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Research Article

Weighted Iterated Radial Composition Operators between Some Spaces of Holomorphic Functions on the Unit Ball

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The boundedness and compactness of weighted iterated radial composition operators from the mixed-norm space to the weighted-type space and the little weighted-type space on the unit ball are characterized here. We also calculate the Hilbert-Schmidt norm of the operator on the weighted Bergman-Hilbert space as well as on the Hardy H^2 space.

1. Introduction

Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$, $\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k$, and $|z| = \sqrt{\langle z, z\rangle}$. Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $\partial\mathbb{B}$ its boundary, and $H(\mathbb{B})$ the class of all holomorphic functions on $\mathbb{B}$.

For an $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let

$$\mathfrak{R}f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta \tag{1.1}$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$ [1]. It is easy to see that

$$\mathfrak{R}f(z) = \langle \nabla f(z), \overline{z}\rangle, \tag{1.2}$$

where ∇f is the complex gradient of function f.

The iterated radial derivative operator $\mathfrak{R}^m f$ is defined inductively by

$$\mathfrak{R}^m f = \mathfrak{R}\left(\mathfrak{R}^{m-1} f\right), \quad m \in \mathbb{N} \setminus \{1\}. \tag{1.3}$$

A positive continuous function v on the interval $[0,1)$ is called normal [2] if there are $\delta \in [0,1)$ and τ and t, $0 < \tau < t$ such that

$$\begin{aligned} &\frac{v(r)}{(1-r)^{\tau}} \text{ is decreasing on } [\delta,1), \quad \lim_{r\to 1}\frac{v(r)}{(1-r)^{\tau}} = 0, \\ &\frac{v(r)}{(1-r)^{t}} \text{ is increasing on } [\delta,1), \quad \lim_{r\to 1}\frac{v(r)}{(1-r)^{t}} = \infty. \end{aligned} \tag{1.4}$$

If we say that a function $v : \mathbb{B} \to [0,\infty)$ is normal, we also assume that it is radial, that is, $v(z) = v(|z|)$, $z \in \mathbb{B}$.

Strictly positive continuous functions on $\mathbb{B}$ are called *weights*.

The weighted-type space $H_\mu^\infty(\mathbb{B}) = H_\mu^\infty$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_\mu^\infty} := \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < \infty, \tag{1.5}$$

where μ is a weight (see, e.g., [3, 4] as well as [5] for a related class of spaces).

The little weighted-type space $H_{\mu,0}^\infty(\mathbb{B}) = H_{\mu,0}^\infty$ is a subspace of H_μ^∞ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0. \tag{1.6}$$

For $0 < p, q < \infty$, and ϕ normal, the mixed-norm space $H(p,q,\phi)(\mathbb{B}) = H(p,q,\phi)$ consists of all functions $f \in H(\mathbb{B})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{1/p} < \infty, \tag{1.7}$$

where

$$M_q(f,r) = \left(\int_{\partial\mathbb{B}} |f(r\zeta)|^q d\sigma(\zeta)\right)^{1/q}, \tag{1.8}$$

and $d\sigma$ is the normalized surface measure on $\partial\mathbb{B}$. For $p = q$, $\phi(r) = (1-r^2)^{(\alpha+1)/p}$, and $\alpha > -1$, the space is equivalent with the weighted Bergman space $A_\alpha^p(\mathbb{B}) = A_\alpha^p$, which is defined as the class of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{A_\alpha^p}^p := \int_{\mathbb{B}} |f(z)|^p \left(1-|z|^2\right)^\alpha dV(z) < \infty, \tag{1.9}$$

where $dV(z)$ is the Lebesgue volume measure on $\mathbb{B}$. Some facts on mixed-norm spaces in various domains in $\mathbb{C}^n$ can be found, for example, in [6–8] (see also the references therein).

For $0 < p < \infty$ the Hardy space $H^p(\mathbb{B}) = H^p$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_{\partial\mathbb{B}} |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} < \infty. \tag{1.10}$$

For $p = 2$ the Hardy and the weighted Bergman space are Hilbert.

Let φ be a holomorphic self-map of $\mathbb{B}$, $u \in H(\mathbb{B})$, and $m \in \mathbb{N}_0$. For $f \in H(\mathbb{B})$, the *weighted iterated radial composition operator* is defined by

$$\mathfrak{R}^m_{u,\varphi}(f)(z) = u(z)\mathfrak{R}^m f(\varphi(z)), \quad z \in \mathbb{B}. \tag{1.11}$$

Note that the operator is the composition of the multiplication, composition and the iterated radial operator, that is

$$\mathfrak{R}^m_{u,\varphi} = M_u \circ C_\varphi \circ \mathfrak{R}^m. \tag{1.12}$$

This is one of the product operators suggested by this author to be investigated at numerous talks (e.g., in [9]). Note that for $m = 0$ the operator $\mathfrak{R}^m_{u,\varphi}$ becomes the weighted composition operator (see, e.g., [4, 8, 10]). It is of interest to provide function-theoretic characterizations for when φ and u induce bounded or compact weighted iterated radial composition operators on spaces of holomorphic functions. Studying products of some concrete linear operators on spaces of analytic functions attracted recently some attention see, for example, [11–32] as well as the related references therein. Some operators on mixed-norm spaces have been studied, for example, in [8, 10, 11, 16, 25, 26, 29, 33] (see also the references therein).

Here we study the boundedness and compactness of weighted iterated radial composition operators from mixed-norm spaces to weighted-type spaces on the unit ball for the case $m \in \mathbb{N}$. We also calculate the Hilbert-Schmidt norm of the operator on the weighted Bergman-Hilbert space $A^2_\alpha(\mathbb{B})$ as well as on the Hardy $H^2(\mathbb{B})$ space.

In this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary Results

In this section we quote several lemmas which are used in the proofs of the main results.

The next characterization of compactness is proved in a standard way, hence we omit its proof (see, e.g., [34]).

Lemma 2.1. *Assume $p, q > 0$, φ is a holomorphic self-map of $\mathbb{B}$, $u \in H(\mathbb{B})$, ϕ is normal and μ is a weight. Then the operator $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is compact if and only if for every bounded sequence $(f_k)_{k\in\mathbb{N}} \subset H(p,q,\phi)$ converging to 0 uniformly on compacts of $\mathbb{B}$ as $k \to \infty$, one has*

$$\lim_{k\to\infty} \left\| \mathfrak{R}^m_{u,\varphi} f_k \right\|_{H^\infty_\mu} = 0. \tag{2.1}$$

The following lemma is a slight modification of Lemma 2.5 in [8] and is proved similar to Lemma 1 in [35].

Lemma 2.2. *Assume μ is a normal weight. Then a closed set K in $H^{\infty}_{\mu,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)\left|f(z)\right| = 0. \tag{2.2}$$

The following lemma is folklore and in the next form it can be found in [36].

Lemma 2.3. *Assume that $0 < p, q < \infty$, ϕ is normal, and $m \in \mathbb{N}$. Then for every $f \in H(\mathbb{B})$ the following asymptotic relationship holds:*

$$\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr \asymp \left|f(0)\right|^p + \int_0^1 M_q^p(\mathfrak{R}^m f,r)(1-r)^{mp}\frac{\phi^p(r)}{1-r}dr. \tag{2.3}$$

Lemma 2.4. *Assume that $m \in \mathbb{N}$, $0 < p, q < \infty$, ϕ is normal and $f \in H(p,q,\phi)$. Then, there is a positive constant C independent of f such that*

$$\left|\mathfrak{R}^m f(z)\right| \le C\left\|f\right\|_{H(p,q,\phi)}\frac{|z|}{\phi(|z|)\left(1-|z|^2\right)^{n/q+m}}. \tag{2.4}$$

Proof. Let $g = \mathfrak{R}^{m-1}f$ and $z \in \mathbb{B}$. By the definition of the radial derivative, the Cauchy-Schwarz inequality and the Chauchy inequality, we have that

$$\left|\mathfrak{R}g(z)\right| \le |z|\left|\nabla g(z)\right| \le C|z|\frac{\sup_{B(z,1-|z|)/4)}\left|g(w)\right|}{1-|z|}. \tag{2.5}$$

From (2.3) with $m \to m-1$ we easily obtain the following inequality (see, e.g., [8, Lemma 2.1]):

$$\left|g(z)\right| \le C\frac{\left\|f\right\|_{H(p,q,\phi)}}{\phi(|z|)\left(1-|z|^2\right)^{n/q+m-1}}. \tag{2.6}$$

From (2.5) and (2.6) and the asymptotic relations

$$1-|w| \asymp 1-|z|, \quad \phi(|z|) \asymp \phi(|w|), \quad \text{for } w \in B\left(z, \frac{1-|z|}{2}\right), \tag{2.7}$$

inequality (2.4) follows. □

Lemma 2.5. *Let*

$$f_{a,s}(z) = \frac{1}{(1-\langle z,a\rangle)^s}, \quad z \in \mathbb{B}. \tag{2.8}$$

Then,

$$\mathfrak{R}^m f_{a,s}(z) = s\frac{P_m(\langle z,a\rangle)}{(1-\langle z,a\rangle)^{s+m}}, \tag{2.9}$$

where

$$P_m(w) = s^{m-1}w^m + p_{m-1}^{(m)}(s)w^{m-1} + \cdots + p_2^{(m)}(s)w^2 + w, \tag{2.10}$$

and where $p_j^{(m)}(s)$, $j = 2,\ldots,m-1$ *are nonnegative polynomials for* $s > 0$.

Proof. We prove the lemma by induction. For $m = 1$,

$$\mathfrak{R} f_{a,s}(z) = s\frac{\langle z,a\rangle}{(1-\langle z,a\rangle)^{s+1}}, \quad z \in \mathbb{B}, \tag{2.11}$$

which is formula (2.9) with $P_1(w) = w$.

Assume (2.9) is true for every $m \in \{1,\ldots,l\}$. Taking the radial derivative operator on equality (2.9) with $m = l$, we obtain

$$\begin{aligned}
\mathfrak{R}^{l+1} f_{a,s}(z) &= s\mathfrak{R}\frac{s^{l-1}\langle z,a\rangle^l + p_{l-1}^{(l)}(s)\langle z,a\rangle^{l-1} + \cdots + p_2^{(l)}(s)\langle z,a\rangle^2 + \langle z,a\rangle}{(1-\langle z,a\rangle)^{s+l}} \\
&= s\frac{(s+l)\left(s^{l-1}\langle z,a\rangle^{l+1} + p_{l-1}^{(l)}(s)\langle z,a\rangle^l + \cdots + p_2^{(l)}(s)\langle z,a\rangle^3 + \langle z,a\rangle^2\right)}{(1-\langle z,a\rangle)^{s+l+1}} \\
&\quad + s\frac{\left(ls^{l-1}\langle z,a\rangle^l + (l-1)p_{l-1}^{(l)}(s)\langle z,a\rangle^{l-1} + \cdots + 2p_2^{(l)}(s)\langle z,a\rangle^2 + \langle z,a\rangle\right)(1-\langle z,a\rangle)}{(1-\langle z,a\rangle)^{s+l+1}}, \\
&= s\frac{s^l\langle z,a\rangle^{l+1} + \cdots + \left[(s+l-2)p_2^{(l)}(s) + 3p_3^{(l)}(s)\right]\langle z,a\rangle^3}{(1-\langle z,a\rangle)^{s+l+1}} \\
&\quad + \frac{\left[(s+l-1) + 2p_2^{(l)}(s)\right]\langle z,a\rangle^2 + \langle z,a\rangle}{(1-\langle z,a\rangle)^{s+l+1}}
\end{aligned} \tag{2.12}$$

from which the inductive proof easily follows. □

3. Boundedness and Compactness of $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ (or $H_{\mu,0}^\infty$)

This section characterizes the boundedness and compactness of $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ (or $H_{\mu,0}^\infty$).

Theorem 3.1. *Assume $m \in \mathbb{N}$, $0 < p, q < \infty$, ϕ is normal, μ is a weight, φ is a holomorphic self-map of $\mathbb{B}$, and $u \in H(\mathbb{B})$. Then $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is bounded if and only if*

$$M_1 := \sup_{z\in\mathbb{B}} \frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}} < \infty. \tag{3.1}$$

Moreover, if $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is bounded, then the following asymptotic relationship holds

$$\left\|\mathfrak{R}_{u,\varphi}^m\right\|_{H(p,q,\phi)\to H_\mu^\infty} \asymp M_1. \tag{3.2}$$

Proof. Assume (3.1) holds. Then by Lemma 2.4 for each $f \in H(p,q,\phi)$, we have that

$$\mu(z)\left|\mathfrak{R}_{u,\varphi}^m f(z)\right| \le C\|f\|_{H(p,q,\phi)} \frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}}. \tag{3.3}$$

Taking the supremum over the unit ball in (3.3) and using (3.1) the boundedness of operator $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ follows and

$$\left\|\mathfrak{R}_{u,\varphi}^m\right\|_{H(p,q,\phi)\to H_\mu^\infty} \le CM_1. \tag{3.4}$$

Now assume that operator $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is bounded. By using the test functions

$$f_j(z) = z_j \in H(p,q,\phi), \quad j = 1,\ldots,n, \tag{3.5}$$

we obtain

$$\mathfrak{R}_{u,\varphi}^m f_j \in H_\mu^\infty, \quad j = 1,\ldots,n, \tag{3.6}$$

that is, for each $j = 1,\ldots,n$, holds

$$\left\|\mathfrak{R}_{u,\varphi}^m f_j\right\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)|u(z)||\varphi_j(z)| \le \|z_j\|_{H(p,q,\phi)} \left\|\mathfrak{R}_{u,\varphi}^m\right\|_{H(p,q,\phi)\to H_\mu^\infty} < \infty, \tag{3.7}$$

which implies that

$$\begin{aligned}\sup_{z\in\mathbb{B}}\mu(z)|u(z)||\varphi(z)| &\le \sum_{j=1}^{n}\sup_{z\in\mathbb{B}}\mu(z)|u(z)||\varphi_j(z)| \\ &\le \left\|\mathfrak{R}_{u,\varphi}^{m}\right\|_{H(p,q,\phi)\to H_\mu^\infty}\sum_{j=1}^{n}\|z_j\|_{H(p,q,\phi)} < \infty.\end{aligned} \tag{3.8}$$

Let

$$f_w(z) = \frac{\left(1-|w|^2\right)^{t+1}}{\phi(w)(1-\langle z,w\rangle)^{n/q+t+1}}. \tag{3.9}$$

It is known that $L_1 := \sup_{w\in\mathbb{B}}\|f_w\|_{H(p,q,\phi)} < \infty$ (see [8, Theorem 3.3]). From this, using the boundedness of $\mathfrak{R}_{u,\varphi}^{m} : H(p,q,\phi) \to H_\mu^\infty$ and by Lemma 2.5, we have that for each $a\in\mathbb{B}$

$$\begin{aligned}L_1\left\|\mathfrak{R}_{u,\varphi}^{m}\right\|_{H(p,q,\phi)\to H_\mu^\infty} &\ge \left\|\mathfrak{R}_{u,\varphi}^{m}(f_{\varphi(a)})\right\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}}\mu(z)|u(z)|\left|\mathfrak{R}^m f_{\varphi(a)}(\varphi(z))\right| \\ &\ge \left(\frac{n}{q}+t+1\right)\frac{\mu(a)|u(a)|P_m\left(|\varphi(a)|^2\right)}{\phi(|\varphi(a)|)\left(1-|\varphi(a)|^2\right)^{n/q+m}} \\ &\ge \left(\frac{n}{q}+t+1\right)\frac{\mu(a)|u(a)||\varphi(a)|^2}{\phi(|\varphi(a)|)\left(1-|\varphi(a)|^2\right)^{n/q+m}}.\end{aligned} \tag{3.10}$$

From (3.10), we have that

$$\begin{aligned}\infty &> \sup_{|\varphi(z)|>1/2}\frac{\mu(z)|u(z)||\varphi(z)|^2}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}} \\ &\ge \sup_{|\varphi(z)|>1/2}\frac{\mu(z)|u(z)||\varphi(z)|}{2\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}}.\end{aligned} \tag{3.11}$$

On the other hand, from (3.8) and since ϕ is normal, we obtain

$$\sup_{|\varphi(z)|\le 1/2}\frac{\mu(\varphi(z))|u(z)||\varphi(z)|^2}{\phi(\varphi(z))\left(1-|\varphi(z)|^2\right)^{n/q+m}} \le C\sup_{z\in\mathbb{B}}\mu(\varphi(z))|u(z)||\varphi(z)| < \infty. \tag{3.12}$$

From (3.8), (3.10), (3.11), and (3.12) condition (3.1) follows, and moreover

$$M_1 \leq C\left\|\mathfrak{R}_{u,\varphi}^m\right\|_{H(p,q,\phi)\to H_\mu^\infty}. \tag{3.13}$$

From (3.4) and (3.13) asymptotic relationship (3.2) follows, finishing the proof of the theorem. □

Theorem 3.2. *Assume $m \in \mathbb{N}$, $0 < p, q < \infty$, ϕ is normal, μ is a weight, φ is a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. Then $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is compact if and only if $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}} = 0. \tag{3.14}$$

Proof. Suppose that $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is compact. Then it is clear that $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is bounded. If $\|\varphi\|_\infty < 1$, then (3.14) is vacuously satisfied. Hence assume that $\|\varphi\|_\infty = 1$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$, and $f_k(z) = f_{\varphi(z_k)}(z), k \in \mathbb{N}$, where f_w is defined in (3.9). Then $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,q,\phi)} < \infty$, $f_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to \infty$ since

$$\lim_{k\to\infty} \frac{\left(1-|\varphi(z_k)|^2\right)^{t+1}}{\phi(|\varphi(z_k)|)} = 0, \tag{3.15}$$

so that

$$\lim_{k\to\infty} \left\|\mathfrak{R}_{u,\varphi}^m f_k\right\|_{H_\mu^\infty} = 0. \tag{3.16}$$

On the other hand, by (3.10), we have

$$\frac{\mu(z_k)|u(z_k)||\varphi(z_k)|^2}{\phi(|\varphi(z_k)|)\left(1-|\varphi(z_k)|^2\right)^{n/q+m}} \leq C\left\|\mathfrak{R}_{u,\varphi}^m f_k\right\|_{H_\mu^\infty}. \tag{3.17}$$

From (3.16) and (3.17), equality (3.14) easily follows.

Conversely, assume that $\mathfrak{R}_{u,\varphi}^m : H(p,q,\phi) \to H_\mu^\infty$ is bounded and (3.14) holds. From the proof of Theorem 3.1 we know that (3.1) holds. On the other hand, from (3.14), we have that, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}} < \varepsilon \tag{3.18}$$

whenever $\delta < |\varphi(z)| < 1$.

Assume $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H(p,q,\phi)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,q,\phi)} \le L$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. Let $K = \{z \in \mathbb{B} : |\varphi(z)| \le \delta\}$. Then from (3.18), and by Lemma 2.4, it follows that

$$\begin{aligned}
\left\|\mathfrak{R}^m_{u,\varphi} f_k\right\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{B}} \mu(z)|u(z)|\left|\mathfrak{R}^m f_k(\varphi(z))\right| \\
&\le \sup_{z\in K} \mu(z)|u(z)|\left|\mathfrak{R}^m f_k(\varphi(z))\right| + \sup_{z\in\mathbb{B}\setminus K} \mu(z)|u(z)|\left|\mathfrak{R}^m f_k(\varphi(z))\right| \\
&\le \sup_{z\in K} \mu(z)|u(z)|\left|\varphi(z)\right|\left|\nabla\mathfrak{R}^{m-1} f_k(\varphi(z))\right| \\
&\quad + C\|f_k\|_{H(p,q,\phi)} \sup_{z\in\mathbb{B}\setminus K} \frac{\mu(z)|u(z)|\left|\varphi(z)\right|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}} \\
&\le K_2 C \sup_{|\zeta|\le\delta}\left|\nabla\mathfrak{R}^{m-1} f_k(\zeta)\right| + C\varepsilon\|f_k\|_{H(p,q,\phi)},
\end{aligned} \tag{3.19}$$

where $K_2 := \sup_{z\in\mathbb{B}}\mu(z)|u(z)||\varphi(z)|$ (see (3.8)). Therefore

$$\left\|\mathfrak{R}^m_{u,\varphi} f_k\right\|_{H^\infty_\mu} \le K_2 \sup_{|\zeta|\le\delta}\left|\nabla\mathfrak{R}^{m-1} f_k(\zeta)\right| + CL\varepsilon. \tag{3.20}$$

Since $(f_k)_{k\in\mathbb{N}}$ converges to zero on compact subsets of $\mathbb{B}$ as $k \to \infty$, by Cauchy's estimates it follows that the sequence $(|\nabla\mathfrak{R}^{m-1} f_k|)_{k\in\mathbb{N}}$ also converges to zero on compact subsets of $\mathbb{B}$ as $k \to \infty$, in particular

$$\lim_{k\to\infty} \sup_{|\zeta|\le\delta}\left|\nabla\mathfrak{R}^{m-1} f_k(\zeta)\right| = 0. \tag{3.21}$$

Using these facts and letting $k \to \infty$ in (3.20), we obtain that

$$\limsup_{k\to\infty}\left\|\mathfrak{R}^m_{u,\varphi} f_k\right\|_{H^\infty_\mu} \le CL\varepsilon\ . \tag{3.22}$$

Since ε is an arbitrary positive number it follows that the last limit is equal to zero. Applying Lemma 2.1, the implication follows. □

Theorem 3.3. *Assume $m \in \mathbb{N}$, $0 < p, q < \infty$, ϕ, μ are normal, φ is a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. Then $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_{\mu,0}$ is bounded if and only if $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and*

$$\lim_{|z|\to 1} \mu(z)|u(z)|\left|\varphi(z)\right| = 0. \tag{3.23}$$

Proof. First assume that $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_{\mu,0}$ is bounded. Then, it is clear that $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded, and as in the proof of Theorem 3.1, by taking the test functions $f_j(z) = z_j$, $j = 1,\ldots,n$, we obtain (3.23).

Conversely, assume that the operator $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and condition (3.23) holds. Then, for each polynomial p, we have

$$\mu(z)\left|\left(\mathfrak{R}^m_{u,\varphi}p\right)(z)\right| \le \mu(z)|u(z)|\left|\left(\mathfrak{R}^m p\right)(\varphi(z))\right| \le \mu(z)|u(z)|\left|\varphi(z)\right|\left\|\nabla\mathfrak{R}^{m-1}p\right\|_\infty, \tag{3.24}$$

from which along with condition (3.23) it follows that $\mathfrak{R}^m_{u,\varphi}p \in H^\infty_{\mu,0}$. Since the set of all polynomials is dense in $H(p,q,\phi)$, we see that for every $f \in H(p,q,\phi)$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\|f - p_k\|_{H(p,q,\phi)} = 0. \tag{3.25}$$

From this and by the boundedness of the operator $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_\mu$, we have that

$$\left\|\mathfrak{R}^m_{u,\varphi}f - \mathfrak{R}^m_{u,\varphi}p_k\right\|_{H^\infty_\mu} \le \left\|\mathfrak{R}^m_{u,\varphi}\right\|_{H(p,q,\phi)\to H^\infty_\mu}\|f - p_k\|_{H(p,q,\phi)} \longrightarrow 0 \tag{3.26}$$

as $k \to \infty$. Hence $\mathfrak{R}^m_{u,\varphi}(H(p,q,\phi)) \subseteq H^\infty_{\mu,0}$, and consequently $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_{\mu,0}$ is bounded. □

Theorem 3.4. *Assume $m \in \mathbb{N}$, $0 < p, q < \infty$, ϕ, μ are normal, φ is a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. Then $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{\mu(z)|u(z)|\left|\varphi(z)\right|}{\phi\left(\left|\varphi(z)\right|\right)\left(1-\left|\varphi(z)\right|^2\right)^{n/q+m}} = 0. \tag{3.27}$$

Proof. From (3.27), we see that (3.1) hold. This fact along with (3.3) implies that the set $\mathfrak{R}^m_{u,\varphi}(\{f \mid \|f\|_{H(p,q,\phi)} \le 1\})$ is bounded in H^∞_μ, moreover in $H^\infty_{\mu,0}$. By taking the supremum in (3.3) over the unit ball in $H(p,q,\phi)$, using (3.27) and applying Lemma 2.2 we obtain that the operator $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_{\mu,0}$ is compact.

If $\mathfrak{R}^m_{u,\varphi} : H(p,q,\phi) \to H^\infty_{\mu,0}$ is compact, then by Theorem 3.2, we have that condition (3.14) holds, which implies that for every $\varepsilon > 0$ there is an $r \in (0,1)$ such that

$$\frac{\mu(z)|u(z)|\left|\varphi(z)\right|}{\phi\left(\left|\varphi(z)\right|\right)\left(1-\left|\varphi(z)\right|^2\right)^{n/q+m}} < \varepsilon, \tag{3.28}$$

for $r < |\varphi(z)| < 1$.

As in Theorem 3.3, we have that (3.23) holds. Thus there is a $\sigma \in (0,1)$ such that

$$\mu(z)|u(z)|\left|\varphi(z)\right| < \varepsilon\left(1-r^2\right)^{n/q+m}\inf_{\left|\varphi(z)\right|\le r}\phi\left(\left|\varphi(z)\right|\right), \tag{3.29}$$

for $\sigma < |z| < 1$.

If $|\varphi(z)| \le r$ and $\sigma < |z| < 1$, then from (3.29), we obtain

$$\frac{\mu(z)|u(z)||\varphi(z)|}{\left(1-|\varphi(z)|^2\right)^{n/q+m}\phi(|\varphi(z)|)} \le \frac{\mu(z)|u(z)||\varphi(z)|}{(1-r^2)^{n/q+m}\inf_{|\varphi(z)|\le r}\phi(|\varphi(z)|)} < \varepsilon. \tag{3.30}$$

Using (3.30) and the fact that from (3.28), we have

$$\frac{\mu(z)|u(z)||\varphi(z)|}{\phi(|\varphi(z)|)\left(1-|\varphi(z)|^2\right)^{n/q+m}} < \varepsilon, \tag{3.31}$$

for $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we get (3.27). □

4. Hilbert-Schmidt Norm of $\mathfrak{R}^m_{u,\varphi} : A^2_\alpha \to A^2_\alpha$ and $\mathfrak{R}^m_{u,\varphi} : H^2 \to H^2$

In this section we calculate Hilbert-Schmidt norm of the operators $\mathfrak{R}^m_{u,\varphi} : A^2_\alpha \to A^2_\alpha$ and $\mathfrak{R}^m_{u,\varphi} : H^2 \to H^2$. For some related results see [37, 38].

If $\mathscr{H}$ is a separable Hilbert space, then the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}}$ of an operator $T : \mathscr{H} \to \mathscr{H}$ is defined by

$$\|T\|_{\mathrm{HS}} = \left(\sum_{n=1}^{\infty}\|Te_n\|^2\right)^{1/2}, \tag{4.1}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis on $\mathscr{H}$. The right-hand side in (4.1) does not depend on the choice of basis. Hence, it is larger than the operator norm $\|T\|_{\mathrm{op}}$ of T.

Let $\langle\cdot\,,\,\cdot\rangle_\alpha$, $\alpha \ge -1$, be the usual scalar product on A^2_α, where we regard that $A^2_{-1} = H_2$. Since for each multi-index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n_0$

$$\int_{\mathbb{B}} \left|z^\beta\right|^2 dv_\alpha(z) = \frac{\beta!\,\Gamma(n+\alpha+1)}{\Gamma(n+|\beta|+\alpha+1)}, \tag{4.2}$$

where $\beta! = \beta_1! \cdots \beta_n!$, and

$$\int_{\mathbb{B}} z^\beta z^\gamma dv_\alpha(z) = 0, \quad \beta \ne \gamma \tag{4.3}$$

(see, e.g., [1]), we have that the vectors

$$e_\beta(z) = \sqrt{\frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)}}\, z^\beta, \quad \beta \in \mathbb{N}^n_0, \tag{4.4}$$

form an orthonormal basis in A^2_α.

Theorem 4.1. *Let $m \in \mathbb{N}_0$. Then Hilbert-Schmidt norm of the operator $\mathfrak{R}^m_{u,\varphi}$ on A^2_α, $\alpha > -1$ is*

$$\left\|\mathfrak{R}^m_{u,\varphi}\right\|_{HS} = \left(\int_{\mathbb{B}} |u(z)|^2 \left(\mathfrak{R}^m \frac{1}{\left(1-\sum_{j=1}^n w_j\right)^{n+\alpha+1}}\right)\Bigg|_{w_j=|\varphi_j(z)|^2} dv_\alpha(z)\right)^{1/2}. \tag{4.5}$$

Proof. By using the definition of the Hilbert-Schmidt norm and the monotone convergence theorem, we have

$$\begin{aligned}
\left\|\mathfrak{R}^m_{u,\varphi}\right\|^2_{\mathrm{HS}} &= \sum_\beta \left\|\mathfrak{R}^m_{u,\varphi}(e_\beta)\right\|^2_{2,\alpha} = \sum_\beta \frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)} \left\|\mathfrak{R}^m_{u,\varphi}(z^\beta)\right\|^2_{2,\alpha} \\
&= \sum_\beta \frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)} |\beta|^m \int_{\mathbb{B}} |u(z)|^2 \prod_{j=1}^n |\varphi_j(z)|^{2\beta_j} dv_\alpha(z) \\
&= \int_{\mathbb{B}} |u(z)|^2 \sum_\beta |\beta|^m \frac{\Gamma(n+|\beta|+\alpha+1)}{\beta!\,\Gamma(n+\alpha+1)} \prod_{j=1}^n |\varphi_j(z)|^{2\beta_j} dv_\alpha(z).
\end{aligned} \tag{4.6}$$

We also have that

$$\begin{aligned}
\left(1-\sum_{j=1}^n w_j\right)^{-(n+\alpha+1)} &= \sum_{k=0}^\infty \left(\sum_{j=1}^n w_j\right)^k \frac{\Gamma(n+\alpha+k+1)}{k!\,\Gamma(n+\alpha+1)} \\
&= \sum_{k=0}^\infty \sum_{|l|=k} \frac{k!}{l!} \prod_{j=1}^n w_j^{l_j} \frac{\Gamma(n+\alpha+k+1)}{k!\,\Gamma(n+\alpha+1)} \\
&= \sum_l \frac{\Gamma(n+\alpha+|l|+1)}{l!\,\Gamma(n+\alpha+1)} \prod_{j=1}^n w_j^{l_j},
\end{aligned} \tag{4.7}$$

from which by taking the radial derivatives it follows that

$$\mathfrak{R}^m \frac{1}{\left(1-\sum_{j=1}^n w_j\right)^{n+\alpha+1}} = \sum_l |l|^m \frac{\Gamma(n+\alpha+|l|+1)}{l!\,\Gamma(n+\alpha+1)} \prod_{j=1}^n w_j^{l_j}. \tag{4.8}$$

From (4.6) and (4.8) the result easily follows. □

Similar to Theorem 4.1 the following result regarding the case of the Hardy space is proved. We omit the proof.

Theorem 4.2. *Let $m \in \mathbb{N}_0$. Then, Hilbert-Schmidt norm of the operator $\mathfrak{R}^m_{u,\varphi}$ on H^2, is*

$$\left\|\mathfrak{R}^m_{u,\varphi}\right\|_{HS} = \sup_{0<r<1}\left(\int_S |u(r\zeta)|^2 \left(\mathfrak{R}^m \frac{1}{\left(1-\sum_{j=1}^n w_j\right)^n}\right)\Bigg|_{w_j=|\varphi_j(r\zeta)|^2} d\sigma(\zeta)\right)^{1/2}. \tag{4.9}$$

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