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Research Article

Normal and Osculating Planes of Δ -Regular Curves

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We present the normal and osculating planes of the curves parameterized by a compact subinterval of a time scale.

1. Introduction

Concept of calculus on time scales (or measure chains) was initiated by Hilger and Aulbach [1,2] in order to unify discrete and continuous analyses. This theory is appealing because it provides a useful tool for modeling dynamical processes. Since a time-scale is a closed subset of the reals [3], curves may have scattered points in multidimensional time scale spaces. Therefore, Δ -differentiation plays a major role in investigation of curves parameterized by an arbitrary time scale.

The results in this paper were motivated by geometric interpretation of the results presented in [4].

In this paper, we consider planes whose normal is Δ -differentiable vector that is each component of the vector is Δ -differentiable (i.e., normal planes) and which contain first and second order Δ -differentiable vectors (i.e., osculating planes). In this study we present the normal and osculating planes of the curves parameterized by a compact subinterval of a time scale. Since we need vector valued functions to study Δ -differentiable vectors of curves, we first define the concept of vector valued functions on time scales in Section 2. In [5] Guseinov and Özylmaz introduced the tangent line for Δ -regular curves in 3-dimensional time scales; then in [4] Bohner and Guseinov obtained the equation of such tangent line. The tangent line can also be studied in the concept of partial Δ -differentiation. In Section 3, we obtain the equations of tangent vectors of planar curves by using partial Δ -differentiation. Then we derive the equation of the normal plane for a Δ -regular curve. In Section 4, we present the basic theorem to construct osculating plane of a curve and obtain the equation of this plane by using first- and-second order Δ -derivatives.

We refer the reader to resources such as [3, 4, 6, 7] and [8, 9] for more detailed discussions on the calculus of time scales and on the differential geometry of curves, respectively.

2. Vector-Valued Functions on Time Scales

Let *n* be fixed. Let \mathbb{T}_i denote a time scale for each $i \in \{1, 2, ..., n\}$. Let us set

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \dots \times \mathbb{T}_n = \{ t = (t_1, \dots, t_n) : t_i \in \mathbb{T}_i \ \forall i \in \{1, 2, \dots, n\} \}.$$
 (2.1)

We call Λ^n an *n*-dimensional time scale. Λ^n is also a complete metric space with

$$d(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} \quad \text{for } x, y \in \Lambda^n.$$
 (2.2)

Let a time-scale parameter t vary in an interval [a,b]. If to each value $t \in [a,b]$ we assign a vector r(t), then we say that a vector-valued function r(t) with argument $t \in [a,b]$ is given. Assume that coordinates x_1, x_2, \ldots, x_n are fixed; then the representation of vector-valued function r(t) is equivalent to the representation of scalar functions $x_1(t), x_2(t), \ldots, x_n(t)$; that is, $r(t) = \{x_1(t), \ldots, x_n(t)\}$.

Definition 2.1. A vector r_0 is called the limit of the vector-valued function r(t) as $t \to t_0$ if the length of the vector $r(t) - r(t_0)$ tends to zero as $t \to t_0$. Here we write

$$\lim_{t \to t_0} r(t) = r(t_0). \tag{2.3}$$

It is clear that the vector-valued function r(t) has a limit if and only if each one of the functions $x_1(t), \ldots, x_n(t)$ has a limit as $t \to t_0$.

Definition 2.2. Δ-Derivative of a vector-valued function can be obtained by Δ-differentiating components $x_1(t), \ldots, x_n(t)$ of r(t); that is,

$$r^{\Delta}(t) = \left\{ x_1^{\Delta}(t), \dots, x_n^{\Delta}(t) \right\}. \tag{2.4}$$

Precisely, for the Δ -derivative $r^{\Delta}(t)$ of the vector-valued function r(t), we call the limit

$$\lim_{s \to t} \frac{r(\sigma(t)) - r(s)}{\sigma(t) - s}.$$
 (2.5)

If this limit exists, then r(t) is called Δ -differentiable.

Proposition 2.3. Let $r_1(t)$ and $r_2(t)$ be vector-valued functions. Then

(i)
$$(r_1(t) + r_2(t))^{\Delta} = r_1^{\Delta}(t) + r_2^{\Delta}(t)$$
,

(ii)
$$(r_1r_2)^{\Delta} = r_1^{\Delta}r_2 + r_1^{\sigma}r_2^{\Delta}$$
.

The Δ -differentiation of the inner products and vector products of vector-valued functions, is computed by the consecutive differentiation of the cofactors.

Proposition 2.4. Let $r_1(t)$ and $r_2(t)$ be vector-valued functions, let \times be Euclidean vector product, \cdot and let Euclidean inner product. Then

(i)
$$(r_1 \cdot r_2)^{\Delta} = r_1^{\Delta} \cdot r_2 + r_1^{\sigma} \cdot r_2^{\Delta}$$
,

(ii)
$$(r_1 \times r_2)^{\Delta} = r_1^{\Delta} \times r_2 + r_1^{\sigma} \times r_2^{\Delta} = r_1^{\Delta} \times r_2^{\sigma} + r_1 \times r_2^{\Delta}$$
.

Definition 2.5 (Taylor's expansion for vector-valued functions). Assume that n times Δ-derivative of the vector-valued function r(t) exist and are rd-continuous, then we can write Taylor's expansions for the components; $x_1(t), \ldots, x_n(t)$ as

$$x_{1}(t) = h_{0}(t, t_{0})x_{1}(t_{0}) + h_{1}(t, t_{0})x_{1}^{\Delta}(t_{0}) + h_{2}(t, t_{0}) \quad x_{1}^{\Delta^{2}}(t_{0}) + \dots + o_{1}(g_{n}(t, t_{0}))$$

$$\vdots$$

$$x_{n}(t) = h_{0}(t, t_{0}) \quad x_{n}(t_{0}) + h_{1}(t, t_{0})x_{n}^{\Delta}(t_{0}) + h_{2}(t, t_{0})x_{n}^{\Delta^{2}}(t_{0}) + \dots + o_{n}(g_{n}(t, t_{0})),$$

$$(2.6)$$

where $h_0(r,s) \equiv 1$, $h_{k+1}(r,s) = \int_s^r h_k(\tau,s) \Delta \tau$ for $k \in \mathbb{N}_0$, and

$$g_n(t,t_0) = \int_{t_0}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) x_i^{\Delta^n}(\tau) \Delta \tau$$
 (2.7)

for $i = \{1, ..., n\}$.

This system of three equations can be written as

$$r(t) = h_0(t, t_0)r(t_0) + h_1(t, t_0)r^{\Delta}(t_0) + h_2(t, t_0)r^{\Delta^2}(t_0) + \dots + o(g_n(t, t_0)),$$
 (2.8)

where $o(g_n(t,t_0))$ denotes a vector whose length is an infinitesimal since $\lim_{t\to t_0} g_n(t,t_0) = 0$.

Remark 2.6. There exists one essential difference between Taylor's expansions of vectorvalued function and scalar function. If we consider Taylor's expansion for a scalar function f(t), then we have

$$o(g_n(t,t_0)) = f^{\Delta^{k+1}}(\xi)h_{k+1}(t,t_0), \tag{2.9}$$

where ξ is a point between $\rho^{n-1}(t)$ and t_0 . For a vector-valued function we cannot write similar formula for the corresponding infinitesimal vector, because in general for different components of the vector $o(g_n(t,t_0))$ the corresponding points ξ are different. However, it is more important to note that the length of the vector $o(g_n(t,t_0))$ is an infinitesimal with respect to $g_n(t,t_0)$.

3. Tangent Line to a Curve

Let \mathbb{T} be a time scale.

Definition 3.1. A Δ -regular curve Γ is defined as a mapping

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t), \quad t \in [a, b]$$
 (3.1)

of the segment $[a,b] \subset \mathbb{T}$, a < b, to the space \mathbb{R}^3 , where f_1, f_2, f_3 are real-valued functions defined on [a,b] and Δ -differentiable on $[a,b]^{\kappa}$ with rd-continuous Δ -derivatives and

$$\left| f_1^{\Delta}(t) \right|^2 + \left| f_2^{\Delta}(t) \right|^2 + \left| f_3^{\Delta}(t) \right|^2 \neq 0.$$
 (3.2)

Definition 3.2. A line \mathcal{L}_0 passing through the point P_0 is called the delta tangent line to the curve Γ at the point P_0 if the following held.

- (i) \mathcal{L}_0 passes also through the point $P_0^{\sigma} = (x(\sigma(t_0)), y(\sigma(t_0)), z(\sigma(t_0)))$.
- (ii) If P_o is not an isolated point of the curve Γ , then

$$\lim_{\substack{P \to P_0 \\ P \neq P_0}} \frac{d(P, \mathcal{L}_0)}{d(P, P_0)},\tag{3.3}$$

where P is the moving point of the curve Γ , $d(P, \mathcal{L}_0)$ is the distance from the point P to the line \mathcal{L}_0 , and $d(P, P_0)$ is the distance from the point P to the point P_0^{σ} .

Theorem 3.3. For any point P_0 of the curve Γ there exists the tangent to Γ at P_0 and the directing vector of the tangent is Δ -differential of its position vector function $r^{\Delta}(t^0)$, where $r(t^0) = P_0$ for $t^0 \in \mathbb{T}$.

Proof. This theorem can be proven as in [5], Theorem 3.3.

Let three functions $x: \mathbb{T} \to \mathbb{R}$, $y: \mathbb{T} \to \mathbb{R}$, and $z: \mathbb{T} \to \mathbb{R}$ be given. Let us set $x(\mathbb{T}) := \mathbb{T}_1$, $y(\mathbb{T}) = \mathbb{T}_2$, and $z(\mathbb{T}) = \mathbb{T}_3$. We will assume that \mathbb{T}_1 , \mathbb{T}_2 , and \mathbb{T}_3 are time scales. Denote by $\sigma_1 \Delta_1$, $\sigma_2 \Delta_2$, $\sigma_3 \Delta_3$ the forward jump operators and delta operators for \mathbb{T}_1 , \mathbb{T}_2 , and \mathbb{T}_3 , respectively.

Under the above assumptions, let functions $\phi : \mathbb{T} \times \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ and $\phi : \mathbb{T} \times \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be given.

Consider a space curve given by two equations.

$$\phi(x, y, z) = 0,$$

$$\varphi(x, y, z) = 0.$$
(3.4)

If x = x(t), y = y(t), z = z(t) is the position vector of the considered curve, then, substituting these three functions into (3.4), we obtain two equalities:

$$\phi(x(t), y(t), z(t)) = 0,$$

$$\varphi(x(t), y(t), z(t)) = 0.$$
(3.5)

If the functions ϕ and ϕ are σ_1 -completely differentiable, then, Δ -differentiation of these two equalities leads

$$\frac{\partial \phi}{\Delta_{1}x} x^{\Delta} + \frac{\partial \phi^{\sigma_{1}}}{\Delta_{2}y} y^{\Delta} + \frac{\partial \phi^{\sigma_{1}}}{\Delta_{3}z} z^{\Delta} = 0,$$

$$\frac{\partial \phi}{\Delta_{1}x} x^{\Delta} + \frac{\partial \phi^{\sigma_{1}}}{\Delta_{2}y} y^{\Delta} + \frac{\partial \phi^{\sigma_{1}}}{\Delta_{3}z} z^{\Delta} = 0.$$
(3.6)

If ϕ and φ are σ_2 -completely differentiable, then Δ -differentiation of (3.5) leads us to obtain the following two equations:

$$\frac{\partial \phi^{\sigma_2}}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi^{\sigma_2}}{\Delta_3 z} z^{\Delta} = 0,$$

$$\frac{\partial \phi^{\sigma_2}}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi^{\sigma_2}}{\Delta_3 z} z^{\Delta} = 0.$$
(3.7)

If ϕ and φ are σ_3 -completely differentiable, then Δ -differentiation of (3.5) leads us to obtain the following two equations:

$$\frac{\partial \phi^{\sigma_3}}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi^{\sigma_3}}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi}{\Delta_3 z} z^{\Delta} = 0,
\frac{\partial \phi^{\sigma_3}}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi^{\sigma_3}}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi}{\Delta_3 z} z^{\Delta} = 0.$$
(3.8)

Other combinations of σ_i -completely differentiability of ϕ and φ can be shown similarly. The components $\{x^{\Delta}, y^{\Delta}, z^{\Delta}\}$ of the tangent vector satisfy the system consisting of two equations: (3.6), (3.7), and (3.8).

Assume that ϕ is σ_1 -completely differentiable planar curve given by the equations $\phi(x,y)=0$, z=0 satisfying the condition $(\partial \phi/\Delta_1 x)^2+(\partial \phi^{\sigma_1}/\Delta_2 y)^2\neq 0$; then the components of the tangent vector $r^\Delta=\{x^\Delta,y^\Delta\}$ are the solution of the linear equation

$$\frac{\partial \phi}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi^{\sigma_1}}{\Delta_2 y} y^{\Delta} = 0. \tag{3.9}$$

Therefore, $\{x^{\Delta}, y^{\Delta}\} = \mu\{-\partial \phi^{\sigma_1}/\Delta_2 y, \partial \phi/\Delta_1 x\}$, and the equation of tangent is

$$\frac{\widetilde{x} - x_0}{-\partial \phi(\sigma_1(x_0), y_0)/\Delta_2 y} = \frac{\widetilde{y} - y_0}{\partial \phi(x_0, y_0)/\Delta_1 x}.$$
(3.10)

If planar curve ϕ is σ_2 -completely differentiable, then equation of tangent plane becomes

$$\frac{(\widetilde{x} - x_0)}{-(\partial \phi(x_0, y_0) / \Delta_2 y)} = \frac{(\widetilde{y} - y_0)}{\partial \phi(x_0, \sigma_2(y_0)) / \Delta_1 x}.$$
(3.11)

Definition 3.4. Let Γ be a smooth and completely differentiable space curve. The plane passing through points $P_0 \in \Gamma$ and orthogonal to the vector tangent to Γ at P_0 is called the plane normal to Γ at P_0 .

Denote by \hat{r} the position vector of the normal plane. Since this plane is orthogonal to the vector r^{Δ} and contains the point with position vector $\hat{r} - r(t_0)$, the equation of the normal plane is

$$(\hat{r} - r(t_0)) \cdot r^{\Delta}(t_0) = 0.$$
 (3.12)

The vectors orthogonal to the tangent are called the vectors normal to Γ .

4. Osculating Plane of a Curve

Let P_0 be a point of a curve Γ . Take two points $Q_1, Q_2 \in \Gamma$ situated right side of P_0^{σ} . If the points Q_1 and Q_2 tend to P_0^{σ} , then the limit position of the plane containing $P_0, P_0^{\sigma}, Q_1, Q_2$ is called the osculating plane of Γ at the point P_0 .

Theorem 4.1. Let Γ be a Δ -regular curve represented as r = r(t). Assume that the vectors r^{Δ} and r^{Δ^2} are not collinear at point P_0 . Then there exists the osculating plane of Γ at P_0 and it is spanned by the vectors r^{Δ} and r^{Δ^2} .

Proof. If $P_0 = P_0^{\sigma}$, that is, P_0 is right-dense point of Γ , then this theorem can be proven as in differential geometry concept.

Let P_0 be a right-scattered point of Γ . Then, the positions vector of $\xrightarrow{P_0Q_1}$ and $\xrightarrow{P_0Q_2}$ are $a_1 = r(t_0 + \tau_1) - r(t_0)$ and $a_2 = r(t_0 + \tau_2) - r(t_0)$, respectively. That is, these vectors, if linearly independent, span the plane E.

This plane is also spanned by the vectors $v^{(i)} = a_i/\tau_i$ for $i \in \{1,2\}$ or by the vectors

$$v^{(1)}, \quad w = \frac{2(v^{(2)} - v^{(1)})}{\tau_2 - \tau_1}.$$
 (4.1)

By the means of Taylor's formula, we have

$$r(t_0 + \tau_i) = h_0(t, t_0)r(t_0) + h_1(t, t_0)r^{\Delta}(t_0) + h_2(t, t_0)r^{\Delta^2}(t_0) + o(g_2(t_0)).$$

$$(4.2)$$

Hence, we obtain

$$v^{(1)} = r^{\Delta}(t_0) + \frac{\tau_1}{2}r^{\Delta^2}(t_0) + o(\tau_1),$$

$$w = r^{\Delta^2}(t_0) + o(1).$$
(4.3)

Consequently, if $\tau_i \to 0$ for $i \in \{1,2\}$, then $v^{(1)} \to r^{\Delta}(t_0)$ and $w \to r^{\Delta^2}(t_0)$.

These vectors, if linearly independent, determine the limiting position of the plane E passing through the points P_0 , P_0^{σ} , Q_1 , Q_2 .

Corollary 4.2. If the vectors $r^{\Delta}(t_0)$ and $r^{\Delta^2}(t_0)$ are collinear, then the limit position of considering plane is not determined. For instance, take a straight line

$$r(t) = a + bt, (4.4)$$

where a, b are constant vectors and $t \in \mathbb{T}$. Then

$$r^{\Delta}(t_0) = b, \qquad r^{\Delta^2}(t_0) = 0,$$
 (4.5)

so the osculating plane of the straight line is not determined uniquely. If $r^{\Delta}(t)$ and $r^{\Delta^2}(t)$ are collinear, then the corresponding point of Γ is called the straightening point of Γ .

Theorem 4.3. The osculating plane of a planar curve coincides with the plane containing this curve.

Proof. Let us consider the Taylor expansion of the position vector r(t) at the neighborhood of P_0 :

$$r(t_0 + \tau) = h_0(t, t_0)r(t_0) + h_1(t, t_0)r^{\Delta}(t_0) + h_2(t, t_0)r^{\Delta^2}(t_0) + o(g_2(t, t_0)).$$
(4.6)

The curve $\overline{\Gamma}$, determined by the expantion,

$$\overline{r} = h_0(t, t_0)r(t_0) + h_1(t, t_0)r^{\Delta}(t_0) + h_2(t, t_0)r^{\Delta^2}(t_0)$$
(4.7)

is situated in the osculating plane of Γ at P_0 ; the difference between the position vectors of Γ and $\overline{\Gamma}$ is a sufficiently small vector

$$r(t_0 + \tau) - \overline{r}(\tau) = o(g_2(t, t_0)). \tag{4.8}$$

Hence a sufficiently small neighborhood of P_0 on the space curve Γ is near to the planar curve $\overline{\Gamma}$ situated in the osculating plane of Γ at P_0 .

Now let us write the equation of the osculating plane of Γ at P_0 . Let \hat{r} be the position vector of the osculating plane. Since r^{Δ} and r^{Δ^2} span the osculating plane, the vector product $r^{\Delta} \times r^{\Delta^2}$ is orthogonal to the osculating plane. The vector $\hat{r} - r(t_0)$ belongs to the osculating plane; therefore, the inner product of these vectors is equal to zero:

$$(\widehat{r} - r(t_0)) \cdot \left(r^{\Delta} \times r^{\Delta^2}\right) = 0. \tag{4.9}$$

With respect to coordinate functions, this equation has the following form:

$$\det\begin{pmatrix} \widehat{x} - x(t_0) & x^{\Delta} & x^{\Delta^2} \\ \widehat{y} - y(t_0) & y^{\Delta} & y^{\Delta^2} \\ \widehat{z} - z(t_0) & z^{\Delta} & z^{\Delta^2} \end{pmatrix} = 0.$$
 (4.10)

References

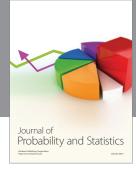
- [1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
- [2] B. Aulbach and S. Hilger, "Linear dynamic processes with inhomogeneous time scale," in *Nonlinear Dynamics and Quantum Dynamical Systems* (*Gaussig*, 1990), vol. 59 of *Math. Res.*, pp. 9–20, Akademie, Berlin, Germany, 1990.
- [3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, Mass, USA, 2001.
- [4] M. Bohner and G. Sh. Guseinov, "Partial differentiation on time scales," *Dynamic Systems and Applications*, vol. 13, no. 3-4, pp. 351–379, 2004.
- [5] G. Sh. Guseinov and E. Özyılmaz, "Tangent lines of generalized regular curves parametrized by time scales," *Turkish Journal of Mathematics*, vol. 25, no. 4, pp. 553–562, 2001.
- [6] F. M. Atici and G. Sh. Guseinov, "On Green's functions and positive solutions for boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.
- [7] M. Bohner and G Sh. Guseinov, "Line integrals and Green's formula on time scales," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1124–1141, 2007.
- [8] E. Kreyszig, Differential Geometry, Dover, New York, NY, USA, 1991.
- [9] R. S. Millman and G. D. Parker, Elements of Differential Geometry, Prentice-Hall, Englewood Cliffs, NJ, USA, 1977.



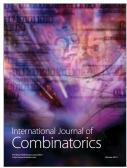








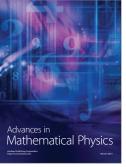


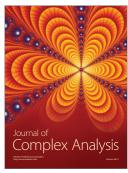




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