Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 601492, 15 pages
doi:10.1155/2010/601492

Research Article

Positive Solutions for Multipoint Boundary Value Problem of Fractional Differential Equations

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Received 28 August 2010; Revised 10 December 2010; Accepted 16 December 2010

Academic Editor: Ferhan M. Atici

We study the existence and multiplicity of positive solutions for the fractional m-point boundary value problem $D_{0+}^{\alpha}u(t)+f(t,u(t))=0$, $0<t<1$, $u(0)=u'(0)=0$, $u'(1)=\sum_{i=1}^{m-2}a_i u'(\xi_i)$, where $2<\alpha<3$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, and $f:[0,1]\times[0,\infty)\mapsto[0,\infty)$ is continuous. Here, $a_i\geqslant 0$ for $i=1,\ldots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, and $\rho=\sum_{i=1}^{m-2}a_i\xi_i^{\alpha-2}$ with $\rho<1$. In light of some fixed point theorems, some existence and multiplicity results of positive solutions are obtained.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. It has been applied to almost every field of science, engineering, and mathematics in the last three decades [1–5]. But the investigation of the theory of fractional differential equations has only been started quite recently.

Among all the researches on the theory of the fractional differential equations, the study of the boundary value problems for fractional differential equations recently has attracted a great deal of attention from many researchers. And some results have been obtained on the existence of solutions (or positive solutions) of the boundary value problems for some specific fractional differential equations [6–11].

More specifically, Bai [12] discussed the existence of positive solutions for the boundary value problem (BVP for short)

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1,\ 1<\alpha\leqslant 2, \\ u(0)=0, \quad u(1)=\beta u(\eta), \end{gathered} \tag{1.1}$$

where $0 < \beta\eta^{\alpha-1} < 1$, $0 < \eta < 1$, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Some existence results of at least one positive solution for the above-mentioned BVP are obtained by the use of fixed point index theory.

In [13], Salem considered the existence of Pseudosolutions for the nonlinear m-point BVP,

$$\begin{gathered} D^{\alpha}x(t) + q(t)f(t, x(t)) = 0, \quad 0 < t < 1, \ \alpha \in (n-1, n], \ n \geqslant 2, \\ x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)} = 0, \quad x(1) = \sum_{i=1}^{m-2} \xi_i x(\eta_i), \end{gathered} \tag{1.2}$$

where x takes values in a reflexive Banach space E. Here, $0 < \eta_1 < \cdots < \eta_{m-2} < 1$, $\xi_i > 0$ with $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-1} < 1$, and $x^{(k)}$ denotes the kth Pseudoderivative of x while D^{α} denotes the Pseudofractional differential operator of order α. In light of the fixed point theorem given by O'Regan, the criteria for the existence of at least one Pseudo solution for the m-point BVP are established.

Very recently, M. El-Shahed [14] considered the existence and nonexistence of positive solutions to the fractional differential equation

$$D_{0+}^{\alpha}u(t) + \lambda f(u(t)) = 0, \quad 0 < t < 1, \ 2 < \alpha < 3, \tag{1.3}$$

subject to the boundary conditions

$$u(0) = u'(0) = 0, \quad u'(1) = 0. \tag{1.4}$$

Their analysis relies on Krasnoselskii's fixed point theorem.

Goodrich [15] then considered the BVP for the higher-dimensional fractional differential equation as follows:

$$\begin{gathered} D^{\alpha}x(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \ \alpha \in (n-1, n], \ n > 3, \\ x^{i}(0) = 0, \quad 0 \leqslant i \leqslant n-2, \\ D^{\nu}x(t)|_{t=1} = 0, \quad 1 \leqslant \nu \leqslant n-2, \end{gathered} \tag{1.5}$$

and a Harnack-like inequality associated with the Green's function related to the above problem is obtained improving the results in [16].

Motivated by the aforementioned results and techniques in coping with those boundary value problems of the fractional differential equations, we then turn to investigate the existence and multiplicity of positive solutions for the following BVP:

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \ 2 < \alpha < 3, \\ u(0) = u'(0) = 0, \quad u'(1) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \end{gathered} \tag{1.6}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α. Here, by a positive solution of BVP (1.6), we mean a function which is positive on $(0,1)$ and satisfies the equation and boundary conditions in (1.6).

Throughout the paper, we will assume that the following conditions hold.

(H1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

(H2) $a_i \geqslant 0$ for $i = 1,\ldots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, and $\rho = \sum_{i=1}^{m-2} a_i \xi_i^{\alpha-2}$ with $\rho < 1$.

There is a vast literature concerning the multipoint BVPs for the integer-order differential equations. An important recent paper was given by Webb and Infante [17]; and they established a new unified method for the existence of multiple positive solutions to a large number of nonlinear nonlocal BVPs for the integer-order differential equations. While in the setting of the fractional-order derivatives, as far as we know, the existence of positive solutions for the multipoint BVP (1.6) has not been discussed in the literature.

The rest of the paper is organized as follows. Section 2 preliminarily provides some definitions and lemmas which are crucial to the following discussion. In Section 3, we obtain the existence and multiplicity results of positive solutions for the BVP (1.6) by means of some fixed point theorems. Finally, we give a concrete example to illustrate the possible application of our analytical results.

2. Preliminaries

In this section, we preliminarily provide some definitions and lemmas which are useful in the following discussion.

Definition 2.1 (see [3]). The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \mapsto \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 (see [3]). The standard Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \mapsto \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, \tag{2.2}$$

where $n = [\alpha] + 1$ provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3 (see [6]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$, then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \tag{2.3}$$

for some $c_i \in \mathbb{R}$, $\quad i = 0,1,\ldots,N$, where N is the smallest integer greater than or equal to α.

By Lemma 2.3, we next present an integral presentation of the solution for the BVP of the linearized equation associated with the BVP (1.6).

Lemma 2.4. *Let $y \in C[0,1]$, then the BVP*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3, \\ u(0) = u'(0) = 0, \quad u'(1) = \sum_{i=1}^{m-2} a_i u'(\xi_i) \end{gathered} \tag{2.4}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) \mathrm{d}s, \tag{2.5}$$

where the Green function G is given by $G(t,s) = G_1(t,s) + G_2(t,s)$, where

$$\begin{gathered} G_1(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \dfrac{(1-s)^{\alpha-2} t^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \\ G_2(t,s) = \frac{1}{(1-\rho)\Gamma(\alpha)} \left(\rho (1-s)^{\alpha-2} - \sum_{i=1}^{m-2} a_i (\xi_i - s)^{\alpha-2} \chi_{E_i}(s) \right) t^{\alpha-1}. \end{gathered} \tag{2.6}$$

Here, $E_i = [0, \xi_i]$, and χ_{E_i} denotes the characteristic function of the set E_i for $i = 1, 2, \ldots, m-2$.

Proof. Lemma 2.3 yields

$$u(t) = -I_{0+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}. \tag{2.7}$$

This, with the condition that $u(0) = 0$, gives $c_3 = 0$. Furthermore, differentiating both sides of the expression of $u(t)$ with respect to t, we obtain

$$u'(t) = -I_{0+}^{\alpha-1} y(t) + c_1(\alpha-1) t^{\alpha-2} + c_2(\alpha-2) t^{\alpha-3}. \tag{2.8}$$

Now, by the conditions that $u'(0) = 0$ and $u'(1) = \sum_{i=1}^{m-2} a_i u'(\xi_i)$, we get that $c_2 = 0$ and

$$c_1 = \frac{1}{(\alpha-1)(1-\rho)} \left(I_{0+}^{\alpha-1} y(1) - \sum_{i=1}^{m-2} a_i I_{0+}^{\alpha-1} y(\xi_i) \right). \tag{2.9}$$

Hence,

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s \\
&\quad + \frac{t^{\alpha-1}}{(1-\rho)\Gamma(\alpha)}\left(\rho\int_0^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s - \sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-2}y(s)\mathrm{d}s\right) \\
&= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s \\
&\quad + \frac{t^{\alpha-1}}{(1-\rho)\Gamma(\alpha)}\left(\rho\int_0^1 (1-s)^{\alpha-2}y(s)\mathrm{d}s - \int_0^1\sum_{i=1}^{m-2}a_i(\xi_i-s)^{\alpha-2}\chi_{E_i}(s)y(s)\mathrm{d}s\right) \\
&= \int_0^1 G(t,s)y(s)\mathrm{d}s.
\end{aligned}
\tag{2.10}
$$

The proof is complete. □

The functions $G_i(t,s)\quad(i=1,2)$ have important properties as follows.

Lemma 2.5. *Assume (H2) holds, then $G_i(1,s)\geqslant G_i(t,s)\geqslant q(t)G_i(1,s)\geqslant 0$ for all $t,s\in[0,1]$ and $i=1,2$, where $q(t)=t^{\alpha-1}$.*

Proof. The asserted relation for $i=1$ is a direct result of Lemma 2.8 in [14]. In addition, the case for $i=2$ follows from a direct application of the definition of $G_2(t,s)$ and the fact that

$$
\begin{aligned}
G_2(t,s) &= \frac{1}{(1-\rho)\Gamma(\alpha)}\left(\sum_{i=1}^{m-2}a_i\xi_i^{\alpha-2}(1-s)^{\alpha-2} - \sum_{i=1}^{m-2}a_i\xi_i^{\alpha-2}\left(1-\frac{s}{\xi_i}\right)^{\alpha-2}\chi_{E_i}(s)\right)t^{\alpha-1} \\
&\geqslant \frac{1}{(1-\rho)\Gamma(\alpha)}\left(\sum_{i=1}^{m-2}a_i\xi_i^{\alpha-2}(1-s)^{\alpha-2} - \sum_{i=1}^{m-2}a_i\xi_i^{\alpha-2}(1-s)^{\alpha-2}\right)t^{\alpha-1} \\
&= 0
\end{aligned}
\tag{2.11}
$$

holds for $t,s\in[0,1]$. The proof is complete. □

Remark 2.6. The definition of $G(t,s)$ and Lemma 2.5 yield

$$G(1,s)\geqslant G(t,s)\geqslant q(t)G(1,s)\geqslant 0,\quad 0\leqslant t,s\leqslant 1. \tag{2.12}$$

Remark 2.7. It is necessary to mention that Bai and Lü [6] showed that their Green's function did not satisfy a classical Harnack-like inequality (HLI) for the homogenous two-point BVP of fractional differential equation with order α in $(1,2]$. They proved that $G(t,s)>\gamma(s)G(s,s)$, where $\gamma(s)\to 0^+$ as $s\to 0^+$, which is a challenge for our seeking positive solutions.

On the other hand, Goodrich [15], for the homogenous BVP of fractional differential equation with order α in $(n-1,n]\quad(n>3)$, established a HLI: $\min_{t\in[1/2,1]}G(t,s)>\gamma G(1,s)$, where γ is a positive constant. Using this inequality, the author obtained sufficient conditions on the existence of positive solutions. In Remark 2.6, we get a generalized HLI contrasting to the result in [15]. In fact, for a constant $\theta \in (0,1]$, it follows from Remark 2.6 that $\min_{t\in[\theta,1]} G(t,s) > \theta^{\alpha-1}G(1,s)$.

From the above discussion, we may infer that it is closely related to both the order of fractional differential equations and boundary conditions that whether or not the Green's function satisfies a traditional HLI, which needs more detailed and rigorous investigations.

Next, we introduce some fixed point theorems which will be adopted to prove the main results in the following section.

Lemma 2.8 (see [18]). *Let $\mathfrak{B}$ be a Banach space, $\mathcal{P} \subseteq \mathfrak{B}$ a cone, and Ω_1, Ω_2 two bounded open balls of $\mathfrak{B}$ centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $\mathcal{A} : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$ is a completely continuous operator such that either*

(B1) $\|\mathcal{A}x\| \leqslant \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|\mathcal{A}x\| \geqslant \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_2$ *or*

(B2) $\|\mathcal{A}x\| \geqslant \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_1$ *and* $\|\mathcal{A}x\| \leqslant \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_2$

Holds, then $\mathcal{A}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2.9. The map θ is said to be a nonnegative continuous concave functional on a cone $\mathcal{P}$ of a real Banach space $\mathfrak{B}$ provided that $\theta : \mathcal{P} \to [0,\infty)$ is continuous and $\theta(tx + (1-t)y) \geqslant t\theta(x) + (1-t)\theta(y)$ for all $x, y \in \mathcal{P}$ and $0 \leqslant t \leqslant 1$.

Lemma 2.10 (see [19]). *Let $\mathcal{P}$ be a cone in a real Banach space $\mathfrak{B}$, $\mathcal{P}_c = \{x \in \mathcal{P} \mid \|x\| < c\}$, θ a nonnegative continuous concave functional on $\mathcal{P}$ such that $\theta(x) \leqslant \|x\|$, for all $x \in \overline{\mathcal{P}}_c$, and $\mathcal{P}(\theta,b,d) = \{x \in \mathcal{P} \mid b \leqslant \theta(x),\ \|x\| \leqslant d\}$. Suppose that $\mathcal{A} : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ is completely continuous and there exist constants $0 < a < b < d \leqslant c$ such that*

(C1) $\{x \in \mathcal{P}(\theta,b,d) \mid \theta(x) > b\} \neq \emptyset$ *and* $\theta(\mathcal{A}x) > b$ *for* $x \in \mathcal{P}(\theta,b,d)$,

(C2) $\|\mathcal{A}x\| < a$ *for* $x \in \overline{\mathcal{P}}_a$,

(C3) $\theta(\mathcal{A}x) > b$ *for* $x \in \mathcal{P}(\theta,b,c)$ *with* $\|\mathcal{A}x\| > d$,

then $\mathcal{A}$ has at least three fixed points x_1, x_2, and x_3 such that

$$\|x_1\| < a, \quad b < \theta(x_2), \quad a < \|x_3\| \quad \text{with } \theta(x_3) < b. \tag{2.13}$$

Remark 2.11. If there holds $d = c$, then condition (C1) of Lemma 2.10 implies condition (C3) of Lemma 2.10.

3. Main Results

In order to apply the fixed point theorems to the BVP (1.6), we first import some notations and operator.

Let $\mathfrak{B} = C[0,1]$ be the classical Banach space with the norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$. Furthermore, define the cone $\mathcal{P}$ by

$$\mathcal{P} = \left\{u \in \mathfrak{B} \mid u(t) \geqslant q(t)\|u\| \text{ for } t \in [0,1]\right\}. \tag{3.1}$$

Notice that $\|u\| = u(1)$ for each $u \in \mathcal{P}$. For a positive number r, define the function space Ω_r by

$$\Omega_r = \{u \in C[0,1] : \|u\| < r\}. \tag{3.2}$$

Also define the operator $\mathcal{A} : \mathcal{P} \to \mathfrak{B}$ by

$$[\mathcal{A}u](t) = \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s. \tag{3.3}$$

Next, we show some properties of the operator $\mathcal{A}$.

Lemma 3.1. *If (H1)-(H2) hold, then $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$.*

Proof. From the definition of the operator $\mathcal{A}$ and Remark 2.6, it follows that, for $u \in \mathcal{P}$,

$$\begin{aligned}
[\mathcal{A}u](t) &\geqslant q(t) \int_0^1 G(1,s) f(s,u(s)) \mathrm{d}s \\
&\geqslant q(t) \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \right\} \\
&= q(t) \|[\mathcal{A}u]\|,
\end{aligned} \tag{3.4}$$

Therefore, $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. This completes the proof. □

Lemma 3.2. *Assume that (H1)-(H2) hold, then the operator $\mathcal{A} : \overline{\Omega}_r \cap \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Lemma 3.1 implies that $\mathcal{A}(\overline{\Omega}_r \cap \mathcal{P}) \subset \mathcal{P}$. Moreover, the uniform continuity of the function $f(t,u)$ on the compact set $[0,1] \times [0,r]$ yields that the operator $\mathcal{A} : \overline{\Omega}_r \cap \mathcal{P} \to \mathcal{P}$ is continuous.

We now show that $\mathcal{A}(\overline{\Omega}_r \cap \mathcal{P})$ is bounded. To this end, let $l = \max\{f(t,u) : 0 \leqslant t \leqslant 1, 0 \leqslant u \leqslant r\}$, then, for $u \in \overline{\Omega}_r \cap \mathcal{P}$, it follows from Remark 2.6 that

$$0 \leqslant [\mathcal{A}u](t) = \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \leqslant l \int_0^1 G(1,s) \mathrm{d}s, \tag{3.5}$$

which implies that $\mathcal{A}(\overline{\Omega}_r \cap \mathcal{P})$ is bounded.

In addition, for each $u \in \overline{\Omega}_r \cap \mathcal{P}$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, writing $F(t_1,t_2) = |[\mathcal{A}u](t_1) - [\mathcal{A}u](t_2)|$, we have the following estimation:

$$\begin{aligned}
F(t_1,t_2) &\leqslant l\int_0^1 |G_1(t_2,s) - G_1(t_1,s)|\mathrm{d}s + l\int_0^1 |G_2(t_2,s) - G_2(t_1,s)|\mathrm{d}s \\
&\leqslant l\int_0^{t_1} |G_1(t_2,s) - G_1(t_1,s)|\mathrm{d}s + l\int_{t_1}^{t_2} |G_1(t_2,s) - G_1(t_1,s)|\mathrm{d}s \\
&\quad + l\int_{t_2}^1 |G_1(t_2,s) - G_1(t_1,s)|\mathrm{d}s + \frac{\rho l}{(\alpha-1)(1-\rho)\Gamma(\alpha)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) \\
&\leqslant \frac{l}{\Gamma(\alpha)}\int_0^{t_1}\left(\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)(1-s)^{\alpha-2} + (t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right)\mathrm{d}s \\
&\quad + \frac{l}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)(1-s)^{\alpha-2} + (t_2-s)^{\alpha-1}\right)\mathrm{d}s \\
&\quad + \frac{l}{\Gamma(\alpha)}\int_{t_2}^1 \left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)(1-s)^{\alpha-2}\mathrm{d}s + \frac{\rho l\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{(\alpha-1)(1-\rho)\Gamma(\alpha)} \\
&\leqslant \frac{l}{(\alpha-1)(1-\rho)\Gamma(\alpha)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + \frac{l}{\alpha\Gamma(\alpha)}\left(t_2^{\alpha} - t_1^{\alpha}\right).
\end{aligned} \tag{3.6}$$

Now, using the fact that the functions $t^{\alpha-1}$ and t^{α} are uniformly continuous on $[0,1]$, we conclude that $\mathcal{A}(\overline{\Omega}_r \cap \mathcal{P})$ is an equicontinuous set on $[0,1]$. It follows from the Arzelà-Ascoli Theorem that $\mathcal{A}(\overline{\Omega}_r \cap \mathcal{P})$ is a relatively compact set. As a consequence, we complete the whole proof. □

Lemma 3.3. *Assume that (H1)-(H2) hold, then $u \in C[0,1]$ is a solution of the BVP* (1.6) *if and only if it is a fixed point of $\mathcal{A}$ in $\mathcal{P}$.*

Proof. If $u \in \mathcal{P}$ and $\mathcal{A}u = u$, then

$$u(t) = -I_{0+}^{\alpha} f(t,u(t)) + \frac{1}{(\alpha-1)(1-\rho)}\left(I_{0+}^{\alpha-1} f(1,u(1)) - \sum_{i=1}^{m-2} a_i I_{0+}^{\alpha-1} f(\xi_i,u(\xi_i))\right)t^{\alpha-1}. \tag{3.7}$$

Thus,

$$D_{0+}^{\alpha} u(t) + f(t,u(t)) = 0, \tag{3.8}$$

and $u(0) = 0$. Furthermore, differentiation of (3.7) with respect to t produces

$$u'(t) = -I_{0+}^{\alpha-1} f(t,u(t)) + \frac{1}{(1-\rho)}\left(I_{0+}^{\alpha-1} f(1,u(1)) - \sum_{i=1}^{m-2} a_i I_{0+}^{\alpha-1} f(\xi_i,u(\xi_i))\right)t^{\alpha-2}. \tag{3.9}$$

This yields $u'(0) = 0$ and

$$u'(1) = \frac{1}{(1-\rho)}\left(\rho I_{0+}^{\alpha-1} f(1,u(1)) - \sum_{i=1}^{m-2} a_i I_{0+}^{\alpha-1} f(\xi_i, u(\xi_i))\right) = \sum_{i=1}^{m-2} a_i u'(\xi_i). \tag{3.10}$$

Therefore, u is a positive solution of the BVP (1.6).

On the other hand, if u is a positive solution of the BVP (1.6), then Lemma 2.4 implies $\mathcal{A}u = u$. Moreover, in view of the proof of Lemma 3.1, we also get $u(t) \geqslant q(t)\|u\|$ for $t \in [0,1]$. Hence, u is a fixed point of $\mathcal{A}$ in $\mathcal{P}$. We consequently complete the proof. □

In the following, fix η in (0,1) and set

$$L = \left(\int_0^1 G_1(1,s)ds + \frac{\rho}{(\alpha-1)(1-\rho)\Gamma(\alpha)}\right)^{-1}, \qquad M = \left(\eta^{\alpha-1}\int_\eta^1 G_1(1,s)ds\right)^{-1}. \tag{3.11}$$

We now present two main results on the existence of positive solutions for the BVP (1.6).

Theorem 3.4. *Assume that (H1)-(H2) hold. In addition, and suppose that one of the following two conditions holds:*

(H3) $\lim_{u\to 0}\min_{t\in[0,1]}(f(t,u)/u) = \infty$, $\lim_{u\to\infty}\max_{t\in[0,1]}(f(t,u)/u) = 0$,

(H4) $\lim_{u\to 0}\min_{t\in[0,1]}(f(t,u)/u) = 0$, $\lim_{u\to\infty}\max_{t\in[0,1]}(f(t,u)/u) = \infty$,

then the BVP (1.6) *has at least one positive solution.*

Proof. Notice that Lemma 3.2 guarantees that the operator $\mathcal{A} : \overline{\Omega}_r \cap \mathcal{P} \to \mathcal{P}$ is completely continuous.

Now, assume that condition (H3) holds. Since $\lim_{u\to 0}\min_{t\in[0,1]}(f(t,u)/u) = \infty$, there exists an $r_1 > 0$ such that

$$f(t,u) \geqslant \varepsilon_1 u \quad \text{for } t \in [0,1],\ 0 \leqslant u \leqslant r_1, \tag{3.12}$$

where the constant $\varepsilon_1 > 0$ is chosen, so that

$$\varepsilon_1 \int_0^1 G(1,s)q(s)ds > 1. \tag{3.13}$$

Thus,

$$f(t,u(t)) \geqslant \varepsilon_1 u(t) \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_{r_1},\ t \in [0,1]. \tag{3.14}$$

This, together with the definitions of $\mathcal{A}$ and Remark 2.6, implies that for any $u \in \mathcal{P} \cap \partial\Omega_{r_1}$,

$$\begin{aligned}\|\mathcal{A}u\| &= \int_0^1 G(1,s) f(s,u(s))\mathrm{d}s \geqslant \varepsilon_1 \int_0^1 G(1,s)u(s)\mathrm{d}s \\ &\geqslant \varepsilon_1 \int_0^1 G(1,s)q(s)\mathrm{d}s\|u\| \geqslant \|u\|.\end{aligned} \tag{3.15}$$

That is, for $u \in \mathcal{P} \cap \partial\Omega_{r_1}$, $\|\mathcal{A}u\| \geqslant \|u\|$.

On the other hand, from $\lim_{u\to\infty}\max_{t\in[0,1]}(f(t,u)/u) = 0$, it follows that there exists a $l_1 > 0$ such that

$$f(t,u) \leqslant \varepsilon_2 u \quad \text{for } t \in [0,1],\ u \geqslant l_1, \tag{3.16}$$

where the constant $\varepsilon_2 > 0$ satisfies

$$\varepsilon_2 \int_0^1 G(1,s)\mathrm{d}s < \frac{1}{2}. \tag{3.17}$$

Now, write $l_2 = \max\{f(t,u) : t \in [0,1],\ u \in [0,l_1]\}$. Then the relation (3.16) yields

$$f(t,u) \leqslant \varepsilon_2 u + l_2 \quad \text{for } t \in [0,1],\ u \geqslant 0. \tag{3.18}$$

Set $r_2 = \max\{2r_1, 2l_2 \int_0^1 G(1,s)\mathrm{d}s\}$ and let $u \in \mathcal{P} \cap \partial\Omega_{r_2}$, then Remark 2.6 and (3.18) imply

$$\begin{aligned}\|\mathcal{A}u\| &= [\mathcal{A}u](1) \\ &= \int_0^1 G(1,s)f(s,u(s))\mathrm{d}s \\ &\leqslant \int_0^1 G(1,s)(\varepsilon_2 u(s) + l_2)\mathrm{d}s \\ &\leqslant \varepsilon_2 \int_0^1 G(1,s)\mathrm{d}s\|u\| + l_2 \int_0^1 G(1,s)\mathrm{d}s \\ &\leqslant \|u\|.\end{aligned} \tag{3.19}$$

Thus, the operator $\mathcal{A}$ satisfies condition (B2) of Lemma 2.8. Consequently, the operator $\mathcal{A}$ has at least one fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is one positive solution of the BVP (1.6).

Next, we suppose that (H4) holds. The proof is similar to that of the case in which assumption (H3) holds and will only be sketched here. Select two positive constants ε_3 and ε_4

with $\varepsilon_3 \int_0^1 G(1,s)\mathrm{d}s < 1$ and $\varepsilon_4 \eta^{\alpha-1} \int_\eta^1 G(1,s)\mathrm{d}s > 1$, respectively. Then, there exist two positive numbers r_3 and l_3, such that

$$f(t,u) \leqslant \varepsilon_3 u \quad \text{for } t \in [0,1],\ 0 \leqslant u \leqslant r_3, \tag{3.20}$$

$$f(t,u) \geqslant \varepsilon_4 u \quad \text{for } t \in [0,1],\ u \geqslant l_3. \tag{3.21}$$

It follows from Remark 2.6 and (3.20) that for $u \in \mathcal{P} \cap \partial\Omega_{r_3}$,

$$\|\mathcal{A}u\| = \int_0^1 G(1,s) f(s,u(s))\mathrm{d}s \leqslant \varepsilon_3 \int_0^1 G(1,s)u(s)\mathrm{d}s \leqslant \|u\|. \tag{3.22}$$

In addition, let $r_4 = \max\{2r_3, l_3\eta^{1-\alpha}\}$. If $u \in \mathcal{P} \cap \partial\Omega_{r_4}$, then

$$u(t) \geqslant q(t)\|u\| \geqslant \eta^{\alpha-1} r_4 \geqslant l_3 \quad \text{for } t \in [\eta, 1], \tag{3.23}$$

and by (3.21),

$$f(t,u(t)) \geqslant \varepsilon_4 u(t) \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_{r_4},\ t \in [\eta, 1]. \tag{3.24}$$

This yields that, for $u \in \partial\Omega_{r_4} \cap \mathcal{P}$,

$$\begin{aligned} \|\mathcal{A}u\| &= [\mathcal{A}u](1) \\ &= \int_0^1 G(1,s) f(s,u(s))\mathrm{d}s \\ &\geqslant \varepsilon_4 \int_\eta^1 G(1,s)u(s)\mathrm{d}s \\ &\geqslant \varepsilon_4 \int_\eta^1 G(1,s)q(s)\|u\|\mathrm{d}s \\ &\geqslant \varepsilon_4 \eta^{\alpha-1} \int_\eta^1 G(1,s)\mathrm{d}s\|u\| \\ &\geqslant \|u\|. \end{aligned} \tag{3.25}$$

Thus, $\|\mathcal{A}u\| \geqslant \|u\|$ for $u \in \partial\Omega_{r_4} \cap \mathcal{P}$. Hence, the operator $\mathcal{A}$ satisfies condition (B1) in Lemma 2.8. As a consequence, the operator $\mathcal{A}$ has at least one fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$. This means that the BVP (1.6) has at least one positive solution u. We complete the whole proof. $\square$

Theorem 3.5. *Assume (H1)-(H2) hold. In addition, suppose that there exist constants* $0 < a < b < c$ *with* $Mb < Lc$*, such that the following assumptions hold:*

(H5) $f(t,u) < La$, *for* $(t,u) \in [0,1] \times [0,a]$,

(H6) $f(t,u) > Mb$, *for* $(t,u) \in [\eta,1] \times [b,c]$,

(H7) $f(t,u) \leqslant Lc$, *for* $(t,u) \in [0,1] \times [0,c]$,

then the BVP (1.6) *has at least three positive solutions.*

Proof. Define the nonnegative continuous concave functional θ on the cone $\mathcal{P}$ by

$$\theta(u) = \min_{\eta \leqslant t \leqslant 1} u(t). \tag{3.26}$$

Next, we intend to verify that all the conditions in Lemma 2.10 hold with respect to the operator $\mathcal{A}$. Lemma 2.10 involves parameters $a,\ b,\ c$, and d with $0 < a < b < d \leqslant c$. Now, let $d = c$, then by Remark 2.11, it is sufficient to verify that the conditions (C1) and (C2) in Lemma 2.10 hold. To this end, let $u \in \overline{\mathcal{P}}_c$, then $\|u\| \leqslant c$. This together with assumption (H7) implies that for $0 \leqslant t \leqslant 1,\ f(t,u(t)) \leqslant Lc$. This relation and Remark 2.6 yield

$$\begin{aligned} \|\mathcal{A}u\| &\leqslant \int_0^1 G(1,s) f(s,u(s))\mathrm{d}s \\ &\leqslant \int_0^1 G_1(1,s) f(s,u(s))\mathrm{d}s + \int_0^1 \frac{\rho}{(1-\rho)\Gamma(\alpha)}(1-s)^{\alpha-2} f(s,u(s))\mathrm{d}s \\ &\leqslant Lc\left[\int_0^1 G_1(1,s)\mathrm{d}s + \frac{\rho}{(\alpha-1)(1-\rho)\Gamma(\alpha)}\right] \\ &= c. \end{aligned} \tag{3.27}$$

This, with Lemma 3.2, clearly manifests that the operator $\mathcal{A} : \overline{\mathcal{P}}_c \to \overline{\mathcal{P}}_c$ is completely continuous. In a similar argument, if $u \in \overline{\mathcal{P}}_a$, then assumption (H5) yields $f(t,u(t)) < La,\ 0 \leqslant t \leqslant 1$. Therefore, condition (C2) of Lemma 2.10 is satisfied.

Moreover, the set $\{x \in \mathcal{P}(\theta,b,c) \mid \theta(x) > b\}$ is not empty, since the constant function $u(t) \equiv (b+c)/2$ is contained in the set $\{x \in \mathcal{P}(\theta,b,c) \mid \theta(x) > b\}$. Now, let $u \in \mathcal{P}(\theta,b,c)$, then $0 \leqslant u(t) \leqslant c$ for $t \in [0,1]$ and $b \leqslant \min_{\eta \leqslant t \leqslant 1} u(t)$. Thus, we obtain

$$b \leqslant u(t) \leqslant c, \quad \eta \leqslant t \leqslant 1. \tag{3.28}$$

Assumption (H6) and (3.28) imply

$$f(t,u(t)) > Mb, \quad \eta \leqslant t \leqslant 1. \tag{3.29}$$

Hence, it follows from (3.29) and Lemma 2.5 that

$$\begin{aligned}\theta(\mathscr{A}u) &= \min_{\eta\leqslant t\leqslant 1}([\mathscr{A}u](t)) \\ &\geqslant \eta^{\alpha-1}\int_{\eta}^{1} G_1(1,s)f(s,u(s))\mathrm{d}s \\ &> Mb\eta^{\alpha-1}\int_{\eta}^{1} G_1(1,s)\mathrm{d}s \\ &= b.\end{aligned} \tag{3.30}$$

Accordingly, the validity of condition (C1) in Lemma 2.10 is verified. Consequently, by virtue of Lemma 2.10 and Remark 2.11, the operator $\mathscr{A}$ has at least three fixed points u_1, u_2, and u_3 satisfying

$$\begin{aligned}&\max_{0\leqslant t\leqslant 1}|u_1(t)| < a, \qquad b < \min_{\eta\leqslant 1\leqslant 1}|u_2(t)|, \\ &a < \max_{0\leqslant t\leqslant 1}|u_3(t)|, \qquad \min_{\eta\leqslant 1\leqslant 1}|u_3(t)| < b.\end{aligned} \tag{3.31}$$

These fixed points are positive solutions for the BVP (1.6). The proof is complete. □

4. Illustrative Example

Consider the BVP

$$\begin{aligned}&D_{0+}^{5/2}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \quad u'(1) = \frac{1}{2}u'\left(\frac{1}{64}\right) + \frac{1}{4}u'\left(\frac{9}{16}\right),\end{aligned} \tag{4.1}$$

where

$$f(t,u) = \begin{cases} \dfrac{1}{t+100} + 52u^3, & 0 \leqslant t \leqslant 1,\ u \leqslant 1, \\ \dfrac{u}{t+100u} + u + 51, & 0 \leqslant t \leqslant 1,\ 1 < u. \end{cases} \tag{4.2}$$

Let $\eta = 3/4$. Clearly, the parameters $\rho = (1/2)(1/64)^{1/2} + (1/4)(9/16)^{1/2} = 1/4 < 1, L = 135\sqrt{\pi}/88 \approx 4.871,\ M = 480\sqrt{\pi}/17\sqrt{3} \approx 51.196$. Choosing $a = 1/10,\ b = 1,\ c = 20$, then

$Mb < Lc$. Now, we can verify the validity of conditions (H5)–(H7) in Theorem 3.5. Indeed, indirect computations yield

$$f(t,u) = \frac{u}{t+100u} + 52u^3 \leqslant 0.01 + 52a^3 < La, \quad \text{for } (t,u) \in [0,1] \times [0,a],$$
$$f(t,u) = \frac{u}{t+100u} + u + 51 \geqslant b + 51 \geqslant Mb, \quad \text{for } (t,u) \in \left[\frac{3}{4},1\right] \times [b,c], \tag{4.3}$$
$$f(t,u) \leqslant \frac{u}{t+100u} + u + 51 \leqslant 0.01 + c + 51 < Lc, \quad \text{for } (t,u) \in [0,1] \times [0,c].$$

Hence, conditions (H5)–(H7) in Theorem 3.5 are satisfied for the above-specified functions and parameters. Therefore, in light of Theorem 3.5, we conclude that the above BVP has at least three positive solutions u_1, u_2, and u_3 defined on $[0,1]$ satisfying $\max_{0\leqslant t\leqslant 1}|u_1(t)| < a,\ b < \min_{\eta\leqslant 1\leqslant 1}|u_2(t)|$, and $a < \max_{0\leqslant t\leqslant 1}|u_3(t)|$ with $\min_{\eta\leqslant 1\leqslant 1}|u_3(t)| < b$.

Acknowledgment

This work is supported by the Scientific Research Foundation of the Higher Education Institutions of Hunan Province, China (Grant no. 10C1125).

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