Research Article

# On the Nevanlinna's Theory for Vector-Valued Mappings 

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The purpose of this paper is to establish the first and second fundamental theorems for an $E$ valued meromorphic mapping from a generic domain $D \subset \mathbb{C}$ to an infinite dimensional complex Banach space $E$ with a Schauder basis. It is a continuation of the work of C. Hu and Q. Hu. For $f(z)$ defined in the disk, we will prove Chuang's inequality, which is to compare the relationship between $T(r, f)$ and $T\left(r, f^{\prime}\right)$. Consequently, we obtain that the order and the lower order of $f(z)$ and its derivative $f^{\prime}(z)$ are the same.

## 1. Introduction

In 1980s, Ziegler [1] established Nevanlinna's theory for the vector-valued meromorphic functions in finite dimensional spaces. After Ziegler some works in finite dimensional spaces were done in 1990s [2-4]. In 2006, C. Hu and Q. Hu [5] considered the case of infinite dimensional spaces and they investigated the $E$-valued meromorphic mappings defined in the disk $C_{r}=\{z:|z|<r\}$. In this article, by using Green function technique, we will consider this theory defined in generic domain $D \subseteq \mathbb{C}$ (see Section 2). In Section 3, motivated by the work of [6-8], we will prove Chuang's inequality, which is to compare the relationship between $T(r, f)$ and $T\left(r, f^{\prime}\right)$. Consequently, we obtain that the order and the lower order of $f(z)$ and its derivative $f^{\prime}(z)$ are the same. This is an extension of an important result for meromorphic functions.

## 2. First and Second Fundamental Theorem in Generic Domains

Let $(E,\|\cdot\|)$ be a complex Banach space with a Schauder basis $\left\{\mathbf{e}_{i}\right\}$ and the norm $\|\cdot\|$. Thus an $E$-valued meromorphic mapping $f(z)$ defined in a domain $D \subseteq \mathbb{C}$ can be written as $f(z)=$
$\left(f_{1}(z), f_{2}(z), \ldots, f_{k}(z), \ldots\right)$. The elements of $E$ are called vectors and are usually denoted by letters from the alphabet: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots$. The symbol $\mathbf{0}$ denotes the zero vector of $E$. We denote vector infinity, complex number infinity, and the norm infinity by $\widehat{\infty}, \infty$, and $+\infty$, respectively. A vector-valued mapping is called holomorphic(meromorphic) if all $f_{j}(z)$ are holomorphic (meromorphic). The $j$ th derivative $(j=1,2, \ldots)$ and the integration of $f(z)$ are defined by

$$
\begin{gather*}
f^{(j)}(z)=\left(f_{1}^{(j)}(z), f_{2}^{(j)}(z), \ldots, f_{k}^{(j)}(z), \ldots\right) \\
\int^{z} f(\zeta) d \zeta=\left(\int^{z} f_{1}(\zeta) d \zeta, \int^{z} f_{2}(\zeta) d \zeta, \ldots, \int^{z} f_{k}(\zeta) d \zeta, \ldots\right) \tag{2.1}
\end{gather*}
$$

respectively. We assume that $f^{(0)}(z)=f(z)$. A point $z_{0} \in D$ is called a "pole" or " $\widehat{\infty}$-point" of $f(z)=\left(f_{1}(z), \ldots, f_{k}(z), \ldots\right)$ if $z_{0}$ is a pole of at least one of the component functions $f_{k}(z)(k=$ $1,2, \ldots)$. We define $\left\|f\left(z_{0}\right)\right\|=+\infty$ when $z_{0}$ is a pole. A point $z_{0} \in D$ is called "zero" of $f(z)$ if all the component functions $f_{k}(z)(k=1,2, \ldots)$ have zeros at $z_{0}$.

Remark 2.1. The integrals are well defined because the set of singularities making $\widehat{\infty}-\widehat{\infty}$ meaningless is zero measurable.

In order to make our statement clear, we first recall some knowledge of Green functions.

Definition 2.2. Let $D$ be a domain surrounded by finitely many piecewise analytic curves. Then for any $a \in D$, there exists a Green function, denoted by $G_{D}(z, a)$, for $D$ with singularity at $a \in D$ which is uniquely determined by the following:
(1) $G_{D}(z, a)$ is harmonic in $D \backslash\{a\}$;
(2) in a neighborhood of $a, G_{D}(z, a)=-\log |z-a|+w(z, a)$ for some function $w(z, a)$ harmonic in $D$;
(3) $G_{D}(z, a) \equiv 0$, on the boundary of $D$.

By $\partial D$ we denote the boundary of $D$ and $\vec{n}$ the inner normal of $\partial D$ with respect to $D$. Using Green function we can establish the following general Poisson formula for the $E$ valued meromorphic mapping, which is similar with [5, Lemma 2.2] (see [9, Theorem 2.1], or [10, Theorem 2.1.1]). We do not give the details here.

Theorem 2.3. Let $f: \bar{D}(\subset \mathbb{C}) \rightarrow E$ be an E-valued meromorphic mapping, which does not reduce to the constant zero element $\mathbf{0} \in E$. Then

$$
\begin{align*}
\log \|f(z)\|= & \frac{1}{2 \pi} \int_{\partial D} \log \|f(\zeta)\| \frac{\partial G_{D}(\zeta, z)}{\partial \vec{n}} d s-\sum_{a_{m} \in D} G_{D}\left(a_{m}, z\right) \\
& +\sum_{b_{n} \in D} G_{D}\left(b_{n}, z\right)-\frac{1}{2 \pi} \int_{D} G_{D}(\zeta, z) \Delta \log \|f(\zeta)\| d x \wedge d y \tag{2.2}
\end{align*}
$$

where $\zeta=x+i y,\left\{a_{m}\right\}$ are the zeros of $f(z)$ and $\left\{b_{n}\right\}$ are the poles of $f(z)$ according to their multiplicities.

Remark 2.4. A simple inspection to the $\mathbb{C}^{2}$-valued case shows that $\log \|f(z)\|$ is not harmonic for a holomorphic (or meromorphic) $E$-valued function. Therefore we have an additional term in formula (2.2).

Following Theorem 2.3, we introduce some notations.

$$
\begin{align*}
& N(D, a, f)=\sum_{b_{n} \in D} G_{D}\left(b_{n}, a\right) \\
& m(D, a, f)=\frac{1}{2 \pi} \int_{\partial D} \log ^{+}\|f(\zeta)\| \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s  \tag{2.3}\\
& V(D, a, f)=\frac{1}{2 \pi} \int_{D} G(\zeta, a) \Delta \log \|f(\zeta)\| d x \wedge d y
\end{align*}
$$

where $a$ is a point in $D$ and $\left\{b_{n}\right\}$ are the poles of $f(z)$ in $D$ appearing according to their multiplicities, $\log ^{+} x=\log \max \{x, 1\}$. Define

$$
\begin{equation*}
T(D, a, f)=m(D, a, f)+N(D, a, f) \tag{2.4}
\end{equation*}
$$

$T(D, a, f)$ is called the Nevanlinna characteristic function of $f(z)$ with the center $a \in D$.
Next, we give the first (FFT) and the second (SFT) fundamental theorems for $f(z)$.
Theorem 2.5 (FFT). Let $f(z)$ be an $E$-valued meromorphic mapping on $\bar{D}$. Then for a fixed vector $\mathbf{b} \in E$ and for any $a \in D$ such that $f(a) \neq \mathbf{b}$, one has

$$
\begin{equation*}
T\left(D, a, \frac{1}{f-\mathbf{b}}\right)=T(D, a, f)-V(D, a, f-\mathbf{b})-\log \|f(a)-\mathbf{b}\|+\varepsilon(\mathbf{b}, D) \tag{2.5}
\end{equation*}
$$

where

$$
|\varepsilon(\mathbf{b}, D)| \leq \begin{cases}\log ^{+}\|\mathbf{b}\|+\log 2, & \mathbf{b} \neq \mathbf{0}  \tag{2.6}\\ 0, & \mathbf{b}=\mathbf{0}\end{cases}
$$

Proof. We can rewrite Theorem 2.3 as follows:

$$
\begin{equation*}
T(D, a, f)=T\left(D, a, \frac{1}{f}\right)+V(D, a, f)+\log \|f(a)\| \tag{2.7}
\end{equation*}
$$

Applying this formula to the function $f(z)-\mathbf{b}$, we can prove the theorem.

Theorem 2.6 (SFT). Let $f(z)$ be an E-valued meromorphic mapping on $\bar{D}$, let $\mathbf{a}^{[j]}(j=1,2, \ldots, q) \in$ $E \bigcup\{\widehat{\infty}\}$ be $q$ distinct vectors, and let $f(a) \neq \mathbf{a}^{[j]}$. Then,

$$
\begin{align*}
(q-2) T(D, a, f) \leq & \sum_{j=1}^{q}\left[N\left(D, a, f=\mathbf{a}^{[j]}\right)+V\left(D, a, f-\mathbf{a}^{[j]}\right)\right]  \tag{2.8}\\
& -V\left(D, a, f^{\prime}\right)-N_{1}(D, a, f)+S(D, a, f)
\end{align*}
$$

where

$$
\begin{align*}
S(D, a, f)= & \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[\sum_{j=1}^{q} \frac{\left\|f^{\prime}(\zeta)\right\|}{\left\|f(\zeta)-\mathbf{a}^{[j]}\right\|}\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \\
& -\log \left\|f^{\prime}(a)\right\|+q \log ^{+} \frac{2 q}{\delta}+\sum_{i=1}^{q} \log \left\|f(a)-\mathbf{a}^{[i]}\right\|,  \tag{2.9}\\
N_{1}(D, a, f)= & 2 N(D, a, f)-N\left(D, a, f^{\prime}\right)+N\left(D, a, \frac{1}{f^{\prime}}\right), \\
\delta= & \min _{i \neq j}\left\|\mathbf{a}^{[i]}-\mathbf{a}^{[j]}\right\|>0 .
\end{align*}
$$

Furthermore, one has the following form:

$$
\begin{align*}
(q-2) T(D, a, f) \leq & \sum_{j=1}^{q}\left[\bar{N}\left(D, a, \frac{1}{f-\mathbf{a}^{[j]}}\right)+V\left(D, a, f-\mathbf{a}^{[j]}\right)\right]  \tag{2.10}\\
& -V\left(D, a, f^{\prime}\right)+S(D, a, f)
\end{align*}
$$

Proof. Set

$$
\begin{equation*}
F(\zeta)=\sum_{j=1}^{q} \frac{1}{\left\|f(\zeta)-\mathbf{a}^{[j]}\right\|} \tag{2.11}
\end{equation*}
$$

According to the property of the logarithm function, we get

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{\partial D} \log ^{+} F(\zeta) \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \\
& \quad \leq m\left(D, a, \frac{1}{f^{\prime}}\right)+\frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[F(\zeta)\left\|f^{\prime}(\zeta)\right\|\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \tag{2.12}
\end{align*}
$$

Denote $\delta=\min _{i \neq j}\left\|a^{[i]}-a^{[j]}\right\|$, and fix $\mu \in\{1,2, \ldots, q\}$. Then we obtain

$$
\begin{equation*}
\left\|f(z)-\mathbf{a}^{[v]}\right\| \geq\left\|\mathbf{a}^{[\mu]}-\mathbf{a}^{[v]}\right\|-\left\|\mathbf{a}^{[\mu]}-f(z)\right\|>\frac{3 \delta}{4} \tag{2.13}
\end{equation*}
$$

for $\mu \neq v$ by

$$
\begin{equation*}
\left\|f(z)-\mathbf{a}^{[\mu]}\right\|<\frac{\delta}{2 q} \leq \frac{\delta}{4} \tag{2.14}
\end{equation*}
$$

So either the set of points on $\partial D$ which is determined by (2.14) is empty or any two of some sets for different $\mu$ have intersection. In any case, on $\partial D$ we have

$$
\begin{align*}
\frac{1}{2 \pi} \int_{\partial D} \log ^{+} F(\zeta) \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s & \geq \frac{1}{2 \pi} \sum_{\mu=1}^{q} \int_{\left\|f-\mathbf{a}^{[\mu]}\right\|<\delta / 2 q} \log ^{+} F(\zeta) \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \\
& \geq \frac{1}{2 \pi} \sum_{\mu=1}^{q} \int_{\left\|f-\mathbf{a}^{[\mu]}\right\|<\delta / 2 q} \log ^{+} \frac{1}{\left\|f(\zeta)-a^{[\mu]}\right\|} \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \tag{2.15}
\end{align*}
$$

Since

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{\left\|f-\mathbf{a}^{[\mu]}\right\|<\delta / 2 q} \log ^{+} \frac{1}{\left\|f(\zeta)-\mathbf{a}^{[\mu]}\right\|} \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \\
& \quad=m\left(D, a, \mathbf{a}^{[\mu]}\right)-\frac{1}{2 \pi} \int_{\left\|f-\mathbf{a}^{[\mu]}\right\|>\delta / 2 q} \log ^{+} \frac{1}{\left\|f(\zeta)-\mathbf{a}^{[\mu]}\right\|} \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s  \tag{2.16}\\
& \quad \geq m\left(D, a, \mathbf{a}^{[\mu]}\right)-\log ^{+} \frac{2 q}{\delta}
\end{align*}
$$

it follows that

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{\partial D} \log ^{+} F(\zeta) \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \geq \sum_{\mu=1}^{q} m\left(D, a, \mathbf{a}^{[\mu]}\right)-q \log ^{+} \frac{2 q}{\delta} \tag{2.17}
\end{equation*}
$$

From (2.12), we get

$$
\begin{equation*}
m\left(D, a, \frac{1}{f^{\prime}}\right) \geq \sum_{\mu=1}^{q} m\left(D, a, \mathbf{a}^{[\mu]}\right)-q \log ^{+} \frac{2 q}{\delta}-\frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[F(\zeta)\left\|f^{\prime}(\zeta)\right\|\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \tag{2.18}
\end{equation*}
$$

Since $f(z)$ is nonconstant vector, $f^{\prime}(z)$ does not reduce to the constant zero element 0 . Applying FFT to $f^{\prime}(z)$, we can obtain

$$
\begin{equation*}
T\left(D, a, f^{\prime}\right)=N\left(D, a, \frac{1}{f^{\prime}}\right)+m\left(D, a, \frac{1}{f^{\prime}}\right)+V\left(D, a, f^{\prime}\right)+\log \left\|f^{\prime}(a)\right\| \tag{2.19}
\end{equation*}
$$

Using this formula, we have

$$
\begin{align*}
T\left(D, a, f^{\prime}\right) \geq & \sum_{\mu=1}^{q} m\left(D, a, \mathbf{a}^{[\mu]}\right)-q \log ^{+} \frac{2 q}{\delta} \\
& -\frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[F(\zeta)\left\|f^{\prime}(\zeta)\right\|\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s  \tag{2.20}\\
& +N\left(D, a, \frac{1}{f^{\prime}}\right)+V\left(D, a, f^{\prime}\right)+\log \left\|f^{\prime}(a)\right\| .
\end{align*}
$$

On the other hand, we have

$$
\begin{align*}
T\left(D, a, f^{\prime}\right) & =m\left(D, a, f^{\prime}\right)+N\left(D, a, f^{\prime}\right) \\
& \leq m(D, a, f)+N\left(D, a, f^{\prime}\right)+\frac{1}{2 \pi} \int_{\partial D} \log \frac{\left\|f^{\prime}(\zeta)\right\|}{\|f(\zeta)\|} \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s . \tag{2.21}
\end{align*}
$$

The two inequalities above give

$$
\begin{align*}
& \sum_{\mu=1}^{q} m\left(D, a, \mathbf{a}^{[\mu]}\right)+V\left(D, a, f^{\prime}\right) \\
& \leq m(D, a, f)+N\left(D, a, f^{\prime}\right)-N\left(D, a, \frac{1}{f^{\prime}}\right)  \tag{2.22}\\
&+\frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[F(\zeta)\left\|f^{\prime}(\zeta)\right\|\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s+\frac{1}{2 \pi} \int_{\partial D} \log \frac{\left\|f^{\prime}(\zeta)\right\|}{\|f(\zeta)\|} \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s \\
&-\log \left\|f^{\prime}(a)\right\|+q \log ^{+} \frac{2 q}{\delta} .
\end{align*}
$$

That is to say,

$$
\begin{align*}
& \sum_{\mu=1}^{q} m\left(D, a, \mathbf{a}^{[\mu]}\right)+V\left(D, a, f^{\prime}\right)  \tag{2.23}\\
& \quad \leq m(D, a, f)+N\left(D, a, f^{\prime}\right)-N\left(D, a, \frac{1}{f^{\prime}}\right)+S(D, a, f)
\end{align*}
$$

Adding $\sum_{\mu=1}^{q} N\left(D, a, f=\mathbf{a}^{[\mu]}\right)$ to the above inequality and applying FFT, we can formulate

$$
\begin{align*}
(q-1) T(D, a, f)< & N(D, a, f)+\sum_{j=1}^{q}\left[N\left(D, a, f=\mathbf{a}^{[j]}\right)+V\left(D, a, f-\mathbf{a}^{[j]}\right)\right]  \tag{2.24}\\
& -N_{1}(D, a, f)+S(D, a, f)
\end{align*}
$$

where

$$
\begin{align*}
S(D, a, f)= & \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[\sum_{j=0}^{q} \frac{\left\|f^{\prime}(\zeta)\right\|}{\left\|f(\zeta)-\mathbf{a}^{[j]}\right\|}\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s  \tag{2.25}\\
& -\log \left\|f^{\prime}(a)\right\|+q \log ^{+} \frac{2 q}{\delta}+\sum_{i=0}^{q} \log \left\|f(a)-\mathbf{a}^{[i]}\right\|, \quad \mathbf{a}^{[0]}=\mathbf{0} .
\end{align*}
$$

Since $N(D, a, f) \leq T(D, a, f),(2.24)$ can be written as

$$
\begin{align*}
(q-2) T(D, a, f)< & \sum_{j=1}^{q}\left[N\left(D, a, f=\mathbf{a}^{[j]}\right)+V\left(D, a, f-\mathbf{a}^{[j]}\right)\right]  \tag{2.26}\\
& -N_{1}(D, a, f)+S(D, a, f)
\end{align*}
$$

If $\left\{\mathbf{a}^{[j]}\right\}$ contains $\widehat{\infty},(2.26)$ also holds. Let $\mathbf{a}^{[q+1]}=\widehat{\infty}$, and substitute $q$ with $q+1$; then we have (2.26), where $\mathbf{a}^{[q]}=\widehat{\infty}$, and

$$
\begin{align*}
S(D, a, f)= & \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left[\sum_{j=0}^{q-1} \frac{\left\|f^{\prime}(\zeta)\right\|}{\left\|f(\zeta)-\mathbf{a}^{[j]}\right\|}\right] \frac{\partial G_{D}(\zeta, a)}{\partial \vec{n}} d s  \tag{2.27}\\
& -\log \left\|f^{\prime}(a)\right\|+q \log ^{+} \frac{2 q}{\delta}+\sum_{i=0}^{q-1} \log \left\|f(a)-\mathbf{a}^{[i]}\right\| .
\end{align*}
$$

Next we establish Hiong King-Lai's inequality for $f(z)$.
Theorem 2.7. Let $f(z)$ be an E-valued meromorphic mapping on $\bar{D}, l \in D$, let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in E$ be three finite vectors, and let $\mathbf{b} \neq \mathbf{0}, \mathbf{c} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{c}, f^{(k)}(l) \neq \mathbf{0}, \mathbf{b}, \mathbf{c}$. Then one has

$$
\begin{align*}
T(D, l, f)< & N(D, l, f=\mathbf{a})+N\left(D, l, f^{(k)}=\mathbf{b}\right)+N\left(D, l, f^{(k)}=\mathbf{c}\right) \\
& +V\left(D, l, f^{(k)}\right)+V\left(D, l, f^{(k)}-\mathbf{b}\right)+V\left(D, l, f^{(k)}-\mathbf{c}\right)-N\left(D, l, \frac{1}{f^{(k+1)}}\right) \\
& +S\left(D, l, f^{(k)}\right) \tag{2.28}
\end{align*}
$$

Proof. First, we have

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left\|\frac{1}{f(\zeta)-\mathbf{a}}\right\| \frac{\partial G_{D}(\zeta, l)}{\partial \vec{n}} d s \\
& \quad \leq \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left\|\frac{1}{f^{(k)}(\zeta)}\right\| \frac{\partial G_{D}(\zeta, l)}{\partial \vec{n}} d s+\frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left\|\frac{f^{(k)}(\zeta)}{f(\zeta)-\mathbf{a}}\right\| \frac{\partial G_{D}(\zeta, l)}{\partial \vec{n}} d s \tag{2.29}
\end{align*}
$$

Applying FFT to $f(z)$ and $f^{(k)}(z)$, respectively, we have

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left\|\frac{1}{f(\zeta)-\mathbf{a}}\right\| \frac{\partial G_{D}(\zeta, l)}{\partial \vec{n}} d s \\
& \quad=T(D, l, f)-N(D, l, f=\mathbf{a})-V(D, l, f-\mathbf{a})-\log \|f(l)-\mathbf{a}\|+\varepsilon(\mathbf{a}, D), \\
& \frac{1}{2 \pi} \int_{\partial D} \log ^{+}\left\|\frac{1}{f^{(k)}(\zeta)}\right\| \frac{\partial G_{D}(\zeta, l)}{\partial \vec{n}} d s  \tag{2.30}\\
& \quad=T\left(D, l, f^{(k)}\right)-N\left(D, l, \frac{1}{f^{(k)}}\right)-V\left(D, l, f^{(k)}\right)-\log \left\|f^{(k)}(l)\right\| .
\end{align*}
$$

Thus we have

$$
\begin{align*}
T(D, l, f) \leq & T\left(D, l, f^{(k)}\right)+N(D, l, f=\mathbf{a})+V(D, l, f-\mathbf{a}) \\
& -N\left(D, l, \frac{1}{f^{(k)}}\right)-V\left(D, l, f^{(k)}\right)+\log \frac{\|f(l)-a\|}{\left\|f^{(k)}(l)\right\|}-\varepsilon(\mathbf{a}, D) . \tag{2.31}
\end{align*}
$$

Applying SFT to $f^{(k)}$ with $\mathbf{0}, \mathbf{b}, \mathbf{c}$, we have

$$
\begin{align*}
T\left(D, l, f^{(k)}\right) \leq & \bar{N}\left(D, l, \frac{1}{f^{(k)}}\right)+\bar{N}\left(D, l, f^{(k)}=\mathbf{b}\right)+\bar{N}\left(D, l, f^{(k)}=\mathbf{c}\right)-N\left(D, l, f^{(k+1)}\right) \\
& +V\left(D, l, f^{(k)}\right)+V\left(D, l, f^{(k)}-\mathbf{b}\right)+V\left(D, l, f^{(k)}-\mathbf{c}\right)-V\left(D, l, f^{(k+1)}\right) \\
& +S\left(D, l, f^{(k)}\right) . \tag{2.32}
\end{align*}
$$

Combining (2.31) with (2.32), we have

$$
\begin{align*}
T(D, l, f) \leq & N(D, l, f=\mathbf{a})+\bar{N}\left(D, l, f^{(k)}=\mathbf{b}\right)+\bar{N}\left(D, l, f^{(k)}=\mathbf{c}\right)-N\left(D, l, f^{(k+1)}\right) \\
& +V(D, l, f-\mathbf{a})+V\left(D, l, f^{(k)}-\mathbf{b}\right)+V\left(D, l, f^{(k)}-\mathbf{c}\right)-V\left(D, l, f^{(k+1)}\right)  \tag{2.33}\\
& +S\left(D, l, f^{(k)}\right) .
\end{align*}
$$

## 3. The Vector-Valued Mapping and Its Derivative

In this section, we will discuss the value distribution theory of $f(z)$ defined in the disk $C_{r}=$ $\{z:|z|<r\}$. We will prove Chuang's inequality. According to (2.3), we have the following terms:

$$
\begin{align*}
& m(r, f)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+}\left\|f\left(r e^{i \theta}\right)\right\| d \theta \\
& N(r, f)=\int_{0}^{r} \frac{n(t, f)-n(0, f)}{t} d t+n(0, f) \log r,  \tag{3.1}\\
& V(r, f)=\frac{1}{2 \pi} \int_{C_{r}} \log \left|\frac{r}{\zeta}\right| \Delta \log \|f(\zeta)\| d x \wedge d y, \quad \zeta=x+i y \\
& T(r, f)=m(r, f)+N(r, f),
\end{align*}
$$

where $n(r, f)$ denotes the number of poles of $f(z)$ in $\{z:|z|<r\}$. The order and the lower order of an $E$-valued meromorphic mapping $f(z)$ are defined by

$$
\begin{align*}
& \lambda(f)=\underset{r \rightarrow \infty}{\limsup } \frac{\log T(r, f)}{\log r}  \tag{3.2}\\
& \mu(f)=\liminf _{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} .
\end{align*}
$$

The following lemma is well known.
Lemma 3.1 (see [11, Boutroux-Cartan Theorem]). Let $\left\{a_{j}\right\}_{j=1}^{n}$ be $n$ complex numbers. Then the set of the point $z$ satisfying

$$
\begin{equation*}
\prod_{j=1}^{n}\left|z-a_{j}\right|<h^{n} \tag{3.3}
\end{equation*}
$$

can be contained in several disks, denoted by $(\gamma)$; the total sum of its radius does not exceed $2 e h$.
The next lemma is a special case of Theorem 2.3.

Lemma 3.2 (see [5]). Let $f: C_{r} \rightarrow E$ be an E-valued meromorphic mapping, which does not reduce to the constant zero element $\mathbf{0} \in E$. Then, for a $z \in C_{r}$, one has

$$
\begin{align*}
\log \|f(z)\|= & \frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left\|f\left(r e^{i \theta}\right)\right\| \frac{r^{2}-t^{2}}{r^{2}-2 r t \cos (\theta-\phi)+t^{2}} d \phi \\
& -\sum_{z_{j}(0) \in C_{r}} \log \left|\frac{r^{2}-\overline{z_{j}(0)} z}{r\left(z-z_{j}(0)\right)}\right|+\sum_{z_{j}(\infty) \in C_{r}} \log \left|\frac{r^{2}-\overline{z_{j}(\widehat{\infty})} z}{r\left(z-z_{j}(\widehat{\infty})\right)}\right|  \tag{3.4}\\
& -\frac{1}{2 \pi} \int_{C_{r}} \log \left|\frac{r^{2}-\bar{\xi} z}{r(z-\xi)}\right| \Delta \log \|f(\xi)\| d x \wedge d y
\end{align*}
$$

Here $z_{j}(\mathbf{0})$ and $z_{j}(\widehat{\infty})$ are all the zeros and poles counting their multiplies of $f$ in $D$.
In order to obtain the relationship between $T(r, f)$ and $T\left(r, f^{\prime}\right)$, we should first establish the following two lemmas.

Lemma 3.3. Let $f: \mathbb{C} \rightarrow E$ be a nonzero E-valued meromorphic mapping, and $f(0) \neq \infty$. If $R$ and $R^{\prime}$ are two positive numbers, and $R<R^{\prime}$, then there exists a $\theta_{0} \in[0,2 \pi)$, such that for any $0 \leq r \leq R$ one has

$$
\begin{equation*}
\log ^{+}\left\|f\left(r e^{i \theta_{0}}\right)\right\| \leq \frac{R^{\prime}+R}{R^{\prime}-R} m\left(R^{\prime}, f\right)+n\left(R^{\prime}, f\right) \log 4+N\left(R^{\prime}, f\right) \tag{3.5}
\end{equation*}
$$

Proof. For $z=r e^{i \theta}, 0 \leq r \leq R$. By Lemma 3.2 we have

$$
\begin{align*}
\log \|f(z)\| \leq & \frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left\|f\left(R^{\prime} e^{i \theta}\right)\right\| \frac{R^{\prime 2}-r^{2}}{R^{\prime 2}-2 R^{\prime} r \cos (\theta-\phi)+r^{2}} d \phi \\
& +\sum_{j=1}^{n} \log \left|\frac{R^{\prime 2}-\overline{b_{j}} z}{R^{\prime}\left(z-b_{j}\right)}\right| \tag{3.6}
\end{align*}
$$

where $\left\{b_{j}\right\}_{j=1}^{n}$ are the poles of $f(z)$ in $|z| \leq R^{\prime}$. Then

$$
\begin{align*}
\log ^{+}\|f(z)\| & \leq \frac{R^{\prime}+r}{R^{\prime}-r} m\left(R^{\prime}, f\right)+\sum_{j=1}^{n} \log \frac{2 R^{\prime}}{\left|z-b_{j}\right|}  \tag{3.7}\\
& \leq \frac{R^{\prime}+r}{R^{\prime}-r} m\left(R^{\prime}, f\right)+\log \frac{\left(2 R^{\prime}\right)^{n}}{\prod_{j=1}^{n}\left|z-b_{j}\right|}
\end{align*}
$$

Writing $b_{j}=\left|b_{j}\right| e^{i \phi_{j}}$, we have

$$
\begin{equation*}
\left|r e^{i \theta}-\left|b_{j}\right| e^{i \phi_{j}}\right| \geq\left|b_{j} \| \sin \left(\theta-\phi_{j}\right)\right| \tag{3.8}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\prod_{j=1}^{n}\left|z-b_{j}\right| \geq\left(\prod_{j=1}^{n}\left|b_{j}\right|\right)\left(\prod_{j=1}^{n}\left|\sin \left(\theta-\phi_{j}\right)\right|\right) \tag{3.9}
\end{equation*}
$$

However,

$$
\begin{equation*}
\int_{0}^{\pi} \log \left|\prod_{j=1}^{n}\right| \sin \left(\theta-\phi_{j}\right)\left|d \theta=n \int_{0}^{\pi} \log \right| \sin \theta \mid d \theta=-n \pi \log 2 \tag{3.10}
\end{equation*}
$$

Hence there exists a real number $\theta_{0}$ such that

$$
\begin{equation*}
\left|\prod_{j=1}^{n} \sin \left(\theta_{0}-\phi_{j}\right)\right|>\frac{1}{2^{n}} \tag{3.11}
\end{equation*}
$$

Combining (3.7) and (3.9) with (3.11), we have

$$
\begin{align*}
\log ^{+}\left\|f\left(r e^{i \theta_{0}}\right)\right\| & \leq \frac{R^{\prime}+R}{R^{\prime}-R} m\left(R^{\prime}, f\right)+n \log 4+\sum_{j=1}^{n} \log \frac{R^{\prime}}{\left|b_{j}\right|}  \tag{3.12}\\
& \leq \frac{R^{\prime}+R}{R^{\prime}-R} m\left(R^{\prime}, f\right)+n \log 4+N\left(R^{\prime}, f\right)
\end{align*}
$$

Lemma 3.4. Let $f: \mathbb{C} \rightarrow E$ be a nonzero E-valued meromorphic mapping, and let $R<R^{\prime}<R^{\prime \prime}$ be three positive numbers. Then there exists a positive number $R \leq \rho \leq R^{\prime}$, and for $|z|=\rho$, one has

$$
\begin{equation*}
\log ^{+}\|f(z)\| \leq \frac{R^{\prime \prime}+R^{\prime}}{R^{\prime \prime}-R^{\prime}} m\left(R^{\prime \prime}, f\right)+n\left(R^{\prime \prime}, f\right) \log \frac{8 e R^{\prime \prime}}{R^{\prime}-R} \tag{3.13}
\end{equation*}
$$

Proof. Let $\left\{b_{j}\right\}_{j=1}^{n}$ be the poles of $f(z)$ in $|z| \leq R^{\prime \prime}$. By Boutroux-Cartan Theorem, we have

$$
\begin{equation*}
\prod_{j=1}^{n}\left|z-b_{j}\right| \geq\left(\frac{R^{\prime}-R}{4 e}\right)^{n} \tag{3.14}
\end{equation*}
$$

except for some points contained in a pack of disks whose radius does not exceed $\left(R^{\prime}-R\right) / 2$. Then there exists a circle $\{z:|z|=\rho\}$ such that $R \leq \rho \leq R^{\prime}$ and $\{|z|=\rho\} \cap(\gamma)=\emptyset$.

Thus (3.14) holds on $\{|z|=\rho\}$. For any $z \in\{z:|z|=\rho\}$, we have

$$
\begin{align*}
\log ^{+}\left\|f\left(r e^{i \theta_{0}}\right)\right\| & \leq \frac{R^{\prime \prime}+\rho}{R^{\prime \prime}-\rho} m\left(R^{\prime \prime}, f\right)+\sum_{j=1}^{n} \log \left|\frac{R^{\prime \prime} 2-\overline{b_{j}} z}{R^{\prime \prime}\left(z-b_{j}\right)}\right|  \tag{3.15}\\
& \leq \frac{R^{\prime \prime}+R^{\prime}}{R^{\prime \prime}-R^{\prime}} m\left(R^{\prime \prime}, f\right)+n \log \frac{8 e R^{\prime \prime}}{R^{\prime}-R} .
\end{align*}
$$

Now we are in the position to establish the following Chuang's inequality.
Theorem 3.5. Let $f: \mathbb{C} \rightarrow E$ be a nonzero E-valued meromorphic mapping and $f(0) \neq \widehat{\infty}$. Then for $\tau>1$ and $0<r<R$, one has

$$
\begin{equation*}
T(r, f)<C_{\tau} T\left(\tau r, f^{\prime}\right)+\log ^{+} \tau r+4+\log ^{+}\|f(0)\|, \tag{3.16}
\end{equation*}
$$

where $C_{\tau}$ is a positive constant.
Proof. Take a $\sigma$ such that $\sigma^{3}=\tau$ and denote $r_{1}=\sigma r, r_{2}=\sigma r_{1}, r_{3}=\sigma r_{2}$. Applying Lemma 3.3 to $f^{\prime}(z)$, we can find a real number $\theta_{0}$ such that $0 \leq t \leq r_{1}$, and we have

$$
\begin{equation*}
\log ^{+}\left\|f^{\prime}\left(t e^{i \theta_{0}}\right)\right\| \leq \frac{r_{2}+r_{1}}{r_{2}-r_{1}} m\left(r_{2}, f^{\prime}\right)+n\left(r_{2}, f^{\prime}\right) \log 4+N\left(r_{2}, f^{\prime}\right) . \tag{3.17}
\end{equation*}
$$

In view of Lemma 3.4, for a fixed $\rho \in\left[r, r_{1}\right]$ we have

$$
\begin{equation*}
\log ^{+}\left\|f^{\prime}(z)\right\| \leq \frac{r_{2}+r_{1}}{r_{2}-r_{1}} m\left(r_{2}, f^{\prime}\right)+n\left(r_{2}, f^{\prime}\right) \log \frac{8 e r_{2}}{r_{1}-r^{\prime}} \tag{3.18}
\end{equation*}
$$

on $\{z:|z|=\rho\}$.
From the origin along the segment $\arg z=\theta_{0}$ to $\rho e^{i \theta_{0}}$, and along $\{z:|z|=\rho\}$ turn a rotation to $\rho e^{i \theta_{0}}$. We denote this curve by $L$, and its length is $(2 \pi+1) \rho$.

We notice that $\varphi(z)=\|f(z)\|$ is continuous on $L$. As in [5], $E_{n}$ is an $n$-dimensional projective space of $E$ with a basis $\left\{\mathbf{e}_{i}\right\}_{i=1}^{n}$. The projection operator $P_{n}: E \rightarrow E_{n}$ is a realization of $E_{n}$ associated to the basis and $P_{n}(f(z))=\left(f_{1}(z), f_{2}(z), \ldots, f_{n}(z)\right)$. We have $P_{n}\left(f^{\prime}(z)\right)=$ $\left(P_{n}(f(z))\right)^{\prime}=\sum_{i=1}^{n} f_{i}^{\prime}(z) \mathbf{e}_{i}$ and $P_{n}(f(z))=P_{n}(f(0))+\sum_{i=1}^{n}\left(\int_{0}^{z} f_{i}^{\prime}(\zeta) d \zeta\right) \mathbf{e}_{i}$. Therefore, since $E_{n}$ is
finite dimensional, there exists $K>0$ (appearing in the inequality $\|\cdot\|_{1} \leq K\|\cdot\|_{2}$, where $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ are any two norms on $E_{n}$ ) such that

$$
\begin{align*}
\left\|P_{n}(f(z))\right\| & \leq\left\|P_{n}(f(0))\right\|+\left\|\sum_{i=1}^{n}\left(\int_{0}^{z} f_{i}^{\prime}(\xi) d \xi\right) e_{i}\right\| \\
& \leq\left\|P_{n}(f(0))\right\|+\frac{1}{K}\left(\sum_{i=1}^{n}\left|\int_{0}^{z} f_{i}^{\prime}(\xi) d \xi\right|^{2}\right)^{1 / 2}  \tag{3.19}\\
& \leq\left\|P_{n}(f(0))\right\|+\frac{1}{K}\left(\sum_{i=1}^{n} \max _{\xi \in L}\left|f_{i}^{\prime}(\xi)\right|^{2}\right)^{1 / 2}(2 \pi+1) \rho \\
& \leq\left\|P_{n}(f(0))\right\|+\frac{K^{\prime}}{K} M_{n}(2 \pi+1) \rho
\end{align*}
$$

where $M_{n}=\max _{z \in L}\left\|P_{n}\left(f^{\prime}(z)\right)\right\|$. Thus, we have

$$
\begin{equation*}
\left\|P_{n}(f(z))\right\| \leq\left\|P_{n}(f(0))\right\|+M_{n}(2 \pi+1) \rho+O(1), \quad|z|=\rho . \tag{3.20}
\end{equation*}
$$

In virtue of [6-8], every meromorphic mapping $f(z)$ with values in a Banach space $E$ with a Schauder basis and the projections $P_{n}(f)$ are convergent in its natural topology; that is, they converge uniformly to $f$ in any compact subset $W$ of $\mathbb{C} \backslash \mathrm{P}_{f}$ ( $P_{f}$ being the set of poles the $f$ in $\mathbb{C}$ ). Thus for $n$ large enough, we have

$$
\begin{equation*}
\left\|P_{n}(f(z))\right\|=\|f(z)\|+O(1), \quad \text { for any } z \in W \subseteq \mathbb{C} \backslash P_{f} \tag{3.21}
\end{equation*}
$$

A similar argument to $f^{\prime}(z)$ implies that for $n$ large enough

$$
\begin{equation*}
\left\|P_{n}\left(f^{\prime}(z)\right)\right\|=\left\|f^{\prime}(z)\right\|+O(1), \quad M_{n} \leq M+O(1) \quad \text { for any } z \in W \subseteq \mathbb{C} \backslash P_{f^{\prime}}, \tag{3.22}
\end{equation*}
$$

where $M=\max _{z \in L}\left\|f^{\prime}(z)\right\|$.
Combining (3.20), (3.21), and (3.22) and the fact that the compact set $\{z:|z|=\rho\} \subseteq$ $L \subseteq \mathbb{C} \backslash P_{f}$, we get

$$
\begin{equation*}
\|f(z)\| \leq\|f(0)\|+M(2 \pi+1) \rho+O(1) \tag{3.23}
\end{equation*}
$$

Then

$$
\begin{equation*}
\log ^{+}\|f(\mathrm{z})\| \leq \log ^{+}\|f(0)\|+\log ^{+} M+\log ^{+} \rho+\log 8 e \pi+O(1) \tag{3.24}
\end{equation*}
$$

In virtue of (3.13) and (3.17), we have

$$
\begin{align*}
\log ^{+} M & \leq \frac{r_{2}+r_{1}}{r_{2}-r_{1}} m\left(r_{2}, f^{\prime}\right)+n\left(r_{2}, f^{\prime}\right) \log \frac{8 e r_{2}}{r_{1}-r}+N\left(r_{2}, f^{\prime}\right) \\
& \leq\left\{\frac{\log \left(8 e r_{2} /\left(r_{1}-r\right)\right)}{\log \left(r_{3} / r_{2}\right)}+\frac{r_{2}+r_{1}}{r_{2}-r_{1}}\right\} T\left(r_{3}, f^{\prime}\right)=C_{\tau}^{\prime} T\left(r_{3}, f^{\prime}\right) \tag{3.25}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
m(\rho, f)<C_{\tau}^{\prime} T\left(r_{3}, f^{\prime}\right)+\log ^{+}(\tau r)+4+\log ^{+}\|f(0)\| \tag{3.26}
\end{equation*}
$$

Thus we have

$$
\begin{align*}
T(r, f) & \leq T(\rho, f)<\left(C_{\tau}^{\prime}+1\right) T\left(r_{3}, f^{\prime}\right)+\log ^{+}(\tau r)+4+\log ^{+}\|f(0)\|  \tag{3.27}\\
& =C_{\tau} T\left(\tau r, f^{\prime}\right)+\log ^{+}(\tau r)+4+\log ^{+}\|f(0)\|
\end{align*}
$$

The following result says that we can also control the $T\left(r, f^{\prime}\right)$ by $T(r, f)$.
Theorem 3.6. Let $f(z)(z \in \mathbb{C})$ be a nonconstant $E$-valued meromorphic mapping. Then one has

$$
\begin{equation*}
T\left(r, f^{\prime}\right) \leq 2 T(r, f)+O\left(\log r+\log ^{+} T(r, f)\right) \tag{3.28}
\end{equation*}
$$

Proof. One has

$$
\begin{align*}
T\left(r, f^{\prime}\right) & =m\left(r, f^{\prime}\right)+N\left(r, f^{\prime}\right) \\
& \leq m(r, f)+N\left(r, f^{\prime}\right)+\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{\left\|f^{\prime}\left(r e^{i \phi}\right)\right\|}{\left\|f\left(r e^{i \phi}\right)\right\|} d \phi \\
& =m(r, f)+N(r, f)+\bar{N}(r, f)+\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{\left\|f^{\prime}\left(r e^{i \phi}\right)\right\|}{\left\|f\left(r e^{i \phi}\right)\right\|} d \phi  \tag{3.29}\\
& \leq 2 T(r, f)+\frac{1}{2 \pi} \int_{0}^{2 \pi} \log ^{+} \frac{\left\|f^{\prime}\left(r e^{i \phi}\right)\right\|}{\left\|f\left(r e^{i \phi}\right)\right\|} d \phi \\
& =2 T(r, f)+O\left(\log r+\log ^{+} T(r, f)\right) .
\end{align*}
$$

From Theorems 3.5 and 3.6, we have the following.
Corollary 3.7. For a nonconstant E-valued meromorphic mapping $f(z)(z \in \mathbb{C})$, One has $\lambda(f)=$ $\lambda\left(f^{\prime}\right), \mu(f)=\mu\left(f^{\prime}\right)$.

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