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Research Article

Normal and Osculating Planes of Δ-Regular Curves

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We present the normal and osculating planes of the curves parameterized by a compact subinterval of a time scale.

1. Introduction

Concept of calculus on time scales (or measure chains) was initiated by Hilger and Aulbach [1, 2] in order to unify discrete and continuous analyses. This theory is appealing because it provides a useful tool for modeling dynamical processes. Since a time-scale is a closed subset of the reals [3], curves may have scattered points in multidimensional time scale spaces. Therefore, Δ-differentiation plays a major role in investigation of curves parameterized by an arbitrary time scale.

The results in this paper were motivated by geometric interpretation of the results presented in [4].

In this paper, we consider planes whose normal is Δ-differentiable vector that is each component of the vector is Δ-differentiable (i.e., normal planes) and which contain first and second order Δ-differentiable vectors (i.e., osculating planes). In this study we present the normal and osculating planes of the curves parameterized by a compact subinterval of a time scale. Since we need vector valued functions to study Δ-differentiable vectors of curves, we first define the concept of vector valued functions on time scales in Section 2. In [5] Guseinov and Özylmaz introduced the tangent line for Δ-regular curves in 3-dimensional time scales; then in [4] Bohner and Guseinov obtained the equation of such tangent line. The tangent line can also be studied in the concept of partial Δ-differentiation. In Section 3, we obtain the equations of tangent vectors of planar curves by using partial Δ-differentiation. Then we derive the equation of the normal plane for a Δ-regular curve. In Section 4, we present the basic theorem to construct osculating plane of a curve and obtain the equation of this plane by using first- and-second order Δ-derivatives.

We refer the reader to resources such as [3, 4, 6, 7] and [8, 9] for more detailed discussions on the calculus of time scales and on the differential geometry of curves, respectively.

2. Vector-Valued Functions on Time Scales

Let n be fixed. Let $\mathbb{T}_i$ denote a time scale for each $i \in \{1, 2, \ldots, n\}$. Let us set

$$\Lambda^n = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : t_i \in \mathbb{T}_i \ \forall i \in \{1, 2, \ldots, n\}\}. \tag{2.1}$$

We call Λ^n an n-dimensional time scale. Λ^n is also a complete metric space with

$$d(x, y) = \left(\sum_{i}^{n} |x_i - y_i|^2\right)^{1/2} \quad \text{for } x, y \in \Lambda^n. \tag{2.2}$$

Let a time-scale parameter t vary in an interval $[a, b]$. If to each value $t \in [a, b]$ we assign a vector $r(t)$, then we say that a vector-valued function $r(t)$ with argument $t \in [a, b]$ is given. Assume that coordinates $x_1, x_2, \ldots, x_n$ are fixed; then the representation of vector-valued function $r(t)$ is equivalent to the representation of scalar functions $x_1(t), x_2(t), \ldots, x_n(t)$; that is, $r(t) = \{x_1(t), \ldots, x_n(t)\}$.

Definition 2.1. A vector r_0 is called the limit of the vector-valued function $r(t)$ as $t \to t_0$ if the length of the vector $r(t) - r(t_0)$ tends to zero as $t \to t_0$. Here we write

$$\lim_{t \to t_0} r(t) = r(t_0). \tag{2.3}$$

It is clear that the vector-valued function $r(t)$ has a limit if and only if each one of the functions $x_1(t), \ldots, x_n(t)$ has a limit as $t \to t_0$.

Definition 2.2. Δ-Derivative of a vector-valued function can be obtained by Δ-differentiating components $x_1(t), \ldots, x_n(t)$ of $r(t)$; that is,

$$r^{\Delta}(t) = \left\{x_1^{\Delta}(t), \ldots, x_n^{\Delta}(t)\right\}. \tag{2.4}$$

Precisely, for the Δ-derivative $r^{\Delta}(t)$ of the vector-valued function $r(t)$, we call the limit

$$\lim_{s \to t} \frac{r(\sigma(t)) - r(s)}{\sigma(t) - s}. \tag{2.5}$$

If this limit exists, then $r(t)$ is called Δ-differentiable.

Proposition 2.3. *Let $r_1(t)$ and $r_2(t)$ be vector-valued functions. Then*

(i) $(r_1(t) + r_2(t))^\Delta = r_1^\Delta(t) + r_2^\Delta(t)$,

(ii) $(r_1 r_2)^\Delta = r_1^\Delta r_2 + r_1^\sigma r_2^\Delta$.

The Δ-differentiation of the inner products and vector products of vector-valued functions, is computed by the consecutive differentiation of the cofactors.

Proposition 2.4. *Let $r_1(t)$ and $r_2(t)$ be vector-valued functions, let $\times$ be Euclidean vector product, $\cdot$ and let Euclidean inner product. Then*

(i) $(r_1 \cdot r_2)^\Delta = r_1^\Delta \cdot r_2 + r_1^\sigma \cdot r_2^\Delta$,

(ii) $(r_1 \times r_2)^\Delta = r_1^\Delta \times r_2 + r_1^\sigma \times r_2^\Delta = r_1^\Delta \times r_2^\sigma + r_1 \times r_2^\Delta$.

Definition 2.5 (Taylor's expansion for vector-valued functions). Assume that n times Δ-derivative of the vector-valued function $r(t)$ exist and are rd-continuous, then we can write Taylor's expansions for the components; $x_1(t), \dots, x_n(t)$ as

$$\begin{aligned} x_1(t) &= h_0(t,t_0)x_1(t_0) + h_1(t,t_0)x_1^\Delta(t_0) + h_2(t,t_0)\quad x_1^{\Delta^2}(t_0) + \cdots + o_1\big(g_n(t,t_0)\big) \\ &\vdots \\ x_n(t) &= h_0(t,t_0)\quad x_n(t_0) + h_1(t,t_0)x_n^\Delta(t_0) + h_2(t,t_0)x_n^{\Delta^2}(t_0) + \cdots + o_n\big(g_n(t,t_0)\big), \end{aligned} \tag{2.6}$$

where $h_0(r,s) \equiv 1$, $h_{k+1}(r,s) = \int_s^r h_k(\tau,s)\Delta\tau$ for $k \in \mathbb{N}_0$, and

$$g_n(t,t_0) = \int_{t_0}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau))x_i^{\Delta^n}(\tau)\Delta\tau \tag{2.7}$$

for $i = \{1, \dots, n\}$.

This system of three equations can be written as

$$r(t) = h_0(t,t_0)r(t_0) + h_1(t,t_0)r^\Delta(t_0) + h_2(t,t_0)r^{\Delta^2}(t_0) + \cdots + o\big(g_n(t,t_0)\big), \tag{2.8}$$

where $o(g_n(t,t_0))$ denotes a vector whose length is an infinitesimal since $\lim_{t\to t_0} g_n(t,t_0) = 0$.

Remark 2.6. There exists one essential difference between Taylor's expansions of vector-valued function and scalar function. If we consider Taylor's expansion for a scalar function $f(t)$, then we have

$$o\big(g_n(t,t_0)\big) = f^{\Delta^{k+1}}(\xi)h_{k+1}(t,t_0), \tag{2.9}$$

where ξ is a point between $\rho^{n-1}(t)$ and t_0. For a vector-valued function we cannot write similar formula for the corresponding infinitesimal vector, because in general for different components of the vector $o(g_n(t,t_0))$ the corresponding points ξ are different. However, it is more important to note that the length of the vector $o(g_n(t,t_0))$ is an infinitesimal with respect to $g_n(t,t_0)$.

3. Tangent Line to a Curve

Let $\mathbb{T}$ be a time scale.

Definition 3.1. A Δ-regular curve Γ is defined as a mapping

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t), \quad t \in [a,b] \tag{3.1}$$

of the segment $[a,b] \subset \mathbb{T}$, $a < b$, to the space $\mathbb{R}^3$, where f_1, f_2, f_3 are real-valued functions defined on $[a,b]$ and Δ-differentiable on $[a,b]^\kappa$ with rd-continuous Δ-derivatives and

$$\left|f_1^\Delta(t)\right|^2 + \left|f_2^\Delta(t)\right|^2 + \left|f_3^\Delta(t)\right|^2 \neq 0. \tag{3.2}$$

Definition 3.2. A line $\mathcal{L}_0$ passing through the point P_0 is called the delta tangent line to the curve Γ at the point P_0 if the following held.

(i) $\mathcal{L}_0$ passes also through the point $P_0^\sigma = (x(\sigma(t_0)), y(\sigma(t_0)), z(\sigma(t_0)))$.

(ii) If P_o is not an isolated point of the curve Γ, then

$$\lim_{\substack{P \to P_0 \\ P \neq P_0}} \frac{d(P, \mathcal{L}_0)}{d(P, P_0)}, \tag{3.3}$$

where P is the moving point of the curve Γ, $d(P, \mathcal{L}_0)$ is the distance from the point P to the line $\mathcal{L}_0$, and $d(P, P_0)$ is the distance from the point P to the point P_0^σ.

Theorem 3.3. *For any point P_0 of the curve Γ there exists the tangent to Γ at P_0 and the directing vector of the tangent is Δ-differential of its position vector function $r^\Delta(t^0)$, where $r(t^0) = P_0$ for $t^0 \in \mathbb{T}$.*

Proof. This theorem can be proven as in [5], Theorem 3.3. □

Let three functions $x : \mathbb{T} \to \mathbb{R}$, $y : \mathbb{T} \to \mathbb{R}$, and $z : \mathbb{T} \to \mathbb{R}$ be given. Let us set $x(\mathbb{T}) := \mathbb{T}_1$, $y(\mathbb{T}) = \mathbb{T}_2$, and $z(\mathbb{T}) = \mathbb{T}_3$. We will assume that $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ are time scales. Denote by σ_1 Δ_1, σ_2 Δ_2, σ_3 Δ_3 the forward jump operators and delta operators for $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$, respectively.

Under the above assumptions, let functions $\phi : \mathbb{T} \times \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ and $\varphi : \mathbb{T} \times \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be given.

Consider a space curve given by two equations.

$$\begin{aligned} \phi(x, y, z) &= 0, \\ \varphi(x, y, z) &= 0. \end{aligned} \tag{3.4}$$

If $x = x(t)$, $y = y(t)$, $z = z(t)$ is the position vector of the considered curve, then, substituting these three functions into (3.4), we obtain two equalities:

$$\begin{aligned} &\phi\left(x(t), y(t), z(t)\right) = 0, \\ &\varphi\left(x(t), y(t), z(t)\right) = 0. \end{aligned} \tag{3.5}$$

If the functions ϕ and φ are σ_1-completely differentiable, then, Δ-differentiation of these two equalities leads

$$\begin{aligned} &\frac{\partial \phi}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi^{\sigma_1}}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi^{\sigma_1}}{\Delta_3 z} z^{\Delta} = 0, \\ &\frac{\partial \varphi}{\Delta_1 x} x^{\Delta} + \frac{\partial \varphi^{\sigma_1}}{\Delta_2 y} y^{\Delta} + \frac{\partial \varphi^{\sigma_1}}{\Delta_3 z} z^{\Delta} = 0. \end{aligned} \tag{3.6}$$

If ϕ and φ are σ_2-completely differentiable, then Δ-differentiation of (3.5) leads us to obtain the following two equations:

$$\begin{aligned} &\frac{\partial \phi^{\sigma_2}}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi^{\sigma_2}}{\Delta_3 z} z^{\Delta} = 0, \\ &\frac{\partial \varphi^{\sigma_2}}{\Delta_1 x} x^{\Delta} + \frac{\partial \varphi}{\Delta_2 y} y^{\Delta} + \frac{\partial \varphi^{\sigma_2}}{\Delta_3 z} z^{\Delta} = 0. \end{aligned} \tag{3.7}$$

If ϕ and φ are σ_3-completely differentiable, then Δ-differentiation of (3.5) leads us to obtain the following two equations:

$$\begin{aligned} &\frac{\partial \phi^{\sigma_3}}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi^{\sigma_3}}{\Delta_2 y} y^{\Delta} + \frac{\partial \phi}{\Delta_3 z} z^{\Delta} = 0, \\ &\frac{\partial \varphi^{\sigma_3}}{\Delta_1 x} x^{\Delta} + \frac{\partial \varphi^{\sigma_3}}{\Delta_2 y} y^{\Delta} + \frac{\partial \varphi}{\Delta_3 z} z^{\Delta} = 0. \end{aligned} \tag{3.8}$$

Other combinations of σ_i-completely differentiability of ϕ and φ can be shown similarly. The components $\{x^{\Delta}, y^{\Delta}, z^{\Delta}\}$ of the tangent vector satisfy the system consisting of two equations: (3.6), (3.7), and (3.8).

Assume that ϕ is σ_1-completely differentiable planar curve given by the equations $\phi(x, y) = 0$, $z = 0$ satisfying the condition $(\partial\phi/\Delta_1 x)^2 + (\partial\phi^{\sigma_1}/\Delta_2 y)^2 \neq 0$; then the components of the tangent vector $r^{\Delta} = \{x^{\Delta}, y^{\Delta}\}$ are the solution of the linear equation

$$\frac{\partial \phi}{\Delta_1 x} x^{\Delta} + \frac{\partial \phi^{\sigma_1}}{\Delta_2 y} y^{\Delta} = 0. \tag{3.9}$$

Therefore, $\{x^{\Delta}, y^{\Delta}\} = \mu\{-\partial\phi^{\sigma_1}/\Delta_2 y, \partial\phi/\Delta_1 x\}$, and the equation of tangent is

$$\frac{\widetilde{x} - x_0}{-\partial\phi(\sigma_1(x_0), y_0)/\Delta_2 y} = \frac{\widetilde{y} - y_0}{\partial\phi(x_0, y_0)/\Delta_1 x}. \tag{3.10}$$

If planar curve ϕ is σ_2-completely differentiable, then equation of tangent plane becomes

$$\frac{(\widetilde{x} - x_0)}{-(\partial\phi(x_0, y_0)/\Delta_2 y)} = \frac{(\widetilde{y} - y_0)}{\partial\phi(x_0, \sigma_2(y_0))/\Delta_1 x}. \tag{3.11}$$

Definition 3.4. Let Γ be a smooth and completely differentiable space curve. The plane passing through points $P_0 \in \Gamma$ and orthogonal to the vector tangent to Γ at P_0 is called the plane normal to Γ at P_0.

Denote by $\hat{r}$ the position vector of the normal plane. Since this plane is orthogonal to the vector r^{Δ} and contains the point with position vector $\hat{r} - r(t_0)$, the equation of the normal plane is

$$(\hat{r} - r(t_0)) \cdot r^{\Delta}(t_0) = 0. \tag{3.12}$$

The vectors orthogonal to the tangent are called the vectors normal to Γ.

4. Osculating Plane of a Curve

Let P_0 be a point of a curve Γ. Take two points $Q_1, Q_2 \in \Gamma$ situated right side of P_0^{σ}. If the points Q_1 and Q_2 tend to P_0^{σ}, then the limit position of the plane containing $P_0, P_0^{\sigma}, Q_1, Q_2$ is called the osculating plane of Γ at the point P_0.

Theorem 4.1. *Let Γ be a Δ-regular curve represented as $r = r(t)$. Assume that the vectors r^{Δ} and r^{Δ^2} are not collinear at point P_0. Then there exists the osculating plane of Γ at P_0 and it is spanned by the vectors r^{Δ} and r^{Δ^2}.*

Proof. If $P_0 = P_0^{\sigma}$, that is, P_0 is right-dense point of Γ, then this theorem can be proven as in differential geometry concept.

Let P_0 be a right-scattered point of Γ. Then, the positions vector of $\underset{P_0Q_1}{\rightarrow}$ and $\underset{P_0Q_2}{\rightarrow}$ are $a_1 = r(t_0 + \tau_1) - r(t_0)$ and $a_2 = r(t_0 + \tau_2) - r(t_0)$, respectively. That is, these vectors, if linearly independent, span the plane E.

This plane is also spanned by the vectors $v^{(i)} = a_i/\tau_i$ for $i \in \{1, 2\}$ or by the vectors

$$v^{(1)}, \quad w = \frac{2\left(v^{(2)} - v^{(1)}\right)}{\tau_2 - \tau_1}. \tag{4.1}$$

By the means of Taylor's formula, we have

$$r(t_0 + \tau_i) = h_0(t, t_0) r(t_0) + h_1(t, t_0) r^{\Delta}(t_0) + h_2(t, t_0) r^{\Delta^2}(t_0) + o\left(g_2(t_0)\right). \tag{4.2}$$

Hence, we obtain

$$v^{(1)} = r^{\Delta}(t_0) + \frac{\tau_1}{2} r^{\Delta^2}(t_0) + o(\tau_1),$$
$$w = r^{\Delta^2}(t_0) + o(1). \tag{4.3}$$

Consequently, if $\tau_i \to 0$ for $i \in \{1,2\}$, then $v^{(1)} \to r^{\Delta}(t_0)$ and $w \to r^{\Delta^2}(t_0)$.

These vectors, if linearly independent, determine the limiting position of the plane E passing through the points $P_0, P_0^{\sigma}, Q_1, Q_2$. □

Corollary 4.2. *If the vectors $r^{\Delta}(t_0)$ and $r^{\Delta^2}(t_0)$ are collinear, then the limit position of considering plane is not determined. For instance, take a straight line*

$$r(t) = a + bt, \tag{4.4}$$

where a, b are constant vectors and $t \in \mathbb{T}$. Then

$$r^{\Delta}(t_0) = b, \qquad r^{\Delta^2}(t_0) = 0, \tag{4.5}$$

so the osculating plane of the straight line is not determined uniquely. If $r^{\Delta}(t)$ and $r^{\Delta^2}(t)$ are collinear, then the corresponding point of Γ is called the straightening point of Γ.

Theorem 4.3. *The osculating plane of a planar curve coincides with the plane containing this curve.*

Proof. Let us consider the Taylor expansion of the position vector $r(t)$ at the neighborhood of P_0:

$$r(t_0 + \tau) = h_0(t, t_0) r(t_0) + h_1(t, t_0) r^{\Delta}(t_0) + h_2(t, t_0) r^{\Delta^2}(t_0) + o\big(g_2(t, t_0)\big). \tag{4.6}$$

The curve $\overline{\Gamma}$, determined by the expantion,

$$\overline{r} = h_0(t, t_0) r(t_0) + h_1(t, t_0) r^{\Delta}(t_0) + h_2(t, t_0) r^{\Delta^2}(t_0) \tag{4.7}$$

is situated in the osculating plane of Γ at P_0; the difference between the position vectors of Γ and $\overline{\Gamma}$ is a sufficiently small vector

$$r(t_0 + \tau) - \overline{r}(\tau) = o\big(g_2(t, t_0)\big). \tag{4.8}$$

Hence a sufficiently small neighborhood of P_0 on the space curve Γ is near to the planar curve $\overline{\Gamma}$ situated in the osculating plane of Γ at P_0. □

Now let us write the equation of the osculating plane of Γ at P_0. Let $\widehat{r}$ be the position vector of the osculating plane. Since r^{Δ} and r^{Δ^2} span the osculating plane, the vector product $r^{\Delta} \times r^{\Delta^2}$ is orthogonal to the osculating plane. The vector $\widehat{r} - r(t_0)$ belongs to the osculating plane; therefore, the inner product of these vectors is equal to zero:

$$(\widehat{r} - r(t_0)) \cdot \left(r^{\Delta} \times r^{\Delta^2}\right) = 0. \tag{4.9}$$

With respect to coordinate functions, this equation has the following form:

$$\det\begin{pmatrix} \widehat{x} - x(t_0) & x^{\Delta} & x^{\Delta^2} \\ \widehat{y} - y(t_0) & y^{\Delta} & y^{\Delta^2} \\ \widehat{z} - z(t_0) & z^{\Delta} & z^{\Delta^2} \end{pmatrix} = 0. \tag{4.10}$$

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