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Research Article

On Semi-Uniform Kadec-Klee Banach Spaces

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Inspired by the concept of U-spaces introduced by Lau, (1978), we introduced the class of semi-uniform Kadec-Klee spaces, which is a uniform version of semi-Kadec-Klee spaces studied by Vlasov, (1972). This class of spaces is a wider subclass of spaces with weak normal structure and hence generalizes many known results in the literature. We give a characterization for a certain direct sum of Banach spaces to be semi-uniform Kadec-Klee and use this result to construct a semi-uniform Kadec-Klee space which is not uniform Kadec-Klee. At the end of the paper, we give a remark concerning the uniformly alternative convexity or smoothness introduced by Kadets et al., (1997).

1. Introduction

Let X be a real Banach space with the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ and the closed unit ball $B_X = \{x \in X : \|x\| \le 1\}$. In this paper, the strong and weak convergences of a sequence $\{x_n\}$ in X to an element $x \in X$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also let

$$\operatorname{sep}\{x_n\} = \inf\{\|x_n - x_m\| : n < m\}. \tag{1.1}$$

Definition 1.1 (see [1]). We say that a Banach space X is a *Kadec-Klee* space if

$$(\text{KK}) : \left.\begin{array}{l} \{x_n\} \subset B_X \\ x_n \rightharpoonup x \\ \operatorname{sep}\{x_n\} > 0 \end{array}\right\} \Longrightarrow \|x\| < 1. \tag{1.2}$$

A uniform version of the KK property is given in the following definition.

Definition 1.2 (see [2]). We say that a Banach space X is *uniform Kadec-Klee* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(\text{UKK}): \left.\begin{array}{l}\{x_n\} \subset B_X \\ x_n \rightharpoonup x \\ \operatorname{sep}\{x_n\} \geq \varepsilon\end{array}\right\} \Longrightarrow \|x\| \leq 1-\delta. \tag{1.3}$$

Two properties above are weaker than the following one.

Definition 1.3 (see [3]). We say that a Banach space X is *uniformly convex* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(\text{UC}): \left.\begin{array}{l}x, y \in B_X \\ \|x-y\| \geq \varepsilon\end{array}\right\} \Longrightarrow \frac{1}{2}\|x+y\| \leq 1-\delta. \tag{1.4}$$

Let us summarize a relationship between these properties in the following implication diagram:

$$\text{UC} \Longrightarrow \text{UKK} \Longrightarrow \text{KK}. \tag{1.5}$$

In the literature, there are some generalizations of UC and KK.

Definition 1.4 (see [4]). We say that a Banach space X is a *U-space* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(U\text{-space}): \left.\begin{array}{l}x, y \in B_X \\ \langle x-y, f\rangle \geq \varepsilon \text{ for some } f \in \nabla_x\end{array}\right\} \Longrightarrow \frac{1}{2}\|x+y\| \leq 1-\delta. \tag{1.6}$$

Here $\nabla_x = \{f \in S_{X^*} : \langle x, f\rangle = \|x\|\}$.

Definition 1.5 (see [5]). We say that a Banach space X is *semi-Kadec-Klee* if

$$(\text{semi-KK}): \left.\begin{array}{l}\{x_n\} \subset S_X \\ x_n \rightharpoonup x \in S_X\end{array}\right\} \Longrightarrow \begin{array}{c}\langle x, f_n\rangle \longrightarrow 1 \quad \forall\{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n} \quad \forall n.\end{array} \tag{1.7}$$

Some interesting results concerning semi-KK property are studied by Megginson [6].

Remark 1.6. It is clear that

$$U\text{-space} \Longrightarrow \text{semi-KK}. \tag{1.8}$$

Remark 1.7. A Banach space X is semi-KK if and only if

$$\left.\begin{array}{l}\{x_n\} \subset B_X \\ x_n \rightharpoonup x \\ \inf\{\langle x_n - x, f_n\rangle : n \in \mathbb{N}\} > 0 \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n},\ \forall n\end{array}\right\} \Longrightarrow \|x\| < 1. \tag{1.9}$$

We now introduce a property lying between U-space and semi-KK.

Definition 1.8. We say that a Banach space X is *semi-uniform Kadec-Klee* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\text{(semi-UKK)}: \left.\begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightharpoonup x \\ \langle x_n - x, f_n\rangle \geq \varepsilon, \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n} \quad \forall n \end{array}\right\} \Longrightarrow \|x\| \leq 1-\delta. \tag{1.10}$$

In this paper, we prove that semi-UKK property is a nice generalization of U-space and semi-KK property. Moreover, every semi-UKK space has weak normal structure. We also give a characterization of the direct sum of finitely many Banach spaces which is semi-KK and semi-UKK. We use such a characterization to construct a Banach space which is semi-UKK but not UKK. Finally we give a remark concerning the uniformly alternative convexity or smoothness introduced by Kadets et al. [7].

2. Results

2.1. Some Implications

For a sequence $\{x_n\} \subset S_X$ and $\{f_n\} \subset S_{X^*}$ satisfying $f_n \in \nabla_{x_n}$ for all n, we let

$$\operatorname{sep}_{\{f_n\}}\{x_n\} = \inf\{\langle x_n - x_m, f_n\rangle : n < m\}. \tag{2.1}$$

It is clear that $\operatorname{sep}_{\{f_n\}}\{x_n\} \leq \operatorname{sep}\{x_n\}$.

Theorem 2.1. *A Banach space X is semi-UKK if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left.\begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightharpoonup x \\ \mathit{sep}_{\{f_n\}}\{x_n\} \geq \varepsilon, \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n}, \ \forall n \end{array}\right\} \Longrightarrow \|x\| \leq 1-\delta. \tag{2.2}$$

The following theorem shows that our new property is well placed.

Theorem 2.2. *The following implication diagram holds:*

$$\begin{array}{ccccc} UC & \Longrightarrow & UKK & \Longrightarrow & KK \\ \Downarrow & & \Downarrow & & \Downarrow \\ U\text{-space} & \Longrightarrow & \text{semi-UKK} & \Longrightarrow & \text{semi-KK}. \end{array} \tag{2.3}$$

Remark 2.3. The implication U -space $\Rightarrow$ semi-UKK strengthens the result of Vlasov. In fact, it was proved by Vlasov ([5, Theorem 7]) that every uniformly smooth Banach space is semi-KK and by Lau ([4, Corollary 2.5]) that every uniformly smooth Banach space is a U-space.

2.2. Sufficient Conditions for Weak Normal Structure

Recall that a Banach space X has *weak normal structure* (*normal structure*, resp.) if for every weakly compact (bounded and closed, resp.) convex subset C of X containing more than one point there exists a point $x_0 \in C$ such that $\sup\{\|x_0 - z\| : z \in C\} < diamC$ (see [8]). It is clear that normal structure and weak normal structure coincide whenever the space is reflexive. It was Kirk [9] who proved that if a Banach space X has weak normal structure, then every nonexpansive self-mapping defined on a weakly compact convex subset of X has a fixed point. In this subsection, we present a new and wider class of Banach spaces with weak normal structure.

Lemma 2.4 (Bollobás [10]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - \langle z, h\rangle < \varepsilon^2/4$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Theorem 2.5. *If a Banach space X has the following property:*

there are two constants $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that

$$\left.\begin{array}{l} \{x_n\} \subset S_X \\ x_n \rightharpoonup x \\ \langle x_n - x, f_n\rangle \geq \varepsilon, \text{ for some } \{f_n\} \subset S_{X^*} \\ \text{satisfying } f_n \in \nabla_{x_n}, \ \forall n \end{array}\right\} \Longrightarrow \|x\| \leq 1 - \delta, \tag{2.4}$$

then X has weak normal structure.

Proof. Suppose that X does not have weak normal structure. Then there exists a sequence $\{x_n\}$ in X such that the following properties are satisfied (see [11]):

(i) $\operatorname{diam}\{x_n\} = 1$;

(ii) $x_n \rightharpoonup 0$;

(iii) $\|x_n - x\| \to 1$ for all $x \in \overline{\operatorname{co}}\{x_n\}$.

In particular, since $0 \in \overline{\operatorname{co}}\{x_n\}$, we have $\|x_n\| \to 1$.

We now show that for each $0 < \varepsilon < 1$ and $0 < \delta < 1$, there are an element $z \in X$ and sequences $\{z_n\} \subset S_X$ and $\{f_n\} \subset S_{X^*}$ such that

(i) $z_n \rightharpoonup z$;

(ii) $\langle z_n - z, f_n\rangle \geq \varepsilon$ and $\langle z_n, f_n\rangle = 1$ for all n;

(iii) $\|z\| > 1 - \delta$.

To see this, let $0 < \delta < 1$ and $0 < \varepsilon < 1$ be given. We may assume that $\|x_1\| > 1 - \delta$. For each n, let $g_n \in \nabla_{x_n - (1/2)x_1}$. This implies $\langle x_n - (1/2)x_1, g_n\rangle = \|x_n - (1/2)x_1\| \to 1$. We observe that

$$\begin{aligned} 1 &= \left\langle x_n - \frac{1}{2}x_1, g_n \right\rangle \\ &= \frac{1}{2}\langle x_n - x_1, g_n\rangle + \frac{1}{2}\langle x_n, g_n\rangle \\ &\leq \frac{1}{2}\|x_n - x_1\| + \frac{1}{2}\|x_n\| \longrightarrow 1. \end{aligned} \tag{2.5}$$

In particular, $\langle x_n - x_1, g_n\rangle \to 1$ and $\langle x_n, g_n\rangle \to 1$.

By Lemma 2.4, there are sequences $\{z_n\} \subset S_X$ and $\{f_n\} \subset S_{X^*}$ such that

$$f_n \in \nabla_{z_n}, \quad \forall n, \quad \|z_n - (x_n - x_1)\| \longrightarrow 0, \qquad \|f_n - g_n\| \longrightarrow 0. \tag{2.6}$$

Put $z = -x_1$. It is clear that $z_n \rightharpoonup z$. Moreover, we have

$$\begin{aligned} &\left|\langle z_n - z, f_n\rangle - \langle x_n, g_n\rangle\right| \\ &\quad \le \left|\langle z_n - z, f_n\rangle - \langle z_n - z, g_n\rangle\right| + \left|\langle z_n - z, g_n\rangle - \langle x_n, g_n\rangle\right| \\ &\quad \le \|z_n - z\| \|f_n - g_n\| + \|z_n - (x_n - x_1)\| \|g_n\| \longrightarrow 0. \end{aligned} \tag{2.7}$$

Consequently, $\langle z_n - z, f_n\rangle \to 1 > \varepsilon$.

By discarding terms from the beginning of the sequence $\{z_n\}$, we obtain a contradiction with the assumption. This finishes the proof. □

Corollary 2.6. *A Banach space X has weak normal structure if X is semi-UKK.*

Corollary 2.7 (see [12]). *Every wUKK space has weak normal structure. Recall that a Banach space X is wUKK if there are two constants $\varepsilon > 0$ and $\delta > 0$ such that*

$$(wUKK): \left.\begin{array}{l} \{x_n\} \subset B_X \\ x_n \rightharpoonup x \\ sep\{x_n\} \ge \varepsilon \end{array}\right\} \Longrightarrow \|x\| \le 1 - \delta. \tag{2.8}$$

Corollary 2.8 (see [13]). *Every U-space has weak normal structure.*

2.3. Stability Results under Taking Finite Direct Sums

In this subsection, we give a necessary and sufficient condition for the direct sum of finitely many Banach spaces to be semi-KK and semi-UKK. Let us recall some definitions.

Let Z be a finite dimensional normed space $(\mathbb{R}^N, \|\cdot\|_z)$, which has a *monotone* norm; that is,

$$\|(a_1, \ldots, a_N)\|_z \le \|(b_1, \ldots, b_N)\|_z \tag{2.9}$$

if $0 \le a_i \le b_i$ for each $i = 1, \ldots, N$. We write $(X_1 \oplus \cdots \oplus X_N)_Z$ for the Z-direct sum of the Banach spaces $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \ldots, x_N)\| = \left\|(\|x_1\|_{X_1}, \ldots, \|x_N\|_{X_N})\right\|_Z, \tag{2.10}$$

where $x_i \in X_i$ for each $i = 1, \ldots, N$.

One should notice that in defining $(X_1 \oplus \cdots \oplus X_N)_Z$, we only need to know the behavior of the Z-norm on $\mathbb{R}^N_+$. Consequently, we can and do assume that the Z-norm is *absolute*; that is,

$$\|(a_1, \ldots, a_N)\|_Z = \|(|a_1|, \ldots, |a_N|)\|_Z, \quad \forall (a_1, \ldots, a_N) \in \mathbb{R}^N. \tag{2.11}$$

The following fact can be proved easily but plays an important role in this paper.

Lemma 2.9. *Suppose that* $X_1, \dots, X_N$ *are Banach spaces. Then each element* f *in the dual* $(X_1 \oplus \cdots \oplus X_N)_Z^*$ *of the Z-direct sum* $(X_1 \oplus \cdots \oplus X_N)_Z$ *is identified with the element* $(x_1^*, \dots, x_N^*)$ *in the* Z^**-direct sum* $(X_1^* \oplus \cdots \oplus X_N^*)_{Z^*}$ *such that*

$$\langle (x_1, \dots, x_n), f \rangle = \langle x_1, x_1^* \rangle + \cdots + \langle x_N, x_N^* \rangle \tag{2.12}$$

for all $(x_1, \dots, x_N) \in (X_1 \oplus \cdots \oplus X_N)_Z$*. Moreover,*

$$\|f\| = \left\| \left(\|x_1^*\|_{X_1^*}, \dots, \|x_N^*\|_{X_N^*} \right) \right\|_{Z^*}. \tag{2.13}$$

Recently, Dowling et al. [14] proved the following theorem.

Theorem 2.10. *Let* $X_1, \dots, X_N$ *be Banach spaces. Then* $(X_1 \oplus \cdots \oplus X_N)_Z$ *is KK (UKK, resp.) if and only if for each* $1 \le j \le N$*,*

(1) X_j *is KK (UKK, resp.), and*

(2) *either* X_j *is Schur or* Z *is strictly monotone in the* j*th coordinate.*

Recall that a Banach space X is a *Schur* space if weak and norm sequential convergences coincide in X, and Z is *strictly monotone in the* j*th coordinate* if

$$\|(a_1, \dots, a_n)\|_Z < \|(b_1, \dots, b_n)\|_Z, \tag{2.14}$$

where $0 \le a_i \le b_i$ for each $i = 1, \dots, n$ and $0 \le a_j < b_j$. Note that by the triangle inequality and the assumption that the Z-norm is absolute, Z is strictly monotone in the jth coordinate if and only if

$$\|(a_1, \dots, a_n)\|_Z < \|(b_1, \dots, b_n)\|_Z, \tag{2.15}$$

where $0 \le a_i = b_i$ for each $i \ne j$ and $0 = a_j < b_j$.

We first define a generalization of Schur spaces.

Definition 2.11. A Banach space X is a *semi-Schur* space if

$$\text{(semi-Schur)}: \left. \begin{array}{l} \{x_n\} \subset X \\ x_n \rightharpoonup x \\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n}, \quad \forall n \end{array} \right\} \Longrightarrow \|x_n\| - \langle x, f_n \rangle \longrightarrow 0. \tag{2.16}$$

The following two propositions follow easily from the definition of semi-Schur spaces and semi-KK spaces.

Proposition 2.12. *A Banach space X is semi-Schur if and only if*

$$\left.\begin{array}{l}\{x_n\} \subset S_X\\ x_n \rightharpoonup x\\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n},\ \forall n\end{array}\right\} \Longrightarrow \|x_n\| - \langle x, f_n\rangle \longrightarrow 0. \tag{2.17}$$

Proposition 2.13. *A Banach space X satisfies semi-KK property if and only if*

$$\left.\begin{array}{l}\{x_n\} \subset X\\ x_n \rightharpoonup x\\ \|x_n\| \longrightarrow \|x\|\\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n},\ \forall n\end{array}\right\} \Longrightarrow \langle x, f_n\rangle \longrightarrow \|x\|. \tag{2.18}$$

We say that Z has *property (S-j)* where $1 \le j \le N$ if whenever $a_1, \dots, a_N, b_1, \dots, b_N \ge 0$ and $0 \le a_j' < a_j$ satisfy

$$\begin{aligned}&\Big\|(a_1, \dots, \overset{j\text{th}}{a_j'}, \dots, a_N)\Big\|_Z = \|(a_1, \dots, a_j, \dots, a_N)\|_Z = 1,\\ &\|(b_1, \dots, b_j, \dots, b_N)\|_{Z^*} = a_1b_1 + \cdots + a_jb_j + \cdots + a_Nb_N = 1,\end{aligned} \tag{2.19}$$

it follows that $b_j = 0$.

Theorem 2.14. *Suppose that* $X_1, \dots, X_N$ *are Banach spaces. Then the direct sum* $(X_1 \oplus \cdots \oplus X_N)_Z$ *is semi-KK if and only if for each* $j = 1, \dots, N$

(a) X_j *is semi-KK and*

(b) *either* X_j *is semi-Schur or* Z *has property (S-j).*

Proof. Sufficiency. Suppose that there exists $j \in \{1, \dots, N\}$ such that X_j is not semi-Schur and Z does not have property (S-j). For convenience, we may assume that $j = 1$. Since X_1 is not semi-Schur, there exist sequences $\{x_n^1\} \subset S_{X_1}$, $\{f_n^1\} \subset S_{X_1^*}$, and a number $\varepsilon_0 > 0$ such that

$$x_n^1 \rightharpoonup x^1 \in B_{X_1}, \quad f_n^1 \in \nabla_{x_n^1}, \quad \left\langle x^1, f_n^1 \right\rangle \le 1 - \varepsilon_0, \quad \forall n. \tag{2.20}$$

Since Z does not have property (S-1), there exist numbers $a_1, \dots, a_N, b_1, \dots, b_N \ge 0$ such that the following properties are satisfied:

(i) $\|(0, a_2, \dots, a_N)\|_Z = \|(a_1, a_2, \dots, a_N)\|_Z = 1$;

(ii) $\|(b_1, \dots, b_N)\|_{Z^*} = a_1b_1 + \cdots + a_Nb_N = 1$;

(iii) $a_1 > 0$ and $b_1 > 0$.

For $j = 2, \ldots, N$, let $x^j \in S_{X_j}$ and $f^j \in \nabla_{x^j}$. Now we put

$$\mathbf{x} = \left(a_1 x^1, a_2 x^2, \ldots, a_N x^N\right),$$
$$\mathbf{x}_n = \left(a_1 x_n^1, a_2 x^2, \ldots, a_N x^N\right), \tag{2.21}$$
$$\mathbf{f}_n = \left(b_1 f_n^1, b_2 f^2, \ldots, b_N f^N\right).$$

It is clear that $\{\mathbf{x}_n\}$ is a sequence of norm one elements converging weakly to $\mathbf{x}$ and $\mathbf{f}_n \in \nabla_{\mathbf{x}_n}$ for all n. Moreover, by the monotonicity of $\|\cdot\|_Z$, we have $\|\mathbf{x}\| = 1$. Finally, we show that $\langle \mathbf{x}, \mathbf{f}_n \rangle \nrightarrow 1$. To see this, we consider

$$\begin{aligned}\langle \mathbf{x}, \mathbf{f}_n \rangle &= a_1 b_1 \left\langle x^1, f_n^1 \right\rangle + a_2 b_2 \left\langle x^2, f^2 \right\rangle + \cdots + a_N b_N \langle x^N, f^N \rangle \\ &\le (1 - \varepsilon_0) a_1 b_1 + a_2 b_2 + \cdots + a_N b_N.\end{aligned} \tag{2.22}$$

It then follows from $a_1 > 0$ and $b_1 > 0$ that $(1 - \varepsilon_0) a_1 b_1 + \cdots + a_N b_N < 1$. This shows that $(X_1 \oplus \cdots \oplus X_N)_Z$ is not semi-KK and hence the first half of the proof is done.

Necessity. Suppose that the conditions (a) and (b) hold. Put

$$A = \{j : X_j \text{ is a semi-Schurs pace}\}, \qquad B = \{j : Z \text{ has property (S-}j)\}. \tag{2.23}$$

Then, by (b), $A \cup B = \{1, \ldots, N\}$. Let $\{\mathbf{x}_n\}$ be a sequence of norm one elements in $(X_1 \oplus \cdots \oplus X_N)_Z$ converging weakly to a norm one element $\mathbf{x} \in (X_1 \oplus \cdots \oplus X_N)_Z$ and $\{\mathbf{f}_n\}$ a sequence of norm one elements in $((X_1 \oplus \cdots \oplus X_N)_Z)^* = (X_1^* \oplus \cdots \oplus X_N^*)_{Z^*}$ such that $f_n \in \nabla_{x_n}$ for all n. For convenience, let us write

$$\mathbf{x} = \left(x^1, \ldots, x^N\right), \qquad \mathbf{x}_n = \left(x_n^1, \ldots, x_n^N\right), \qquad \mathbf{f}_n = \left(f_n^1, \ldots, f_n^N\right), \tag{2.24}$$

where $x^j, x_n^j \in X_j$ and $f_n^j \in X_j^*$ for all j and n. We prove that

$$\left\langle x^1, f_n^1 \right\rangle + \cdots + \langle x^N, f_n^N \rangle \longrightarrow 1. \tag{2.25}$$

Notice that

$$\begin{aligned}1 &= \left\| \left(x_n^1, \ldots, x_n^N\right) \right\| = \langle \mathbf{x}_n, \mathbf{f}_n \rangle = \langle x_n^1, f_n^1 \rangle + \cdots + \langle x_n^N, f_n^N \rangle \\ &\le \left\|x_n^1\right\| \left\|f_n^1\right\| + \cdots + \left\|x_n^N\right\| \left\|f_n^N\right\| \le \left\| \left(x_n^1, \ldots, x_n^N\right) \right\| \left\| \left(f_n^1, \ldots, f_n^N\right) \right\|^* = 1.\end{aligned} \tag{2.26}$$

Then $\langle x_n^j, f_n^j \rangle = \|x_n^j\| \|f_n^j\|$ and

$$\left\|x_n^1\right\| \left\|f_n^1\right\| + \cdots + \left\|x_n^N\right\| \left\|f_n^N\right\| = 1 \tag{2.27}$$

for all n. In order to show that (2.25) holds, it suffices to show that

$$\langle x^j, f_n^j \rangle - \left\| x_n^j \right\| \left\| f_n^j \right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty \tag{2.28}$$

for all $j = 1, \ldots, N$.

Let us note the following facts:

(i) $f_n^j / \| f_n^j \| \in \nabla_{x_n^j}$ provided that $f_n^j \neq 0$;

(ii) $x_n^j \rightharpoonup x^j$ as $n \to \infty$ for all j.

We first prove that (2.28) holds for all $j \in A$. To see this, we note that if $f_n^j = 0$, then $\langle x^j, f_n^j \rangle = \| x_n^j \| \| f_n^j \| = 0$. Now, we assume that $f_n^j \neq 0$ for all n. It follows then that $f_n^j / \| f_n^j \| \in \nabla_{x_n^j}$ and hence from the semi-Schur property of X_j that $\langle x^j, f_n^j / \| f_n^j \| \rangle - \| x_n^j \| \to 0$.

Passing to a subsequence, we may assume that the following limits:

$$\lim_k \left\| x_{n_k}^j \right\|, \quad \lim_k \left\| f_{n_k}^j \right\| \text{ exist.} \tag{2.29}$$

Notice that

$$\begin{gathered} \left\| \left(\lim_k \left\| x_{n_k}^1 \right\|, \ldots, \lim_k \left\| x_{n_k}^N \right\| \right) \right\|_Z = \left\| \left(\lim_k \left\| f_{n_k}^1 \right\|, \ldots, \lim_k \left\| f_{n_k}^N \right\| \right) \right\|_{Z^*} = 1, \\ \lim_k \left\| x_{n_k}^1 \right\| \lim_k \left\| f_{n_k}^1 \right\| + \cdots + \lim_k \left\| x_{n_k}^N \right\| \lim_k \left\| f_{n_k}^N \right\| = 1. \end{gathered} \tag{2.30}$$

We next show that (2.28) holds for all $j \in B$. Let us split the proof into two cases.

Case 1. There exists $j \in B$ such that

$$\left\| x^j \right\| < \lim_k \left\| x_{n_k}^j \right\|. \tag{2.31}$$

In this case, it follows from the property (S-j) that

$$\lim_k \left\| f_{n_k}^j \right\| = 0. \tag{2.32}$$

This implies that $\langle x^j, f_{n_k}^j \rangle - \| x_{n_k}^j \| \| f_{n_k}^j \| \to 0$.

Case 2. $\| x^j \| = \lim_k \| x_{n_k}^j \|$ for all $j \in B$. Again, if $f_{n_k}^j = 0$, then $\langle x^j, f_n^j \rangle = \| x^j \| \| f_n^j \| = 0$. Now we may assume that $f_{n_k}^j \neq 0$ for all k. This implies that $f_{n_k}^j / \| f_{n_k}^j \| \in \nabla_{x_{n_k}^j}$ and hence it follows from the semi-KK property of X_j that $\langle x^j, f_{n_k}^j / \| f_{n_k}^j \| \rangle - \| x^j \| \to 0$. In particular, $\langle x^j, f_{n_k}^j \rangle - \| x_{n_k}^j \| \| f_{n_k}^j \| \to 0$.

From both cases, we have proved that every subsequence of the sequence $\{ \langle \mathbf{x}, \mathbf{f}_n \rangle \}$ has a further subsequence $\{ \langle \mathbf{x}, \mathbf{f}_{n_k} \rangle \}$ such that $\langle \mathbf{x}, \mathbf{f}_{n_k} \rangle \to 1$. Hence $\langle \mathbf{x}, \mathbf{f}_n \rangle \to 1$, as desired. □

Using the proof of the preceding theorem and the fact that the property (S-j) is a uniform property, we obtain the following result.

Theorem 2.15. *Suppose that $X_1, \ldots, X_N$ are Banach spaces. Then $(X_1 \oplus \cdots \oplus X_N)_Z$ is semi-UKK if and only if for each $j = 1, \ldots, N$,*

(a) *X_j is semi-UKK and*

(b) *either X_j is semi-Schur or Z has property (S-j).*

Finally, we use the characterization above and Theorem 2.10 to construct a Banach space which is semi-UKK but not UKK.

Example 2.16 (A Banach space which is semi-UKK but not UKK). Let Z be a two-dimensional space $\mathbb{R}^2$ equipped with the norm

$$\left|(\alpha,\beta)\right| = \begin{cases} |\beta| & \text{if } |\alpha| \le |\beta|, \\ \dfrac{\alpha^2+\beta^2}{2|\alpha|}, & \text{if } |\alpha| > |\beta|. \end{cases} \tag{2.33}$$

It follows that Z is a (uniformly) smooth space, and its unit sphere consists of

(i) two half unit circles: the first one is a right half centered at $(1,0)$ and the second is a left half centered at $(-1,0)$

(ii) two horizontal line segments joining the points $(-1,1)$ and $(1,1)$ and the points $(-1,-1)$ and $(1,-1)$, respectively.

Furthermore, Z has properties (S-1) and (S-2) but is not strictly monotone in the first-coordinate. Let $X = (\ell_2 \oplus \mathbb{R})_Z$. Then X is semi-UKK but not UKK. The latter follows since Z is not strictly monotone in the first coordinate and ℓ_2 does not have the Schur property.

3. U-Spaces and Uniformly Alternatively Convex or Smooth Spaces

In this section, we discuss some properties of uniformly alternatively convex or smooth spaces which was introduced by Kadets et al. [7].

Definition 3.1. A Banach space X is *uniformly alternatively convex or smooth* if

$$\text{(UACS)}: \left.\begin{array}{l} \{x_n\},\{y_n\} \subset S_X \\ \|x_n + y_n\| \longrightarrow 2 \\ \langle x_n, f_n\rangle \longrightarrow 1 \text{ for some } \{f_n\} \subset S_{X^*} \end{array}\right\} \Longrightarrow \langle y_n, f_n\rangle \longrightarrow 1. \tag{3.1}$$

Remark 3.2. It is not hard to see that X is UACS if and only if

$$\left.\begin{array}{l}\{x_n\}, \{y_n\} \subset S_X \\ \|x_n + y_n\| \longrightarrow 2 \\ \{f_n\} \subset S_{X^*} \text{ satisfying } f_n \in \nabla_{x_n}, \ \forall n\end{array}\right\} \Longrightarrow \langle y_n, f_n\rangle \longrightarrow 1. \tag{3.2}$$

Consequently, X is UACS if and only if it is a U-space.

It is proved by Gao and Lau ([13, Theorem 4.4]) that every UACS space has uniform normal structure which is the same result of Theorem 3.1 of Sirotkin [15]. In fact, this result is recently strengthened by Saejung in [16]. Recall that a Banach space X has *uniform normal structure* if there exists a constant $0 < c < 1$ such that for every bounded closed convex subset C of X containing more than one point there exists a point $x_0 \in C$ such that $\sup\{\|x_0 - z\| : z \in C\} < c \cdot \operatorname{diam} C$.

Moreover, it was Lau ([4, Theorem 2.4]) who proved that X is UACS if and only if its dual space X^* is UACS. By Sirotkin's result ([15, Theorem 2.3]), we have the following theorem.

Theorem 3.3. *Let (S, Σ, μ) be a complete measure space and X be a Banach space. Then the following statements are equivalent:*

(i) *$L^p(\mu, X)$ is UACS for some (and hence all) $1 < p < \infty$;*

(ii) *$L^p(\mu, X^*)$ is UACS for some (and hence all) $1 < p < \infty$;*

(iii) *X is UACS;*

(iv) *X^* is UACS.*

In particular, if X is UACS, then both $L^p(\mu, X)$ and $L^p(\mu, X^*)$ have uniform normal structure.

Recall that $L^p(\mu, X)$, where $1 < p < \infty$, is the Lebesgue-Bochner function space of μ-equivalence classes of strongly measurable functions $f : S \to X$ with $\int_S \|f(t)\|^p d\mu < \infty$, endowed with the norm $\|f\| = \left(\int_S \|f(t)\|^p d\mu\right)^{1/p}$ (for more detail, see [17, 18]).

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