{n \to \infty} s_n = 0$.

Lemma 2.3. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle \tag{2.15}$$

for all $x, y \in H$.

Lemma 2.4 (see [22]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\tilde{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\tilde{\gamma}$.*

Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \leq \xi_i \leq 1$ for every $i \in N$. For any $n \in N$, define a mapping W_n of C into C as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \xi_n S_n U_{n,n+1} + (1 - \xi_n) I, \\ U_{n,n-1} &= \xi_{n-1} S_{n-1} U_{n,n} + (1 - \xi_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \xi_k S_k U_{n,k+1} + (1 - \xi_k) I, \\ U_{n,k-1} &= \xi_{k-1} S_{k-1} U_{n,k} + (1 - \xi_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \xi_2 S_2 U_{n,3} + (1 - \xi_2) I, \\ W_n = U_{n,1} &= \xi_1 S_1 U_{n,2} + (1 - \xi_1) I. \end{aligned} \tag{2.16}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$; see [28, 29].

Lemma 2.5 (see [28]). *Let C be a nonempty, closed, and convex subset of a Banach space E. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Then W_n is asymptotically regular and nonexpansive. Further, if E is strict convex, then $F(W_n) = \bigcap_{i=1}^{n} \mathrm{Fix}(S_i)$.*

Lemma 2.6 (see [29]). *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. Then for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Remark 2.7. Using Lemma 2.6, one can define mappings $U_{\infty,k}$ and W of C into itself as follows:

$$U_{\infty,k} x = \lim_{n\to\infty} U_{n,k} x, \tag{2.17}$$

and $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ for every $x \in C$. Such a mapping W is called the W-mapping generated by $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|. \tag{2.18}$$

If $\{x_n\}$ is a bounded sequence in C, then we have

$$\lim_{n\to\infty} \|Wx - W_n x\| = 0. \tag{2.19}$$

Lemma 2.8 (see [29]). *Let C be a nonempty, closed and convex subset of a strictly convex Banach space E. Let $S_1, S_2, \ldots$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq d < 1$ for every $i \in N$. Then $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$.*

3. Strong Convergence Theorem

In this section, we prove strong convergence theorem which solve the problem of finding a common element of the solutions set of a system of equilibrium problems, fixed-points set of a family of infinitely nonexpansive mappings, and the solutions set of a variational inequality for a relaxed cocoercive mapping in Hilbert space.

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of H. Let $F_1, F_2, \ldots, F_m$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A5). Let $A : C \to H$ be relaxed (u, v)-cocoercive and μ-Lipschitz continuous and B a strongly positive linear bounded operator on H with coefficient $\tilde{\gamma} > 0$. Assume that $0 < \gamma < \tilde{\gamma}/\alpha$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap \mathrm{VI}(C, A) \cap \bigcap_{k=1}^{m} \mathrm{EP}(F_k) \neq \emptyset$ and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i \leq \delta < 1$ for every $i \in N$, and let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \ldots, S_1$ and $\xi_n, \xi_{n-1}, \ldots, \xi_1$. Let $f : C \to C$ be a contraction with coefficient α $(0 < \alpha < 1)$ and $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered} x_1 = x \in H, \\ u_n = T_{\beta_n}^{F_m} T_{\beta_n}^{F_{m-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} x_n, \\ y_n = P_C(I - s_n A) u_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B) W_n P_C (I - r_n A) y_n \end{gathered} \tag{3.1}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{s_n\}$ are sequences of numbers which satisfy the conditions:

(C1) *$\{\alpha_n\} \subset [0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;*

(C2) *$\{r_n\} \subset [a, b]$ and $\{s_n\} \subset [a, b]$ for some a, b with $0 \leq a \leq b \leq 2(v - u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;*

(C3) *$\liminf_{n\to\infty} \beta_n > 0$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.*

Then, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ converge strongly to a point $q \in \Omega$ which solves the following variational inequality:

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \Omega. \tag{3.2}$$

Equivalently, one has $q = P_\Omega(\gamma f + (I - B))(q)$.

Proof. Since $\alpha_n \to 0$ from condition (C1), we may assume, with no loss of generality, that $\alpha_n \leq \|B\|^{-1}$ for all n. Lemma 2.4 implies $\|I - \alpha_n B\| \leq 1 - \alpha_n \tilde{\gamma}$. Next, we will assume that $\|I - B\| \leq 1 - \tilde{\gamma}$. Now, we show that the mappings $I - s_n A$ and $I - r_n A$ are nonexpansive. Indeed, from the relaxed (u, v)-cocoercivity and μ-Lipschitz continuity of A and condition (C2), we have

$$\begin{aligned}
\left\|(I - s_n A)x - (I - s_n A)y\right\|^2 &= \left\|(x - y) - s_n(Ax - Ay)\right\|^2 \\
&= \|x - y\|^2 - 2s_n\langle x - y, Ax - Ay\rangle + s_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2s_n\left[-u\|Ax - Ay\|^2 + v\|x - y\|^2\right] + s_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + 2s_n\mu^2 u\|x - y\|^2 - 2s_n v\|x - y\|^2 + \mu^2 s_n^2\|x - y\|^2 \\
&= \left(1 + 2s_n\mu^2 u - 2s_n v + \mu^2 s_n^2\right)\|x - y\|^2 \\
&\leq \|x - y\|^2,
\end{aligned} \tag{3.3}$$

which implies the mapping $I - s_n A$ is nonexpansive, so is $I - r_n A$.

For $k \in \{0, 1, 2, \ldots, m\}$, and for any positive integer number n, we define the operator $\Theta^k_{\beta_n} : H \to C$ as follows:

$$\begin{aligned}
\Theta^0_{\beta_n} x &= x, \\
\Theta^k_{\beta_n} x &= T^{F_k}_{\beta_n} T^{F_{k-1}}_{\beta_n} \cdots T^{F_2}_{\beta_n} T^{F_1}_{\beta_n} x, \quad k = 1, 2, \ldots, m.
\end{aligned} \tag{3.4}$$

Next, we show that the sequence $\{x_n\}$ is bounded. Let $p \in \Omega$. Then from Lemma 2.1(3), we know that for $k \in \{1, 2, \ldots, m\}$, $T^{F_k}_{\beta_n}$ is nonexpansive and $p = T^{F_k}_{\beta_n} p$, and

$$\|u_n - p\| = \left\|\Theta^m_{\beta_n} x_n - p\right\| = \left\|\Theta^m_{\beta_n} x_n - \Theta^m_{\beta_n} p\right\| \leq \|x_n - p\| \tag{3.5}$$

for all $n = 1, 2, \ldots$. By $p = P_C(I - s_n A)p$ and (3.5), we have

$$\begin{aligned}
\|y_n - p\| &= \|P_C(I - s_n A)u_n - P_C(I - s_n A)p\| \\
&\leq \|(I - s_n A)u_n - (I - s_n A)p\| \leq \|u_n - p\| \leq \|x_n - p\|.
\end{aligned} \tag{3.6}$$

Since $x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - r_n A)y_n$ and $p = W_n p$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(\gamma f(W_n x_n) - Bp) + (I - \alpha_n B)(W_n P_C(I - r_n A)y_n - p)\| \\
&\leq \alpha_n\|\gamma f(W_n x_n) - Bp\| + (1 - \alpha_n\tilde{\gamma})\|P_C(I - r_n A)y_n - p\| \\
&\leq \alpha_n\gamma\|f(W_n x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\tilde{\gamma})\|y_n - p\| \\
&\leq [1 - \alpha_n(\tilde{\gamma} - \alpha\gamma)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|.
\end{aligned} \tag{3.7}$$

By inductions, we have

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{\|\gamma f(p) - Bp\|}{\tilde{\gamma} - \alpha\gamma} \right\}, \tag{3.8}$$

which proves that the sequence $\{x_n\}$ is bounded. It follows from (3.5) and (3.6) that $\{y_n\}$ and $\{u_n\}$ are also bounded.

Since $\Theta^k_{\beta_n} x_n = T^{F_k}_{\beta_n} \Theta^{k-1}_{\beta_n} x_n$ and $\Theta^k_{\beta_{n+1}} x_{n+1} = T^{F_k}_{\beta_{n+1}} \Theta^{k-1}_{\beta_{n+1}} x_{n+1}$ for each $k = 1, 2, \dots, m$, by Lemma 2.1, we have

$$F_k\left(\Theta^k_{\beta_n} x_n, y\right) + \frac{1}{\beta_n}\left\langle y - \Theta^k_{\beta_n} x_n, \Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n \right\rangle \ge 0 \quad \forall y \in C, \tag{3.9}$$

$$F_k\left(\Theta^k_{\beta_{n+1}} x_{n+1}, y\right) + \frac{1}{\beta_{n+1}}\left\langle y - \Theta^k_{\beta_{n+1}} x_{n+1}, \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1} \right\rangle \ge 0 \quad \forall y \in C, \tag{3.10}$$

Setting $y = \Theta^k_{\beta_{n+1}} x_{n+1}$ in (3.9) and $y = \Theta^k_{\beta_n} x_n$ in (3.10), we have

$$\begin{aligned} &F_k\left(\Theta^k_{\beta_n} x_n, \Theta^k_{\beta_{n+1}} x_{n+1}\right) + \frac{1}{\beta_n}\langle \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n, \Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n \rangle \ge 0, \\ &F_k\left(\Theta^k_{\beta_{n+1}} x_{n+1}, \Theta^k_{\beta_n} x_n\right) + \frac{1}{\beta_{n+1}}\left\langle \Theta^k_{\beta_n} x_n - \Theta^k_{\beta_{n+1}} x_{n+1}, \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1} \right\rangle \ge 0. \end{aligned} \tag{3.11}$$

Adding the two inequalities and from the monotonicity of F, we get

$$\left\langle \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n, \frac{\Theta^k_{\beta_n} x_n - \Theta^{k-1}_{\beta_n} x_n}{\beta_n} - \frac{\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}}{\beta_{n+1}} \right\rangle \ge 0 \tag{3.12}$$

and hence

$$\begin{aligned} &\left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n\right\|^2 \\ &\quad \le \left\langle \Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n, \left(\Theta^{k-1}_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_n} x_n\right) + \left(1 - \frac{\beta_n}{\beta_{n+1}}\right)\left(\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}\right) \right\rangle. \end{aligned} \tag{3.13}$$

Without loss of generality, let us assume that there exists a real number d such that $\beta_n > d > 0$ for all $n = 1, 2, \dots$. Hence, for each $k = 1, 2, \dots, m$ we have

$$\begin{aligned} \left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^k_{\beta_n} x_n\right\| &\le \left\|\Theta^{k-1}_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_n} x_n\right\| + \frac{1}{\beta_{n+1}}\left|\beta_{n+1} - \beta_n\right| \left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}\right\| \\ &\le \left\|\Theta^{k-1}_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_n} x_n\right\| + \frac{1}{d}\left|\beta_{n+1} - \beta_n\right| M_0, \end{aligned} \tag{3.14}$$

where M_0 is an approximate constant such that

$$M_0 \geq \max\left\{\sup_{n\geq 1}\left\{\left\|\Theta^k_{\beta_{n+1}} x_{n+1} - \Theta^{k-1}_{\beta_{n+1}} x_{n+1}\right\|\right\},\ k = 1, 2, \ldots, m\right\}. \tag{3.15}$$

It follows from (3.14) that

$$\|u_{n+1} - u_n\| = \left\|\Theta^m_{\beta_{n+1}} x_{n+1} - \Theta^m_{\beta_n} x_n\right\| \leq \|x_{n+1} - x_n\| + \frac{m}{d}\left|\beta_{n+1} - \beta_n\right| M_0. \tag{3.16}$$

Put $\rho_n = P_C(I - r_n A)y_n$. We have

$$\begin{aligned}\left\|y_n - y_{n+1}\right\| &= \|P_C(I - s_n A)u_n - P_C(I - s_{n+1}A)u_{n+1}\| \\ &\leq \|(I - s_n A)u_n - (I - s_{n+1}A)u_{n+1}\| \\ &= \|(u_n - s_n A u_n) - (u_{n+1} - s_n A u_{n+1}) + (s_{n+1} - s_n)A u_{n+1}\| \\ &\leq \|u_n - u_{n+1}\| + |s_{n+1} - s_n| M_1,\end{aligned} \tag{3.17}$$

where M_1 is an approximate constant such that $M_1 \geq \max\{\sup_{n\geq 1}\{\|Au_n\|\}, M_0\}$.

Substituting (3.16) into (3.17), we have

$$\left\|y_n - y_{n+1}\right\| \leq \|x_{n+1} - x_n\| + \left[\frac{m}{d}\left|\beta_{n+1} - \beta_n\right| + |s_{n+1} - s_n|\right] M_1. \tag{3.18}$$

It follows from (3.18) that

$$\begin{aligned}\left\|\rho_n - \rho_{n+1}\right\| &= \left\|P_C(I - r_n A)y_n - P_C(I - r_{n+1}A)y_{n+1}\right\| \\ &\leq \left\|(I - r_n A)y_n - (I - r_{n+1}A)y_{n+1}\right\| \\ &= \left\|(y_n - r_n A y_n) - (y_{n+1} - r_n A y_{n+1}) + (r_{n+1} - r_n)A y_{n+1}\right\| \\ &\leq \left\|y_n - y_{n+1}\right\| + |r_{n+1} - r_n| M_2 \\ &\leq \|x_n - x_{n+1}\| + \left[\frac{m}{d}\left|\beta_{n+1} - \beta_n\right| + |s_{n+1} - s_n| + |r_{n+1} - r_n|\right] M_2,\end{aligned} \tag{3.19}$$

where M_2 is an approximate constant such that $M_2 \geq \max\{M_1, \sup_{n\geq 1}\{\|Ay_{n+1}\|\}\}$.

Observe that

$$\begin{aligned}x_{n+1} &= \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n \rho_n, \\ x_{n+2} &= \alpha_{n+1}\gamma f(W_{n+1}x_{n+1}) + (I - \alpha_{n+1}B)W_{n+1}\rho_{n+1},\end{aligned} \tag{3.20}$$

we have

$$\begin{aligned}x_{n+2} - x_{n+1} = {}& \alpha_{n+1}\gamma\left[f(W_{n+1}x_{n+1}) - f(W_n x_n)\right] + (I - \alpha_{n+1}B)\left(W_{n+1}\rho_{n+1} - W_n\rho_n\right) \\ &+ (\alpha_{n+1} - \alpha_n)\left[\gamma f(W_n x_n) - BW_n\rho_n\right].\end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\leq \alpha_{n+1}\gamma\alpha\|W_{n+1}x_{n+1}-W_nx_n\| + \left(1-\alpha_{n+1}\widetilde{\gamma}\right)\left\|W_{n+1}\rho_{n+1}-W_n\rho_n\right\| \\ &\quad + |\alpha_{n+1}-\alpha_n|\left\|\gamma f(W_nx_n)-BW_n\rho_n\right\| \\ &\leq \alpha_{n+1}\gamma\alpha(\|x_{n+1}-x_n\| + \|W_{n+1}x_n-W_nx_n\|) \\ &\quad + \left(1-\alpha_{n+1}\widetilde{\gamma}\right)\left(\left\|\rho_{n+1}-\rho_n\right\| + \left\|W_{n+1}\rho_n-W_n\rho_n\right\|\right) \\ &\quad + |\alpha_{n+1}-\alpha_n|\left\|\gamma f(W_nx_n)-BW_n\rho_n\right\|.\end{aligned} \tag{3.22}$$

Next we estimate $\|W_{n+1}x_n-W_nx_n\|$ and $\|W_{n+1}\rho_n-W_n\rho_n\|$. It follows from the definition of W_n and nonexpansiveness of S_i that

$$\begin{aligned}\|W_{n+1}x_n-W_nx_n\| &= \|U_{n+1,1}x_n-U_{n,1}x_n\| \\ &= \|\xi_1S_1U_{n+1,2}x_n+(1-\xi_1)x_n-\{\xi_1S_1U_{n,2}x_n+(1-\xi_1)x_n\}\| \\ &= \xi_1\|S_1U_{n+1,2}x_n-S_1U_{n,2}x_n\| \\ &\leq \xi_1\|U_{n+1,2}x_n-U_{n,2}x_n\| \\ &= \xi_1\|\xi_2S_2U_{n+1,3}x_n+(1-\xi_2)x_n-\{\xi_2S_2U_{n,3}x_n+(1-\xi_2)x_n\}\| \\ &= \xi_1\xi_2\|S_2U_{n+1,3}x_n-S_2U_{n,3}x_n\| \\ &\leq \xi_1\xi_2\|U_{n+1,3}x_n-U_{n,3}x_n\| \\ &\ \ \vdots \\ &\leq \Pi_{i=1}^{n}\xi_i\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\| \\ &= \Pi_{i=1}^{n}\xi_i\|\xi_{n+1}S_{n+1}x_n+(1-\xi_{n+1})x_n-x_n\| \\ &= \Pi_{i=1}^{n+1}\xi_i\|S_{n+1}x_n-x_n\| \\ &\leq \Pi_{i=1}^{n+1}\xi_iM_3,\end{aligned} \tag{3.23}$$

where M_3 is an approximate constant such that

$$M_3 \geq \max\left\{M_2, \sup_{n\geq1}\{\|S_{n+1}x_n-x_n\|\}, \sup_{n\geq1}\{\left\|S_{n+1}\rho_n-\rho_n\right\|\}\right\}. \tag{3.24}$$

Similarly, we have

$$\left\|W_{n+1}\rho_n-W_n\rho_n\right\| \leq \Pi_{i=1}^{n+1}\xi_iM_3. \tag{3.25}$$

Substituting (3.19), (3.23), and (3.25) into (3.22) yields that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&\quad\le \alpha_{n+1}\gamma\alpha\Big(\|x_{n+1}-x_n\| + \Pi_{i=1}^{n+1}\xi_i M_3\Big) \\
&\qquad + \left(1-\alpha_{n+1}\widetilde{\gamma}\right)\Big(\|x_{n+1}-x_n\| + \Big[\frac{m}{d}\big|\beta_{n+1}-\beta_n\big| + |s_{n+1}-s_n| + |r_{n+1}-r_n|\Big]M_2 + \Pi_{i=1}^{n+1}\xi_i M_3\Big) \\
&\qquad + |\alpha_{n+1}-\alpha_n|\big\|\gamma f(W_n x_n) - BW_n\rho_n\big\| \\
&\quad\le \left[1-\alpha_{n+1}(\widetilde{\gamma}-\gamma\alpha)\right]\|x_{n+1}-x_n\| + M_4\Big(\frac{m}{d}\big|\beta_{n+1}-\beta_n\big| + |s_{n+1}-s_n| + |r_{n+1}-r_n| + |\alpha_{n+1}-\alpha_n|\Big), \\
&\qquad + \Pi_{i=1}^{n+1}\xi_i
\end{aligned} \tag{3.26}$$

where M_4 is an approximate constant such that

$$M_4 \ge \max\left\{ M_3, \sup_{n\ge 1}\left\{ \big\|\gamma f(W_n x_n) - BW_n\rho_n\big\| \right\}\right\}. \tag{3.27}$$

It follows from conditions (C1)–(C3) and $\Pi_{i=1}^{n+1}\xi_i \le \delta^{n+1}$ and Lemma 2.2 that

$$\|x_{n+1}-x_n\| \longrightarrow 0. \tag{3.28}$$

Observe that

$$x_{n+1} - W_n\rho_n = \alpha_n\left(\gamma f(W_n x_n) - BW_n\rho_n\right), \tag{3.29}$$

it follows from (C1) that

$$\lim_{n\to\infty}\big\|W_n\rho_n - x_{n+1}\big\| = 0. \tag{3.30}$$

For $p \in \Omega$, we have

$$\begin{aligned}
\big\|y_n - p\big\|^2 &= \big\|P_C(I-s_nA)u_n - P_C(I-s_nA)p\big\|^2 \\
&\le \big\|(u_n-p) - s_n(Au_n - Ap)\big\|^2 \\
&= \big\|u_n-p\big\|^2 - 2s_n\langle u_n-p, Au_n-Ap\rangle + s_n^2\big\|Au_n-Ap\big\|^2 \\
&\le \big\|x_n-p\big\|^2 - 2s_n\Big[-u\big\|Au_n-Ap\big\|^2 + v\big\|u_n-p\big\|^2\Big] + s_n^2\big\|Au_n-Ap\big\|^2 \\
&\le \big\|x_n-p\big\|^2 + \left(2s_nu + s_n^2 - \frac{2s_nv}{\mu^2}\right)\big\|Au_n-Ap\big\|^2.
\end{aligned} \tag{3.31}$$

Similarly, we have

$$\|\rho_n - p\|^2 \leq \|x_n - p\|^2 + \left(2r_n u + r_n^2 - \frac{2r_n v}{\mu^2}\right)\|Ay_n - Ap\|^2. \tag{3.32}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Bp) + (I - \alpha_n B)(W_n \rho_n - p)\|^2 \\
&\leq \left(\alpha_n \|\gamma f(W_n x_n) - Bp\| + (1 - \alpha_n \widetilde{\gamma})\|\rho_n - p\|\right)^2 \\
&\leq \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|\rho_n - p\|^2 + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.33}$$

Substituting (3.32) into (3.33), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq{}& \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 + \left(2r_n u + r_n^2 - \frac{2r_n v}{\mu^2}\right)\|Ay_n - Ap\|^2 \\
&+ 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.34}$$

It follows from condition (C2) that

$$\begin{aligned}
&\left(\frac{2av}{\mu^2} - 2bu - b^2\right)\|Ay_n - Ap\|^2 \\
&\quad \leq \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 \\
&\qquad - \|x_{n+1} - p\|^2 + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\| \\
&\quad \leq \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \left(\|x_n - p\| + \|x_{n+1} - p\|\right)\|x_{n+1} - x_n\| \\
&\qquad + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.35}$$

As $\|x_{n+1} - x_n\| \to 0$ and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} \|Ay_n - Ap\| = 0 \tag{3.36}$$

It is easy to see that $\|\rho_n - p\| \leq \|y_n - p\|$. Using (3.33) again, we have

$$\|x_{n+1} - p\|^2 \leq \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|y_n - p\|^2 + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|. \tag{3.37}$$

Substituting (3.31) into (3.37), we can obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq{}& \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 + \left(2s_n u + s_n^2 - \frac{2s_n v}{\mu^2}\right)\|Au_n - Ap\|^2 \\
&+ 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|.
\end{aligned} \tag{3.38}$$

It follows from (C2) that

$$\begin{aligned}\left(\frac{2av}{\mu^2}-2bv-b^2\right)\|Au_n-Ap\|^2 &\leq \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\ &\leq \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|-\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\ &\quad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned} \tag{3.39}$$

As $\|x_{n+1}-x_n\| \to 0$ and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|Au_n-Ap\|=0. \tag{3.40}$$

Observe that

$$\begin{aligned}\|\rho_n-p\|^2 &= \|P_C(I-r_nA)y_n-P_C(I-r_nA)p\|^2\\ &\leq \langle (I-r_nA)y_n-(I-r_nA)p,\rho_n-p\rangle\\ &= \frac{1}{2}\Big\{\|(I-r_nA)y_n-(I-r_nA)p\|^2+\|\rho_n-p\|^2\\ &\qquad -\|(I-r_nA)y_n-(I-r_nA)p-(\rho_n-p)\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\|y_n-p\|^2+\|\rho_n-p\|^2-\|(y_n-\rho_n)-r_n(Ay_n-Ap)\|^2\Big\}\\ &\leq \frac{1}{2}\Big\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|y_n-\rho_n\|^2-r_n^2\|Ay_n-Ap\|^2\\ &\qquad +2r_n\langle y_n-\rho_n,Ay_n-Ap\rangle\Big\},\end{aligned} \tag{3.41}$$

which yields that

$$\|\rho_n-p\|^2 \leq \|x_n-p\|^2-\|y_n-\rho_n\|^2+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|. \tag{3.42}$$

Substituting (3.42) into (3.33) we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|y_n-\rho_n\|^2\\ &\quad+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|,\end{aligned} \tag{3.43}$$

which implies that

$$\begin{aligned}\|y_n-\rho_n\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\&\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\&\quad+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned}\tag{3.44}$$

It follows from (C1), $\|x_{n+1}-x_n\| \to 0$, and $\|Ay_n-Ap\| \to 0$ that $\lim_{n\to\infty}\|y_n-\rho_n\|=0$.

For $p\in\Omega$, we have

$$\begin{aligned}&\|y_n-p\|^2\\&=\|P_C(I-s_nA)u_n-P_C(I-s_nA)p\|^2\\&\le\langle P_C(I-s_nA)u_n-P_C(I-s_nA)p,(I-s_nA)u_n-(I-s_nA)p\rangle\\&=\langle y_n-p,(I-s_nA)u_n-(I-s_nA)p\rangle\\&=\frac{1}{2}\left(\|y_n-p\|^2+\|(I-s_nA)u_n-(I-s_nA)p\|^2-\|(y_n-p)-[u_n-p-s_n(Au_n-Ap)]\|^2\right)\\&\le\frac{1}{2}\left(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-p-[u_n-p-s_n(Au_n-Ap)]\|^2\right)\\&=\frac{1}{2}\left(\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\langle y_n-u_n,Au_n-Ap\rangle-s_n^2\|Au_n-Ap\|^2\right).\end{aligned}\tag{3.45}$$

This implies that

$$\begin{aligned}\|y_n-p\|^2&\le\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\langle y_n-u_n,Au_n-Ap\rangle-s_n^2\|Au_n-Ap\|^2\\&\le\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\|y_n-u_n\|\|Au_n-Ap\|.\end{aligned}\tag{3.46}$$

By (3.46), (3.37), and (3.5), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|u_n-p\|^2-\|y_n-u_n\|^2+2s_n\|y_n-u_n\|\|Au_n-Ap\|\\&\quad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\&\le\alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|y_n-u_n\|^2\\&\quad+2s_n\|y_n-u_n\|\|Au_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned}\tag{3.47}$$

It follows that

$$\begin{aligned}\|y_n-u_n\|^2 &\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2s_n\|y_n-u_n\|\|Au_n-Ap\|\\&\quad+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|\\&\le \alpha_n\|\gamma f(W_nx_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-x_{n+1}\|)\\&\quad+2s_n\|y_n-u_n\|\|Au_n-Ap\|+2\alpha_n\|\rho_n-p\|\|\gamma f(W_nx_n)-Bp\|.\end{aligned} \tag{3.48}$$

It follows from (C1), $\|Au_n-Ap\|\to 0$, and $\|x_{n+1}-x_n\|\to 0$ that $\|y_n-u_n\|\to 0$. It follows from $\|\rho_n-u_n\|\le\|\rho_n-y_n\|+\|y_n-u_n\|$ that $\lim_{n\to\infty}\|u_n-\rho_n\|=0$.

We now show that

$$\lim_{n\to\infty}\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|=0,\quad k=1,2,\ldots,m. \tag{3.49}$$

Indeed, let $p\in\Omega$, it follows from the firmly nonexpansiveness of $T^{F_k}_{\beta_n}$, we have for each $k\in\{1,2,\ldots,m\}$,

$$\begin{aligned}\left\|\Theta^k_{\beta_n}x_n-p\right\|^2&=\left\|T^{F_k}_{\beta_n}\Theta^{k-1}_{\beta_n}x_n-T^{F_k}_{\beta_n}p\right\|^2\le\langle\Theta^k_{\beta_n}x_n-p,\Theta^{k-1}_{\beta_n}x_n-p\rangle\\&=\frac12\left(\left\|\Theta^k_{\beta_n}x_n-p\right\|^2+\left\|\Theta^{k-1}_{\beta_n}x_n-p\right\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2\right).\end{aligned} \tag{3.50}$$

Thus, we get

$$\left\|\Theta^k_{\beta_n}x_n-p\right\|^2\le\left\|\Theta^{k-1}_{\beta_n}x_n-p\right\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2,\quad k=1,2,\ldots,m. \tag{3.51}$$

This implies that for each $k\in\{1,2,\ldots,m\}$,

$$\begin{aligned}\left\|\Theta^k_{\beta_n}x_n-p\right\|^2&\le\left\|\Theta^0_{\beta_n}x_n-p\right\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2\\&\quad-\left\|\Theta^{k-1}_{\beta_n}x_n-\Theta^{k-2}_{\beta_n}x_n\right\|^2-\cdots-\left\|\Theta^2_{\beta_n}x_n-\Theta^1_{\beta_n}x_n\right\|^2-\left\|\Theta^1_{\beta_n}x_n-\Theta^0_{\beta_n}x_n\right\|^2.\end{aligned} \tag{3.52}$$

It follows from $u_n=\Theta^m_{\beta_n}x_n$ that for each $k=1,2,\ldots,m$

$$\|u_n-p\|^2\le\|x_n-p\|^2-\left\|\Theta^k_{\beta_n}x_n-\Theta^{k-1}_{\beta_n}x_n\right\|^2. \tag{3.53}$$

By (3.37), (3.6), and (3.53), we have that for each $k = 1, 2, \ldots, m$

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|u_n - p\|^2 + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\| \\ &\le \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 - \left\|\Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n\right\|^2 \\ &\quad + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|.\end{aligned} \tag{3.54}$$

Thus, we have that for each $k = 1, 2, \ldots, m$

$$\begin{aligned}\left\|\Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n\right\|^2 &\le \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\| \\ &\le \alpha_n \|\gamma f(W_n x_n) - Bp\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\ &\quad + 2\alpha_n \|\rho_n - p\| \|\gamma f(W_n x_n) - Bp\|.\end{aligned} \tag{3.55}$$

It follows from (C1) and $\|x_{n+1} - x_n\| \to 0$ that for each $k = 1, 2, \ldots, m$

$$\left\|\Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n\right\| \longrightarrow 0. \tag{3.56}$$

Since

$$\begin{aligned}\|W_n \rho_n - \rho_n\| &\le \|W_n \rho_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \left\|x_n - \Theta_{\beta_n}^1 x_n\right\| + \left\|\Theta_{\beta_n}^1 x_n - \Theta_{\beta_n}^2 x_n\right\| \\ &\quad + \cdots + \left\|\Theta_{\beta_n}^{m-1} x_n - \Theta_{\beta_n}^m x_n\right\| + \|u_n - y_n\| + \|y_n - \rho_n\|.\end{aligned} \tag{3.57}$$

It follows from (3.56) that

$$\lim_{n \to \infty} \|W_n \rho_n - \rho_n\| = 0. \tag{3.58}$$

Observe that

$$\|W \rho_n - \rho_n\| \le \|W \rho_n - W_n \rho_n\| + \|W_n \rho_n - \rho_n\|. \tag{3.59}$$

It follows from Remark 2.7 that

$$\lim_{n \to \infty} \|W \rho_n - \rho_n\| = 0. \tag{3.60}$$

We show that $P_{\Omega}(\gamma f+(I-B))$ is a contraction. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}
&\left\|P_{\Omega}(\gamma f+(I-B))(x)-P_{\Omega}(\gamma f+(I-B))(y)\right\| \\
&\quad \leq\|(\gamma f+(I-B))(x)-(\gamma f+(I-B))(y)\| \\
&\quad \leq \gamma\|f(x)-f(y)\|+\|I-B\|\|x-y\| \\
&\quad \leq \gamma \alpha\|x-y\|+(1-\tilde{\gamma})\|x-y\| \\
&\quad =(\gamma \alpha+1-\tilde{\gamma})\|x-y\| .
\end{aligned} \tag{3.61}$$

The Banach's Contraction Mapping Principle guarantees that $P_{\Omega}(\gamma f+(I-B))$ has a unique fixed point, say $q \in H$. That is, $q=P_{\Omega}(\gamma f+(I-B))(q)$.

Next, we show that

$$\limsup_{n \to \infty}\langle\gamma f(q)-B q, x_n-q\rangle \leq 0. \tag{3.62}$$

To show that, we choose a subsequence $\{x_{n_i}\}$ of x_n such that

$$\limsup_{n \to \infty}\langle\gamma f(q)-B q, x_n-q\rangle=\lim_{i \to \infty}\langle\gamma f(q)-B q, x_{n_i}-q\rangle. \tag{3.63}$$

As $\{x_{n_i}\}$ is bounded, we know that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to p. We may assume, without loss of generality, that $x_{n_i} \rightharpoonup p$. From $\|\Theta_{\beta_n}^k x_n-\Theta_{\beta_n}^{k-1} x_n\| \rightarrow 0$ for each $k=1,2, \ldots, m$, we obtain that $\Theta_{\beta_{n_i}}^k x_{n_i} \rightharpoonup p$ for $k=1,2, \ldots, m$. From $\|u_n-\rho_n\| \rightarrow 0$, we also obtain that $\rho_{n_i} \rightharpoonup p$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $p \in C$.

Now we show that $p \in \Omega$. Indeed, let us first show that $p \in \mathrm{VI}(C, A)$. Put

$$T w_1= \begin{cases}A w_1+N_C w_1 & \text{if } w_1 \in C, \\ \emptyset & \text{if } w_1 \notin C.\end{cases} \tag{3.64}$$

Since A is relaxed (u, v)-cocoercive, we have

$$\langle A x-A y, x-y\rangle \geq(-u)\|A x-A y\|^2+v\|x-y\|^2 \geq\left(v-u \mu^2\right)\|x-y\|^2 \geq 0, \tag{3.65}$$

which yields that A is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2-A w_1 \in N_C w_1$ and $\rho_n \in C$, we have

$$\langle w_1-\rho_n, w_2-A w_1\rangle \geq 0. \tag{3.66}$$

On the other hand, from $\rho_n=P_C(I-r_n A) y_n$, we have

$$\langle w_1-\rho_n, \rho_n-(I-r_n A) y_n\rangle \geq 0 \tag{3.67}$$

and hence

$$\left\langle w_1 - \rho_n, \frac{\rho_n - y_n}{r_n} + Ay_n \right\rangle \geq 0. \tag{3.68}$$

It follows that

$$\begin{aligned}\langle w_1 - \rho_{n_i}, w_2 \rangle &\geq \langle w_1 - \rho_{n_i}, Aw_1 \rangle \\ &\geq \langle w_1 - \rho_{n_i}, Aw_1 \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} + Ay_{n_i} \right\rangle \\ &\geq \left\langle w_1 - \rho_{n_i}, Aw_1 - \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} - Ay_{n_i} \right\rangle \\ &= \langle w_1 - \rho_{n_i}, Aw_1 - A\rho_{n_i} \rangle + \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Ay_{n_i} \rangle \\ &\quad - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} \right\rangle \\ &\geq \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Ay_{n_i} \rangle - \left\langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - y_{n_i}}{r_{n_i}} \right\rangle,\end{aligned} \tag{3.69}$$

which implies that $\langle w_1 - p, w_2 \rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in \mathrm{VI}(C, A)$.

We next show that $p \in \bigcap_{k=1}^{m} \mathrm{EP}(F_k)$. Indeed, by Lemma 2.1, we have that for each $k = 1, 2, \ldots, m$,

$$F_k\left(\Theta_{\beta_n}^k x_n, y\right) + \frac{1}{\beta_n}\left\langle y - \Theta_{\beta_n}^k x_n, \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\rangle \geq 0, \quad \forall y \in C \tag{3.70}$$

.

It follows from (A2) that

$$\frac{1}{\beta_n}\left\langle y - \Theta_{\beta_n}^k x_n, \Theta_{\beta_n}^k x_n - \Theta_{\beta_n}^{k-1} x_n \right\rangle \geq F_k\left(y, \Theta_{\beta_n}^k x_n\right), \quad \forall y \in C. \tag{3.71}$$

Hence,

$$\left\langle y - \Theta_{\beta_{n_i}}^k x_{n_i}, \frac{\Theta_{\beta_{n_i}}^k x_{n_i} - \Theta_{\beta_{n_i}}^{k-1} x_{n_i}}{\beta_{n_i}} \right\rangle \geq F_k\left(y, \Theta_{\beta_{n_i}}^k x_{n_i}\right), \quad \forall y \in C. \tag{3.72}$$

It follows from (A4), (A5), $(\Theta_{\beta_{n_i}}^k x_{n_i} - \Theta_{\beta_{n_i}}^{k-1} x_{n_i})/\beta_{n_i} \to 0$, and $\Theta_{\beta_{n_i}}^k x_{n_i} \rightharpoonup p$ that for each $k = 1, 2, \ldots, m$,

$$F_k(y, p) \leq 0, \quad \forall y \in C. \tag{3.73}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Since $y \in C$ and $p \in C$, we obtain $y_t \in C$ and hence $F_k(y_t, p) \leq 0$. So by (A4), we have

$$0 = F_k(y_t, y_t) \leq tF_k(y_t, y) + (1-t)F_k(y_t, p) \leq tF_k(y_t, y). \tag{3.74}$$

Dividing by t, we get that for each $k = 1, 2, \ldots, m$,

$$F_k(y_t, y) \geq 0. \tag{3.75}$$

Letting $t \to 0$, it follows from (A3) that for each $k = 1, 2, \ldots, m$,

$$F_k(p, y) \geq 0 \tag{3.76}$$

for all $y \in C$ and hence $p \in \mathrm{EP}(F_k)$ for $k = 1, 2, \ldots, m$. That is, $p \in \bigcap_{k=1}^{m} \mathrm{EP}(F_k)$.

We now show that $p \in \mathrm{Fix}(W)$. Assume that $p \notin \mathrm{Fix}(W)$. Since $\rho_{n_i} \rightharpoonup p$ and $p \neq Wp$, from (3.60) and the Opial condition we have

$$\begin{aligned}\liminf_{i\to\infty}\|\rho_{n_i} - p\| &< \liminf_{i\to\infty}\|\rho_{n_i} - Wp\| \\ &\leq \liminf_{i\to\infty}\{\|\rho_{n_i} - W\rho_{n_i}\| + \|W\rho_{n_i} - Wp\|\} \\ &\leq \liminf_{i\to\infty}\|\rho_{n_i} - p\|,\end{aligned} \tag{3.77}$$

which is a contradiction. So, we get $p \in \mathrm{Fix}(W) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. This implies that $p \in \Omega$.

Since $q = P_\Omega(\gamma f + (I - B))(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle &= \lim_{i\to\infty}\langle \gamma f(q) - Bq, x_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Bq, p - q\rangle \leq 0.\end{aligned} \tag{3.78}$$

That is, (3.62) holds. Next, we consider

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\alpha_n(\gamma f(W_n x_n) - Bq) + (I - \alpha_n B)(W_n\rho_n - q)\|^2 \\ &\leq (1 - \alpha_n\widetilde{\gamma})^2\|W_n\rho_n - q\|^2 + 2\alpha_n\langle \gamma f(W_n x_n) - Bq, x_{n+1} - q\rangle \\ &\leq (1 - \alpha_n\widetilde{\gamma})^2\|x_n - q\|^2 + 2\alpha_n\gamma\langle f(W_n x_n) - f(q), x_{n+1} - q\rangle \\ &\quad + 2\alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\ &\leq (1 - \alpha_n\widetilde{\gamma})^2\|x_n - q\|^2 + \alpha_n\gamma\alpha\left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\ &\quad + 2\alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle\end{aligned} \tag{3.79}$$

So, we can obtain

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \frac{(1-\alpha_n\tilde{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Bq, x_{n+1}-q\rangle \\
&= \frac{1-2\alpha_n\tilde{\gamma}+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2+\frac{\alpha_n^2\tilde{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\leq \left[1-\frac{2\alpha_n(\tilde{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\right]\|x_n-q\|^2 \\
&\quad+\frac{2\alpha_n(\tilde{\gamma}-\alpha\gamma)}{1-\alpha_n\gamma\alpha}\left[\frac{1}{\tilde{\gamma}-\alpha\gamma}\langle\gamma f(q)-Bq, x_{n+1}-q\rangle+\frac{\alpha_n\tilde{\gamma}^2}{2(\tilde{\gamma}-\alpha\gamma)}M\right],
\end{aligned} \tag{3.80}$$

where M is an approximate constant such that $M \geq \sup_{n\geq 1}\{\|x_n-q\|^2\}$.

Put $l_n = 2\alpha_n(\tilde{\gamma}-\alpha\gamma)/(1-\alpha_n\gamma\alpha)$ and $t_n = (1/(\tilde{\gamma}-\alpha\gamma))\langle\gamma f(q)-Bq, x_{n+1}-q\rangle+(\alpha_n\tilde{\gamma}^2/2(\tilde{\gamma}-\alpha\gamma))M$. That is,

$$\|x_{n+1}-q\|^2 \leq (1-l_n)\|x_n-q\|+l_nt_n. \tag{3.81}$$

From condition (C1) and Lemma 2.2, we concluded that $x_n \to q \in \Omega$. It is easy to see that $u_n \to q$ and $y_n \to q$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfies conditions (A1)–(A5). Let $A : C \to H$ be relaxed (u,v)-cocoercive and μ-Lipschitz continuous and B a strongly positive linear bounded operator on H with coefficient $\tilde{\gamma}>0$. Assume that $0<\gamma<\tilde{\gamma}/\alpha$. Let $S_1, S_2, \ldots$ be a family of infinitely nonexpansive mappings of C into itself such that $\Gamma = \bigcap_{i=1}^{\infty}\operatorname{Fix}(S_i)\cap VI(C,A)\cap EP(F)\neq\emptyset$, let $\xi_1, \xi_2, \ldots$ be real numbers such that $0<\xi_i\leq\delta<1$ for every $i\in N$ and W_n be the W-mapping of C into itself generated by $S_n, S_{n-1},\ldots,S_1$ and $\xi_n, \xi_{n-1},\ldots,\xi_1$. Let $f : C \to C$ be a contraction with coefficient α $(0<\alpha<1)$ and $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}
x_1 = x \in H, \\
F(u_n, y)+\frac{1}{\beta_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in C, \\
y_n = P_C(I-s_nA)u_n, \\
x_{n+1} = \alpha_n\gamma f(W_nx_n)+(I-\alpha_nB)W_nP_C(I-r_nA)y_n
\end{gathered} \tag{3.82}$$

for every $n = 1, 2, \ldots$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ *and* $\{s_n\}$ *are sequences of numbers satisfying the conditions:*

(C1) $\{\alpha_n\} \subset [0,1]$ *with* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a,b]$ *and* $\{s_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v-u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(C3) $\liminf_{n\to\infty}\beta_n > 0$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in \Gamma$, *which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p - q\rangle \le 0, \quad \forall p \in \Gamma. \tag{3.83}$$

Proof. Let $m = 1$, by Theorem 3.1, we obtain the desired result. □

Corollary 3.3. *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $A : C \to H$ *be relaxed* (u,v)*-cocoercive and* μ*-Lipschitz continuous and let* B *be a strongly positive linear bounded operator on* H *with coefficient* $\tilde{\gamma} > 0$. *Assume that* $0 < \gamma < \tilde{\gamma}/\alpha$. *Let* $S_1, S_2, \ldots$ *be a family of infinitely nonexpansive mappings of* C *into itself such that* $\Delta = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i) \cap VI(C,A) \neq \emptyset$, *let* $\xi_1, \xi_2, \ldots$ *be real numbers such that* $0 < \xi_i \le \delta < 1$ *for every* $i \in N$, *and let* W_n *be the* W*-mapping of* C *into itself generated by* $S_n, S_{n-1}, \ldots, S_1$ *and* $\xi_n, \xi_{n-1}, \ldots, \xi_1$. *Let* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and* $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *be sequences generated by*

$$\begin{gathered} x_1 = x \in C, \\ y_n = P_C(I - s_n A)x_n, \\ x_{n+1} = \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B) W_n P_C (I - r_n A) y_n \end{gathered} \tag{3.84}$$

for every $n = 1, 2, \ldots$, *where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, *and* $\{s_n\}$ *are sequences of numbers satisfying the conditions:*

(C1) $\{\alpha_n\} \subset [0,1]$ *with* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C2) $\{r_n\} \subset [a,b]$ *and* $\{s_n\} \subset [a,b]$ *for some* a, b *with* $0 \le a \le b \le 2(v-u\mu^2)/\mu^2$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ *and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$.

Then, $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $q \in \Delta$, *which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p - q\rangle \le 0, \quad \forall p \in \Delta. \tag{3.85}$$

Proof. Let $F(x,y) = 0$ for $x, y \in C$, by Corollary 3.2 we obtain the desired result. □

Corollary 3.4. *Let* C *be a nonempty, closed and convex subset of* H. *Let* $F_1, F_2, \ldots, F_m$ *be bifunctions from* $C \times C$ *to* $\mathbb{R}$ *satisfies conditions (A1)–(A5). Let* $A : C \to H$ *be relaxed* (u,v)*-cocoercive and* μ*-Lipschitz continuous and* B *a strongly positive linear bounded operator on* H *with coefficient* $\tilde{\gamma} > 0$

such that $\Xi = \bigcap_{k=1}^{m} EP(F_k) \cap VI(C, A) \neq \emptyset$*. Let* $f : C \to C$ *be a contraction with coefficient* α $(0 < \alpha < 1)$ *and* $\{x_n\}$*,* $\{u_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{aligned} x_1 &= x \in H, \\ u_n &= T_{\beta_n}^{F_m} T_{\beta_n}^{F_{m-1}} \cdots T_{\beta_n}^{F_2} T_{\beta_n}^{F_1} x_n, \\ y_n &= P_C(I - s_n A) u_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n B) P_C (I - r_n A) y_n \end{aligned} \tag{3.86}$$

for every $n = 1, 2, \ldots,$ *where* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{r_n\}$ *and* $\{s_n\}$ *are sequences of numbers satisfying the conditions:*

(C1) $\{\alpha_n\} \subset [0,1]$ *with* $\lim_{n\to\infty}\alpha_n = 0$*,* $\sum_{n=1}^{\infty} \alpha_n = \infty$*, and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$*;*

(C2) $\{r_n\} \subset [a, b]$ *and* $\{s_n\} \subset [a, b]$ *for some* a, b *with* $0 \leq a \leq b \leq 2(v - u\mu^2)/\mu^2$*,* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*, and* $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$*;*

(C3) $\liminf_{n\to\infty}\beta_n > 0$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$*.*

Then, $\{x_n\}$*,* $\{y_n\}$ *and* $\{u_n\}$ *converge strongly to* $q \in \Omega$*, which solves the following variational inequality:*

$$\langle \gamma f q - Bq, p - q \rangle \leq 0, \quad \forall p \in \Xi. \tag{3.87}$$

Remark 3.5. (i) If $s_n = 0$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 2.1 in [9]. If $s_n = 0$ and $S_i = I$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 2.1 in [8] with $S = I$. If $s_n = 0$, $r_n = 0$ and $S_i = I$ for all $n \geq 0$, by Corollary 3.2, we get Theorem 3.1 in [6] with $S = I$ and Theorem 3.3 in [7] with $S = I$ and $C = H$.

(ii) Corollary 3.3 extends, generalizes and improves the main results in [21, 22, 24].

(iii) It is easy to see that Theorem 3.1 is different from the main results in [1–4].

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