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Research Article

Spectral Properties and Finite Pole Assignment of Linear Neutral Systems in Banach Spaces

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We will consider a pole assignment problem for a class of linear neutral functional differential equations in Banach spaces. We will think of the neutral system studied as that of involving no time delays and reduce the study of adjoint semigroups and spectral properties of neutral equations to those of Cauchy problems. Under the assumption that both the control and eigenspace of pole are finite dimensional, we establish the rank conditions for finite pole assignability.

1. Introduction

Consider the linear system on some Banach space X

$$\frac{dy(t)}{dt} = Ay(t), \quad t \geq 0,\ y(0) = x \in X, \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup e^{tA}, $t \geq 0$. A *mild solution* of (1.1) is defined as $y(t) = e^{tA}x \in X$ for any $t \geq 0$. The null solution of (1.1) is said to be *(exponentially) stable* if, for any initial $x \in X$, the corresponding mild solution, is $y(t) \to 0$ as $t \to \infty$. It may be shown that the null solution is stable if and only if there exist positive numbers $\alpha \geq 1$, $\mu > 0$ such that, for all $t \geq 0$, $\|e^{tA}\| \leq \alpha e^{-\mu t}$. If the null solution of system (1.1) is unstable,then it is important to consider stabilizability problem of its linear control system

$$\frac{dy(t)}{dt} = Ay(t) + Mu(t), \quad t \geq 0,\ y(0) = x \in X, \tag{1.2}$$

where M is a bounded linear operator from some Banach space U of control parameters into X. The mild solution of (1.2) is well defined for every locally integrable control function $u(t)$, $t \geq 0$, and is given by the form

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A} Mu(s)ds, \quad t \geq 0. \tag{1.3}$$

System (1.2) is said to be *feedback (exponentially) stabilizable* if there exists a bounded linear operator K from X into U such that the system

$$\frac{dy(t)}{dt} = (A + MK)y(t), \quad t \geq 0, \; y(0) = x \in X, \tag{1.4}$$

is exponentially stable.

Stability and feedback stabilization problems of the above systems and relevant nonlinear extensions, which play an important role in control theory and related topics, have been studied extensively by many researchers over the last two decades. The reader is referred to, for instance, the monograph of Luo et al.in [1] for a comprehensive statement about this topic and its applications.

If we incorporate extra structure into A, the stability and stabilizability problem would become complicated. One of the most important situations is to perturb A appropriately by a time-delay term so as that a strongly continuous family of bounded linear operators $G(t)$ satisfying proper quasisemigroup properties completely describes the dynamics of the system studied. This idea therefore leads to the consideration of a class of linear time-delay systems

$$\frac{dy(t)}{dt} = Ay(t) + \int_{-r}^{0} d\eta(\theta)y(\theta + t), \quad t \geq 0, \tag{1.5}$$

where $r > 0$, A generates a C_0-semigroup e^{tA}, $t \geq 0$, and η is the Stieltjes measures given by

$$\eta(\tau) = -\sum_{i=1}^{m} \chi_{(-\infty,-r_i]}(\tau)A_i - \int_{\tau}^{0} A_0(\theta)d\theta, \quad \tau \in [-r, 0]. \tag{1.6}$$

Here A_i, $i = 1, \ldots, m$, and $A_0(\theta)$, $\theta \in [-r, 0]$, are properly defined linear, bounded operators from X into X (cf., Wu [2]).

To our knowledge, very little paper has been done on feedback stabilization of infinite-dimensional control systems with memory. The only papers in this area are those by Yamamoto [3], Nakagiri and Yamamoto [4], Da Prato and Lunardi [5], and Jeong [6], all of which are devoted to retarded systems. In [4], the rank condition for exponential stabilizability in terms of eigenvectors and controllers was established.

In the present paper, we will study the finite pole assignability problem for a class of neutral linear control system

$$\frac{d}{dt}\left[y(t) - \int_{-r}^{0} d\zeta(\theta)y(\theta+t)\right] = Ay(t) + \int_{-r}^{0} d\eta(\theta)y(\theta+t) + Mu(t), \quad t \geq 0,$$
$$y(0) = \phi_0, \qquad y_0(\cdot) = \phi_1(\cdot), \tag{1.7}$$

where $\phi = (\phi_0, \phi_1)$ is some initial datum to be identified later. Generally speaking, for neutral systems as above it is quite difficult to study stabilizability problem and there are few satisfactory results in this respect. The reason is that, as pointed out in the study by Salamon in [7], it is generally required a memory feedback involving derivative terms for the purposes of stabilization of (1.7) even in finite-dimensional cases. Thus we shall study in this work a weaker concept, finite pole assignability, for (1.7) by means of state feedback law which does not necessarily contain derivative terms. To this end, the whole paper is divided into five sections. After reviewing some useful notions and notations, we will establish in Section 2 a semigroup theory which enables us to reduce the neutral systems (1.7) to a class of control systems involving no delays in an appropriate infinite-dimensional space. In order to formulate systems (1.7) in the L^2 product space setting, we restrict ourselves to the case that the neutral delay term on the left-hand side of (1.7) does not involve discrete delays. The associated semigroup is well defined by a solution state $(y(t), y_t)$, where y_t denotes a t-segment of solutions, a situation which is different from that in-Burns et al. in [8]. The infinitesimal generator of this semigroup is explicitly described and its relationship with neutral resolvent operators is explored. In Section 3, we will establish an adjoint theory which will play an important role in the study of the usual controllability and stabilizability. Sections 4 and 5 are devoted to the investigation of spectral properties and pole assignability, respectively. Under suitable conditions such as the finite dimensionality of spectral modes, we will establish useful criteria of finite pole assignability.

The real and complex number vector spaces are denoted by $\mathbb{R}^n$ and $\mathbb{C}^n$, $n \geq 1$, respectively. Also, $\mathbb{R}_+$ denotes the set of all nonnegative numbers. For any $\lambda \in \mathbb{C}^1$, the symbols $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ denote the real and imaginary parts of complex number λ, respectively. Let X and U be complex, separable Banach spaces and X^*, U^* their adjoint spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_U$ and the dual pairings $\langle\cdot,\cdot\rangle_{X,X^*}$ and $\langle\cdot,\cdot\rangle_{U,U^*}$, respectively. We use $\mathcal{L}(U,X)$ to denote the space consisting of all bounded linear operators T from U into X with domain U. When $X = U$, $\mathcal{L}(X,X)$ is denoted by $\mathcal{L}(X)$. Each operator norm is simply denoted by $\|\cdot\|$ when there is no danger of confusion. For any operator T, we employ $\mathfrak{D}(T)$ to denote the domain of T, and the symbols $\operatorname{Ker} T$ and $\operatorname{Im} T$ will be used to denote the kernel and image of operator T, respectively. For any fixed constant $r > 0$, we denote by $L_r^2 = L^2([-r,0];X)$ the space of all X-valued equivalence classes of measurable functions which are squarely integrable on $[-r,0]$. Let $\mathcal{X}$ denote the Banach space $X \times L_r^2$ with the norm

$$\|\phi\|_{\mathcal{X}} = \sqrt{\|\phi_0\|_X^2 + \|\phi_1\|_{L_r^2}^2}, \quad \forall\phi = (\phi_0, \phi_1) \in \mathcal{X}. \tag{1.8}$$

Let $W^{1,2}([-r,0];X)$ denote the Sobolev space of X-valued functions $x(t)$ on $[-r,0]$ such that $x(t)$ and its distributional derivative belong to $L^2([-r,0];X)$.

2. Neutral Control Systems

Consider the following neutral linear functional differential equation on the Banach space X:

$$\begin{gathered}\frac{d}{dt}\left[y(t)-\int_{-r}^{0}d\zeta(\theta)y(\theta+t)\right]=Ay(t)+\int_{-r}^{0}d\eta(\theta)y(\theta+t),\quad t\geq 0,\\ y(0)=\phi_0,\quad y_0(\cdot)=\phi_1(\cdot),\quad \phi=(\phi_0,\phi_1)\in\mathcal{X},\end{gathered}\tag{2.1}$$

where A generates a C_0-semigroup e^{tA}, $t\geq 0$, $y_0(\theta)=y(\theta)$, $\theta\in[-r,0]$, and η,ζ are the Stieltjes measures given by

$$\begin{gathered}\eta(\tau)=-\sum_{i=1}^{m}\chi_{(-\infty,-r_i]}(\tau)A_i-\int_{\tau}^{0}A_0(\theta)d\theta,\quad \tau\in[-r,0],\\ \zeta(\tau)=-\int_{\tau}^{0}B_0(\theta)d\theta,\quad \tau\in[-r,0].\end{gathered}\tag{2.2}$$

Here $0<r_1<r_2<\cdots<r_m\leq r$, $A_i\in\mathcal{L}(X)$, $i=1,\ldots,m$, $A_0(\cdot)\in L^2([-r,0];\mathcal{L}(X))$, and $B_0(\cdot)\in L^2([-r,0];\mathcal{L}(X))$. Unless otherwise specified, we always use $\int_{-r}^{0}d\eta(\theta)y(t+\theta):\ y\in L^2([-r,T];X)\to X$, $T\geq 0$, to denote the bounded, linear extension of the mapping

$$\int_{-r}^{0}d\eta(\theta)y(t+\theta)=\sum_{i=1}^{m}A_iy(t-r_i)+\int_{-r}^{0}A_0(\theta)y(t+\theta)d\theta,\quad y\in C([-r,T];X),\tag{2.3}$$

for any $t\geq 0$, and the same remark applies to $\int_{-r}^{0}d\zeta(\theta)y(t+\theta):\ y\in L^2([-r,T];X)\to X$ in an obvious way.

We also wish to consider the hereditary neutral controlled system of (2.1) on X:

$$\begin{gathered}\frac{d}{dt}\left[y(t)-\int_{-r}^{0}d\zeta(\theta)y(\theta+t)\right]=Ay(t)+\int_{-r}^{0}d\eta(\theta)y(\theta+t)+Mu(t),\quad t\geq 0,\\ y(0)=\phi_0,\quad y_0(\cdot)=\phi_1(\cdot),\quad \phi=(\phi_0,\phi_1)\in\mathcal{X},\qquad u\in L^2([0,\infty);U),\end{gathered}\tag{2.4}$$

where $M\in\mathcal{L}(U,X)$. A *mild solution* $y(t,\phi,u)$ of (2.4) is defined as the unique solution of the following integral equation on the Banach space X,

$$\begin{aligned}y(t,\phi,u)=&\int_{-r}^{0}d\zeta(\theta)y(t+\theta,\phi,u)+e^{tA}\left[\phi_0-\int_{-r}^{0}d\zeta(\theta)\phi_1(\theta)\right]\\ &+\int_{0}^{t}e^{(t-s)A}\left[\int_{-r}^{0}d\eta(\theta)y(s+\theta,\phi,u)+\int_{-r}^{0}A\,d\zeta(\theta)y(s+\theta,\phi,u)\right]ds\\ &+\int_{0}^{t}e^{(t-s)A}Mu(s)ds,\quad \forall t>0,\end{aligned}\tag{2.5}$$

with $y(0,\phi,u)=\phi_0$, $y_0(\cdot,\phi,u)=\phi_1(\cdot)$, and $\phi=(\phi_0,\phi_1)\in\mathcal{X}$.

To ensure the uniqueness and existence of mild solutions, we further assume that, for each i, $i = 1, \dots, m$, and $\theta \in [-r, 0]$, $\mathrm{Im}(B_0(\theta)) \subset \mathfrak{D}(A)$ such that $AB_0(\cdot) \in L^2([-r,0]; \mathcal{L}(X))$. Under these conditions, it has been shown in the study by Liu in [9] that there exists a unique mild solution $y(t, \phi, u)$ for (2.4) with $y(0, \phi, u) = \phi_0$ and $y_0(\cdot, \phi, u) = \phi_1(\cdot)$.

Note that, for any $\phi \in \mathcal{X}$, the mild solution $y(t, \phi)$ is continuous for $t > 0$. To see this, it suffices to notice that, for any $s, t > 0$,

$$\begin{aligned} &\left\| \int_{-r}^{0} d\zeta(\theta) y(\theta + s, \phi) d\theta - \int_{-r}^{0} d\zeta(\theta) y(\theta + t, \phi) d\theta \right\|_x \\ &\quad \le \|B_0\|_{L^2([-r,0];\mathcal{L}(X))} \|y_s(\cdot) - y_t(\cdot)\|_{L^2([-r,0];X)}. \end{aligned} \tag{2.6}$$

We define a mapping $\mathcal{S}(t)$ on $\mathcal{X}$, $t \ge 0$, by

$$\mathcal{S}(t)\phi = (y(t, \phi), y_t(\cdot, \phi)), \quad t \ge 0, \tag{2.7}$$

where $y_t(\cdot, \phi) = y(t + \cdot, \phi)$ for any $t \ge 0$. It turns out that $\mathcal{S}(t)$, $t \ge 0$, is a strongly continuous semigroup on $\mathcal{X}$.

Proposition 2.1. *For any $t \ge s \ge 0$ and $\phi \in \mathcal{X}$, the following relation holds:*

$$\mathcal{S}(t - s)(y(s, \phi), y_s(\cdot, \phi)) = (y(t, \phi), y_t(\cdot, \phi)). \tag{2.8}$$

That is,

$$\mathcal{S}(t - s)\mathcal{S}(s)\phi = \mathcal{S}(t)\phi. \tag{2.9}$$

Moreover, $\mathcal{S}(t)$ is a C_0-semigroup of bounded linear operators on $\mathcal{X}$.

Proof. The linearity of $\mathcal{S}(t)$ is obvious. Strong continuity of $\mathcal{S}(t)$ on $\mathcal{X}$ follows from the fact that $y(t, \phi) \to \phi_0$ in X as $t \to 0+$ by virtue of (2.5) and (2.6), and on the other hand, it is easy to see that $y_t(\cdot, \phi) \to \phi_1$ in $L^2([-r, 0]; X)$ as $t \to 0+$. In order to show the semigroup property (2.8), let $t \ge s$ and

$$\Phi(s) = \mathcal{S}(s)\phi = (y(s, \phi), y_s(\cdot, \phi)) \in \mathcal{X}. \tag{2.10}$$

Then from (2.5), it is easy to verify that

$$\begin{aligned}
&y(t-s,\Phi(s))-\int_{-r}^{0}d\zeta(\theta)y(t-s+\theta,\Phi(s))\\
&=e^{(t-s)A}\left(y(s,\phi)-\int_{-r}^{0}d\zeta(\theta)y(\theta+s,\phi)\right)\\
&\quad+\int_{s}^{t}e^{(t-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u-s+\theta,\Phi(s))+\int_{-r}^{0}A\,d\zeta(\theta)y(u-s+\theta,\Phi(s))\right]du\\
&=e^{(t-s)A}\Bigg[e^{sA}\left(\phi_0-\int_{-r}^{0}d\zeta(\theta)\phi_1(\theta)\right)\\
&\qquad+\int_{0}^{s}e^{(s-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u+\theta,\phi)+\int_{-r}^{0}A\,d\zeta(\theta)y(u+\theta,\phi)\right]du\\
&\qquad+\int_{s}^{t}e^{(t-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u-s+\theta,\Phi(s))+\int_{-r}^{0}A\,d\zeta(\theta)y(u-s+\theta,\Phi(s))\right]\Bigg]du\\
&=e^{tA}\left(\phi_0-\int_{-r}^{0}d\zeta(\theta)\phi_1(\theta)\right)+\int_{0}^{s}e^{(t-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u+\theta,\phi)+\int_{-r}^{0}A\,d\zeta(\theta)y(u+\theta,\phi)\right]du\\
&\quad+\int_{s}^{t}e^{(t-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u-s+\theta,\Phi(s))+\int_{-r}^{0}A\,d\zeta(\theta)y(u-s+\theta,\Phi(s))\right]du.
\end{aligned}\tag{2.11}$$

On the other hand, we have for $t\ge s$ that

$$\begin{aligned}
y(t,\phi)&-\int_{-r}^{0}d\zeta(\theta)y(t+\theta,\phi)\\
&=e^{tA}\left(\phi_0-\int_{-r}^{0}d\zeta(\theta)\phi_1(\theta)\right)\\
&\quad+\int_{0}^{s}e^{(t-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u+\theta,\phi)+\int_{-r}^{0}A\,d\zeta(\theta)y(u+\theta,\phi)\right]du\\
&\quad+\int_{s}^{t}e^{(t-u)A}\left[\int_{-r}^{0}d\eta(\theta)y(u+\theta,\phi)+\int_{-r}^{0}A\,d\zeta(\theta)y(u+\theta,\phi)\right]du.
\end{aligned}\tag{2.12}$$

Thus, by the uniqueness of solutions of (2.5) with $M=0$, it implies that

$$y(t-s,\Phi(s))=y(t,\phi),\quad \text{for almost all } t\ge s.\tag{2.13}$$

Hence, $[\mathcal{S}(t-s)\mathcal{S}(s)\phi]_0=[\mathcal{S}(t)\phi]_0$ for all $t\ge s$, and so $[\mathcal{S}(t-s)\mathcal{S}(s)\phi]_1=[\mathcal{S}(t)\phi]_1$ in $L_r^2([-r,0];X)$. The semigroup property (2.8) is thus proved and the proof is complete. □

Let $\mathscr{A}$ be the infinitesimal generator of $\mathcal{S}(t)$ and denote $\mathcal{S}(t)$ simply by $e^{t\mathscr{A}}$. The next theorem explicitly describes the operator $\mathscr{A}$.

Theorem 2.2. *The infinitesimal generator $\mathscr{A}$ of $e^{t\mathscr{A}}$ is described by*

$$
\begin{aligned}
\mathfrak{D}(\mathscr{A}) &= \left\{ \phi = (\phi_0, \phi_1) \in \mathscr{X} : \phi_1 \in W^{1,2}([-r,0];X), \phi_0 = \phi_1(0) \in \mathfrak{D}(A) \right\}, \\
\mathscr{A}\phi &= \left(A\phi_0 + \int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta) + \int_{-r}^{0} d\eta(\theta)\phi_1(\theta), \phi_1'(\theta) \right),
\end{aligned}
\tag{2.14}
$$

for any $\phi = (\phi_0, \phi_1) \in \mathfrak{D}(\mathscr{A})$.

Proof. We denote by $\tilde{\mathscr{A}}$ and $\mathfrak{D}(\tilde{\mathscr{A}})$ the infinitesimal generator of $e^{t\mathscr{A}}$ and its domain, respectively. Let $\phi = (\phi_0, \phi_1) \in \mathfrak{D}(\tilde{\mathscr{A}})$ and

$$
\tilde{\mathscr{A}}\phi = (\psi_0, \psi_1). \tag{2.15}
$$

Since the second coordinate of $e^{t\mathscr{A}}\phi$ is the t-shift $y(t+\cdot)$, it follows immediately that

$$
\begin{aligned}
y(\theta) &= \phi_1(\theta) \in W^{1,2}([-r,0];X), \quad \theta \in [-r,0], \\
[7pt] \frac{d^+}{d\theta} y(\theta) &= \phi_1'(\theta) = \psi_1(\theta), \quad \text{in } L^2([-r,0];X), \ \theta \in [-r,0],
\end{aligned}
\tag{2.16}
$$

where $d^+/d\theta$ denotes the right-hand derivative. By redefining on the set of measure zero, we can suppose that $y(\theta) = \phi_1(\theta)$ is absolutely continuous from $[-r,0]$ to X by Theorem 2.2, p. 19, of [10]. Since $y(0) = \phi_0$, this implies that $\phi_1(0) = \phi_0$ and $y(\cdot) : [-r,\infty) \to X$ is strongly continuous. Then the functions $\int_{-r}^{0} d\eta(\theta) y(t+\theta)$ and $\int_0^t A\, d\zeta(\theta) y(t+\theta)$ are strongly continuous in $t \geq 0$ such that

$$
\begin{aligned}
\lim_{t \to 0+} \frac{1}{t} \int_0^t e^{(t-s)A} \int_{-r}^{0} d\eta(\theta) y(s+\theta) ds &= \int_{-r}^{0} d\eta(\theta)\phi_1(\theta), \\
\lim_{t \to 0+} \frac{1}{t} \int_0^t e^{(t-s)A} \int_{-r}^{0} A\zeta(\theta) y(s+\theta) ds &= \int_{-r}^{0} A\, d\zeta(\theta)\phi_1(\theta).
\end{aligned}
\tag{2.17}
$$

Applying (2.17) to the first coordinate of (2.15), we obtain that

$$\begin{aligned}
\varphi_0 &= \lim_{t\to 0+}\frac{1}{t}(y(t,\phi)-\phi_0)\\
&= \lim_{t\to 0+}\frac{1}{t}\left\{\int_{-r}^{0} d\zeta(\theta)y(t+\theta,\phi)+e^{tA}\left(\phi_0-\int_{-r}^{0} d\zeta(\theta)\phi_1(\theta)\right)\right.\\
&\qquad\left.+\int_0^t e^{(t-s)A}\left[\int_{-r}^{0} d\eta(\theta)y(s+\theta,\phi)+\int_{-r}^{0} A\,d\zeta(\theta)y(s+\theta,\phi)\right]ds-\phi_0\right\}\\
&= \lim_{t\to 0+}\frac{1}{t}\left\{\left[\int_{-r}^{0} d\zeta(\theta)y(t+\theta,\phi)-\int_{-r}^{0} d\zeta(\theta)\phi_1(\theta)\right]\right.\\
&\qquad-\left[e^{tA}\int_{-r}^{0} d\zeta(\theta)\phi_1(\theta)-\int_{-r}^{0} d\zeta(\theta)\phi_1(\theta)\right]\\
&\qquad\left.+\int_0^t e^{(t-s)A}\left[\int_{-r}^{0} d\eta(\theta)y(s+\theta,\phi)+\int_{-r}^{0} A\,d\zeta(\theta)y(s+\theta,\phi)\right]ds+\left[e^{tA}\phi_0-\phi_0\right]\right\}\\
&= \int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta)-\int_{-r}^{0} A\,d\zeta(\theta)\phi_1(\theta)+\int_{-r}^{0} d\eta(\theta)\phi_1(\theta)\\
&\quad+\int_{-r}^{0} A\,d\zeta(\theta)\phi_1(\theta)+\lim_{t\to 0+}\frac{1}{t}\left[e^{tA}\phi_0-\phi_0\right]\\
&= \int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta)+\int_{-r}^{0} d\eta(\theta)\phi_1(\theta)+\lim_{t\to 0+}\frac{1}{t}\left[e^{tA}\phi_0-\phi_0\right].
\end{aligned} \tag{2.18}$$

Hence, $\lim_{t\to 0+}t^{-1}(e^{tA}\phi_0-\phi_0)$ exists in X; that is, $\phi_0\in\mathfrak{D}(A)$, and

$$\varphi_0 = A\phi_0+\int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta)+\int_{-r}^{0} d\eta(\theta)\phi_1(\theta), \tag{2.19}$$

which shows that

$$\mathfrak{D}\left(\tilde{\mathcal{A}}\right)\subset\mathfrak{D}(\mathcal{A}),\quad \mathcal{A}\phi=\tilde{\mathcal{A}}\phi,\quad \text{for } \phi\in\mathfrak{D}\left(\tilde{\mathcal{A}}\right). \tag{2.20}$$

Next we will show the reverse inclusion. Let $\phi=(\phi_0,\phi_1)\in\mathfrak{D}(\mathcal{A})$; then it is easy to see that $y(\cdot,\phi)\in W^{1,2}([-r,T];X)$ for any $T>0$, from which (2.17) follow. Combining this with $\phi_0=\phi_1(0)\in\mathfrak{D}(A)$, we see that

$$\lim_{t\to 0+}\frac{1}{t}(y(t,\phi)-\phi_0)=A\phi_0+\int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta)+\int_{-r}^{0} d\eta(\theta)\phi_1(\theta). \tag{2.21}$$

Noting that

$$\begin{aligned}\frac{y_t(\theta,\phi)-\phi_1(\theta)}{t}-\phi_1'(\theta) &= \frac{y(t+\theta,\phi)-y(\theta,\phi)}{t}-y'(\theta,\phi)\\ &= \left(\frac{1}{t}\right)\int_0^t \left(y'(s+\theta,\phi)-y'(\theta,\phi)\right)ds\end{aligned} \tag{2.22}$$

for $\theta \in [-r,0]$, we obtain by using Hölder inequality that

$$\left\|\frac{1}{t}(y_t(\cdot,\phi)-\phi_1)-\phi_1'\right\|_{L_r^2}^2 \leq \frac{1}{t}\int_0^t\left[\int_{-r}^0 \|y'(s+\theta,\phi)-y'(\theta,\phi)\|_X^2 d\theta\right]ds. \tag{2.23}$$

This implies that $\lim_{t\to 0+}t^{-1}(y_t(\cdot,\phi)-\phi_1)$ exists in $L^2([-r,0];X)$ and equals ϕ_1'. Therefore, we prove that $\mathfrak{D}(\mathcal{A}) \subset \mathfrak{D}(\widetilde{\mathcal{A}})$ and $\mathcal{A}\phi = \widetilde{\mathcal{A}}\phi$ for $\phi \in \mathfrak{D}(\mathcal{A})$, and (2.14) are shown. □

For each $\lambda \in \mathbb{C}^1$, define the densely defined, closed linear operator $\Delta(\lambda, A, \eta, \zeta)$ by

$$\Delta(\lambda, A,\eta,\zeta) = \lambda I - A - \int_{-r}^0 e^{\lambda\theta}d\eta(\theta) - \int_{-r}^0 \lambda e^{\lambda\theta}d\zeta(\theta). \tag{2.24}$$

The *neutral resolvent set* $\rho(A,\eta,\zeta)$ is defined as the set of all values λ in $\mathbb{C}^1$ for which the operator $\Delta(\lambda, A,\eta,\zeta)$ has a bounded inverse on X.

Proposition 2.3. *For any $\lambda \in \mathbb{C}^1$, the relation*

$$(\lambda I - \mathcal{A})(\phi_1(0),\phi_1) = (\psi_0,\psi_1) \in \mathcal{X}, \tag{2.25}$$

is equivalent to

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau \in W^{1,2}([-r,0];X), \tag{2.26}$$

$$\begin{aligned}\Delta(\lambda, A,\eta,\zeta)\phi_1(0) = \ &\psi_0 + \int_{-r}^0 d\eta(\theta)\int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau + \int_{-r}^0 \lambda\, d\zeta(\theta)\int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau\\ &-\int_{-r}^0 d\zeta(\theta)\psi_1(\theta)d\theta.\end{aligned} \tag{2.27}$$

In particular, the resolvent set $\rho(\mathcal{A})$ is equal to $\rho(A,\eta,\zeta)$.

Proof. Note that the relation (2.25) is equivalent to

$$\psi_0 = \lambda\phi_1(0) - A\phi_1(0) - \int_{-r}^{0} d\eta(\theta)\phi_1(\theta) - \int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta), \tag{2.28}$$

$$\psi_1(\theta) = \lambda\phi_1(\theta) - \frac{d\phi_1(\theta)}{d\theta}, \quad \theta \in [-r, 0]. \tag{2.29}$$

The variation of constants formula for the ordinary differential equation (2.29) on $[-r, 0]$ shows that

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_{\theta}^{0} e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau \in W^{1,2}([-r, 0]; X). \tag{2.30}$$

In order to show (2.27), note that, from (2.28), we have

$$(\lambda I - A)\phi_1(0) = \psi_0 + \int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta) + \int_{-r}^{0} d\eta(\theta)\phi_1(\theta). \tag{2.31}$$

Substituting (2.30) into (2.31) immediately yields the desired (2.27). The equality of resolvent sets $\rho(\mathcal{A})$ and $\rho(A, \eta, \zeta)$ is easily seen according to the equivalence of (2.25) and (2.26), (2.27). The proof is now complete. □

3. Adjoint Systems

In the remainder of this work, unless otherwise specified, we always assume that X is reflexive. As indicated in the study by Hale in [11], the adjoint theory of neutral linear functional differential equations in $C([-r, 0]; X)$ is quite complicated. However, for the control equation (2.4), it is possible to construct an elementary adjoint theory for $\mathcal{S}(t)$.

Let $\psi^* = (\psi_0^*, \psi_1^*) \in \mathcal{X}^*$ and define a "formal" transposed neutral system of (2.1) on X^* by

$$\begin{gathered} \frac{d}{dt}\left[y(t) - \int_{-r}^{0} d\zeta^*(\theta)y(t+\theta)\right] = A^*y(t) + \int_{-r}^{0} d\eta^*(\theta)y(t+\theta), \quad t > 0, \\ y(0) = \psi_0^*, \quad y_0(t) = \psi_1^*(t), \quad t \in [-r, 0), \end{gathered} \tag{3.1}$$

where $\eta^*(\theta)$, $\zeta^*(\theta)$ and A^* denote the adjoint operators of $\eta(\theta)$, $\zeta(\theta)$, and A, respectively. It is well known that A^* generates a C_0-semigroup $S^*(t)$ on X^* which is the adjoint of $S(t)$, $t \geq 0$. For any $\lambda \in \mathbb{C}^1$, define

$$\Delta(\lambda, A^*, \eta^*, \zeta^*) = \lambda I - A^* - \int_{-r}^{0} e^{\lambda\theta}d\eta^*(\theta) - \int_{-r}^{0} \lambda e^{\lambda\theta}d\zeta^*(\theta). \tag{3.2}$$

Proposition 3.1. *For any $\lambda \in \mathbb{C}^1$, the equation*

$$\left(\overline{\lambda} I - \mathcal{A}^*\right)(\phi_0^*, \phi_1^*) = (\psi_0^*, \psi_1^*(\cdot)) \tag{3.3}$$

is equivalent to

$$\Delta\left(\overline{\lambda}, A^*, \eta^*, \zeta^*\right)\phi_0^* = \psi_0^* + \int_{-r}^{0} e^{\overline{\lambda}\theta}\psi_1^*(\theta)d\theta, \tag{3.4}$$

$$\phi_1^*(\theta) = \int_{-r}^{\theta} e^{\overline{\lambda}(\tau-\theta)}\psi_1^*(\tau)d\tau - B_0^*(\theta)\phi_0^* + \int_{-r}^{\theta} e^{\overline{\lambda}(\tau-\theta)}d\eta^*(\tau)\phi_0^* + \int_{-r}^{\theta} \overline{\lambda} e^{\overline{\lambda}(\tau-\theta)}B_0^*(\tau)d\tau\phi_0^* \tag{3.5}$$

almost everywhere for any $\psi^* = (\psi_0^*, \psi_1^*) \in \mathcal{X}^*$.

Proof. Note that $(\overline{\lambda} I - \mathcal{A}^*)^{-1} = ((\lambda I - \mathcal{A})^{-1})^*$, so we may calculate the adjoint operator of $(\lambda I - \mathcal{A})^{-1}$. In view of (2.26), it is not difficult to see that, for any $\psi = (\psi_0, \psi_1) \in \mathcal{X}$,

$$\begin{aligned}
&\left\langle(\psi_0, \psi_1), ((\lambda I - \mathcal{A})^{-1})^*(\psi_0^*, \psi_1^*)\right\rangle_{\mathcal{X},\mathcal{X}^*} \\
&\quad= \left\langle(\lambda I - \mathcal{A})^{-1}(\psi_0, \psi_1), (\psi_0^*, \psi_1^*)\right\rangle_{\mathcal{X},\mathcal{X}^*} \\
&\quad= \langle\phi_1(0), \psi_0^*\rangle_{X,X^*} + \int_{-r}^{0}\left\langle e^{\lambda\theta}\phi_1(0) - \int_{0}^{\theta} e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \psi_1^*(\theta)\right\rangle_{X,X^*} d\theta \\
&\quad= \langle\phi_1(0), \psi_0^*\rangle_{X,X^*} + \int_{-r}^{0}\left\langle e^{\lambda\theta}\phi_1(0), \psi_1^*(\theta)\right\rangle_{X,X^*} d\theta - \int_{-r}^{0}\int_{0}^{\theta}\left\langle e^{\lambda(\theta-\tau)}\psi_1(\tau), \psi_1^*(\theta)\right\rangle_{X,X^*} d\tau\, d\theta \\
&\quad= \left\langle\phi_1(0), \psi_0^* + \int_{-r}^{0} e^{\overline{\lambda}\theta}\psi_1^*(\theta)d\theta\right\rangle_{X,X^*} + \int_{-r}^{0}\int_{\theta}^{0}\left\langle e^{\lambda(\theta-\tau)}\psi_1(\tau), \psi_1^*(\theta)\right\rangle_{X,X^*} d\tau\, d\theta.
\end{aligned} \tag{3.6}$$

We reformulate the expression in (3.6), starting with the last term:

$$\begin{aligned}
\int_{-r}^{0}\int_{\theta}^{0}\left\langle e^{\lambda(\theta-\tau)}\psi_1(\tau), \psi_1^*(\theta)\right\rangle_{X,X^*} d\tau\, d\theta &= \int_{-r}^{0}\int_{-r}^{\tau}\left\langle e^{\lambda(\theta-\tau)}\psi_1(\tau), \psi_1^*(\theta)\right\rangle_{X,X^*} d\theta\, d\tau \\
&= \int_{-r}^{0}\left\langle\psi_1(\tau), \int_{-r}^{\tau} e^{\overline{\lambda}(\theta-\tau)}\psi_1^*(\theta)d\theta\right\rangle_{X,X^*} d\tau.
\end{aligned} \tag{3.7}$$

Letting $\kappa^* = \psi_0^* + \int_{-r}^0 e^{\overline{\lambda}\theta}\psi_1^*(\theta)d\theta \in X^*$ and applying Proposition 2.3 to such a $\phi = (\phi_1(0), \phi_1)$ yield that

$$\begin{aligned}
&\left\langle \phi_1(0), \psi_0^* + \int_{-r}^0 e^{\overline{\lambda}\theta}\psi_1^*(\theta)d\theta \right\rangle_{X,X^*} \\
&\quad = \langle \phi_1(0), \kappa^* \rangle_{X,X^*} \\
&\quad = \left\langle \Delta(\lambda, A, \eta, \zeta)^{-1}\left[\psi_0 - \int_{-r}^0 B_0(\theta)\psi_1(\theta)d\theta + \int_{-r}^0 B_0(\tau)\int_\tau^0 \lambda e^{\lambda(\tau-\theta)}\psi_1(\theta)d\theta d\tau \right.\right. \\
&\qquad\qquad \left.\left. + \int_{-r}^0 d\eta(\tau)\int_\tau^0 e^{\lambda(\tau-\theta)}\psi_1(\theta)d\theta\right], \kappa^* \right\rangle_{X,X^*} \\
&\quad = \left\langle \psi_0, \left(\Delta(\lambda, A, \eta, \zeta)^{-1}\right)^*\kappa^* \right\rangle_{X,X^*} - \int_{-r}^0 \left\langle \psi_1(\theta), B_0^*(\theta)\left(\Delta(\lambda, A, \eta, \zeta)^{-1}\right)^*\kappa^* \right\rangle_{X,X^*} d\theta \\
&\qquad + \int_{-r}^0 \left\langle \psi_1(\theta), \int_{-r}^\theta \left(e^{\overline{\lambda}(\tau-\theta)}d\eta^*(\tau) + \overline{\lambda}e^{\overline{\lambda}(\tau-\theta)}B_0^*(\tau)d\tau\right)\left(\Delta(\lambda, A, \eta, \zeta)^{-1}\right)^*\kappa^* \right\rangle_{X,X^*} d\theta.
\end{aligned} \tag{3.8}$$

If we combine these equalities and use the fact that $\phi_0^* = (\Delta(\lambda, A, \eta, \zeta)^*)^{-1}\kappa^*$, then we obtain

$$\begin{aligned}
&\left\langle (\psi_0, \psi_1), \left((\lambda I - \mathcal{A})^{-1}\right)^*(\psi_0^*, \psi_1^*) \right\rangle_{\mathcal{X},\mathcal{X}^*} \\
&\quad = \langle \psi_0, \phi_0^* \rangle_{X,X^*} - \int_{-r}^0 \langle \psi_1(\theta), B_0^*(\theta)\phi_0^* \rangle_{X,X^*} d\theta \\
&\qquad + \int_{-r}^0 \left\langle \psi_1(\theta), \int_{-r}^\theta \left(e^{\overline{\lambda}(\tau-\theta)}d\eta^*(\tau) + \overline{\lambda}e^{\overline{\lambda}(\tau-\theta)}B_0^*(\tau)d\tau\right)\phi_0^* + \int_{-r}^\theta e^{\overline{\lambda}(\tau-\theta)}\psi_1^*(\tau)d\tau \right\rangle_{X,X^*} d\theta,
\end{aligned} \tag{3.9}$$

and this proves the desired result. The proof is now complete. □

The following corollary which characterizes the infinitesimal generator $\mathcal{A}^*$ of the semigroup $\mathcal{S}^*(t)$ on $\mathcal{X}^*$ is a direct result of Proposition 3.1.

Corollary 3.2. *The infinitesimal generator $\mathcal{A}^*$ of $\mathcal{S}^*(t)$ is given by*

$$\begin{aligned}
\mathfrak{D}(\mathcal{A}^*) = \Bigg\{ &\phi^* = (\phi_0^*, \phi_1^*) \in \mathcal{X}^* : \ \phi_0^* \in \mathfrak{D}(A^*), \ \phi_1^*(\theta) = \int_{-r}^\theta d\eta^*(\tau)\phi_0^* - B_0^*(\theta)\phi_0^* - \varphi^*(\theta), \\
&\text{a.e. } \ \theta \in [-r, 0] \ \ \text{where} \ \ \varphi^*(\cdot) \in W^{1,2}([-r, 0]; X^*) \Bigg\},
\end{aligned} \tag{3.10}$$

and moreover

$$\mathcal{A}^*\phi^* = \left(A^*\phi_0^* + \phi_1^*(0+) + B_0^*(0+)\phi_0^*, \frac{d}{d\theta}\left[\int_{-r}^{\theta} d\eta^*(\tau)\phi_0^* - \phi_1^*(\theta) - B_0^*(\theta)\phi_0^*\right]\right),$$
$$\phi^* = (\phi_0^*, \phi_1^*) \in \mathfrak{D}(\mathcal{A}^*), \tag{3.11}$$

where $\phi_1^*(0+) + B_0^*(0+)\phi_0^*$ *is given by the limit* $\lim_{\theta\to 0+}(\phi_1^*(\theta) + B_0^*(\theta)\phi_0^*)$.

Proof. Let $\lambda = 0$ in (3.4) and (3.5),then it follows that

$$\left(-A^* - \int_{-r}^{0} d\eta^*(\theta)\right)\phi_0^* = \psi_0^* + \int_{-r}^{0}\psi_1^*(\theta)d\theta, \tag{3.12}$$

$$\phi_1^*(\theta) = \int_{-r}^{\theta}\psi_1^*(\tau)d\tau - B_0^*(\theta)\phi_0^* + \int_{-r}^{\theta} d\eta^*(\tau)\phi_0^*, \quad a.e.\ \ \theta \in [-r, 0]. \tag{3.13}$$

Since the term $\int_{-r}^{\theta} d\eta^*(\tau)\phi_0^*$ in (3.13) is left continuous at $\theta = 0$, then the sum $\phi_1^*(\theta) + B_0^*(\theta)\phi_0^*$ is also left continuous at $\theta = 0$. Then we see from (3.13) that the limit $\lim_{\theta\to 0+}(\phi_1^*(\theta) + B_0^*(\theta)\phi_0^*)$ exists in X and

$$\phi_1^*(0+) + B_0^*(0+)\phi_0^* = \int_{-r}^{0}\psi_1^*(\tau)d\tau + \int_{-r}^{0} d\eta^*(\tau)\phi_0^*. \tag{3.14}$$

Substituting (3.14) into (3.12) yield

$$-\psi_0^* = A^*\phi_0^* + \phi_1^*(0+) + B_0^*(0+)\phi_0^*, \tag{3.15}$$

and letting $\lambda = 0$ in (3.5) and further taking derivative with respect to $\theta \in [-r, 0]$ yields

$$-\psi_1^*(\theta) = \frac{d}{d\theta}\left[\int_{-r}^{\theta} d\eta^*(\tau)\phi_0^* - \phi_1^*(\theta) - B_0^*(\theta)\phi_0^*\right] \tag{3.16}$$

from which the desired results are easily obtained. The proof is now complete. □

The *adjoint neutral resolvent set* $\rho(A^*, \eta^*, \zeta^*)$ is defined similarly as the set of all values λ in $\mathbb{C}^1$ for which the operator $\Delta(\lambda, A^*, \eta^*, \zeta^*)$ has a bounded inverse on X^*. Then by applying the adjoint version of Proposition 3.1, we see that $\rho(A^*) = \rho(A^*, \eta^*, \zeta^*)$.

4. Spectral Properties

In this section we investigate the spectral properties of operators $\mathcal{A}$ and $\mathcal{A}^*$ by means of $\Delta(\lambda, A, \eta, \zeta)$ and $\Delta(\overline{\lambda}, A^*, \eta^*, \zeta^*)$ in preceding sections. In the remainder of this paper, we denote $\Delta(\lambda, A, \eta, \zeta)$ by $\Delta(\lambda)$. Also recall that the *neutral spectrum* $\sigma(\Delta(\lambda, A, \eta, \zeta))$, or simply $\sigma(\Delta)$, is defined by $\sigma(\Delta) = \mathbb{C}^1 \setminus \rho(A, \eta, \zeta)$. The *spectrum* $\sigma(\Delta)$ of $\Delta(\lambda, A, \eta, \zeta)$ can be divided into three disjoint subsets in the following manner. The *continuous* spectrum $\sigma_C(\Delta)$ is the set of values of λ for which $\Delta(\lambda, A, \eta, \zeta)$ has an unbounded inverse with dense domain in X. The *residual* spectrum $\sigma_R(\Delta)$ is the set of values of λ for which $\Delta(\lambda, A, \eta, \zeta)$ has an inverse whose domain is not dense in X. The *point* spectrum $\sigma_P(\Delta)$ is the set of values of λ for which no inverse of $\Delta(\lambda, A, \eta, \zeta)$ exists. Define the subset $\sigma_d(\Delta)$ of $\sigma_P(\Delta)$ by

$$\sigma_d(\Delta) = \{\lambda : \lambda \in \sigma_P(\Delta) \text{ and } \dim \operatorname{Ker} \Delta(\lambda) \text{ is finite}\}. \tag{4.1}$$

Throughout this paper we suppose that $\sigma_R(\Delta) = \emptyset$ and $\sigma_d(\Delta)$ is a denumerable nonempty set. Further we suppose on $\sigma_d(\Delta)$ that, for each pair $\lambda_1, \lambda_2 \in \sigma_d(\Delta)$, there exists a continuous rectifiable arc $C \subset \rho(\Delta)$ joining λ_1 and λ_2. This condition implies that for any finite set Λ in $\sigma_d(\Delta)$ there exists a continuous rectifiable arc $C_\Lambda \subset \rho(\Delta)$ which surrounds Λ inside and contains no other points in $\sigma(\Delta)$.

We also denote $\Delta(\lambda, A^*, \eta^*, \zeta^*)$ by $\Delta^*(\lambda)$ and define the spectrum sets $\sigma(\Delta^*)$, $\sigma_P(\Delta^*)$ and $\sigma_d(\Delta^*)$, in a similar way to those for $\Delta(\lambda, A, \eta, \zeta)$. The proposition below shows some identical relations between the neutral point spectrum of $\mathcal{A}$, $\mathcal{A}^*$ and Δ, Δ^*.

Proposition 4.1. *The neutral point spectrum of $\mathcal{A}$ (resp., $\mathcal{A}^*$) satisfies that $\sigma_P(\mathcal{A}) = \sigma_P(\Delta)$ (resp., $\sigma_P(\mathcal{A}^*) = \sigma_P(\Delta^*)$) and $\sigma_d(\mathcal{A}) = \sigma_d(\Delta)$ (resp., $\sigma_d(\mathcal{A}^*) = \sigma_d(\Delta^*)$).*

Proof. Recall that, by Proposition 2.3, for any $\lambda \in \mathbb{C}^1$ the relation $(\lambda I - \mathcal{A})\phi = \psi$, $\phi \in \mathfrak{D}(\mathcal{A})$, $\psi \in \mathcal{X}$ is equivalent to the relation $\Delta(\lambda)\phi_1(0) = G_\lambda(\psi)$, $\phi_0 = \phi_1(0) \in \mathfrak{D}(A)$, where $G_\lambda(\psi)$ is given by

$$\begin{aligned} G_\lambda(\psi) &= \psi_0 + \int_{-r}^0 d\eta(\theta) \int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau \\ &\quad + \int_{-r}^0 \lambda d\zeta(\theta) \int_\theta^0 \lambda e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau - \int_{-r}^0 d\zeta(\theta)\psi_1(\theta)d\theta, \\ \phi_1(\theta) &= e^{\lambda\theta}\phi_1(0) + \int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad \theta \in [-r, 0] \end{aligned} \tag{4.2}$$

If we substitute $\psi = 0$ in the above equalities, we have that $\operatorname{Ker}(\lambda I - \mathcal{A}) = \{0\}$ is equivalent to $\operatorname{Ker}\Delta(\lambda) = \{0\}$, and hence $\operatorname{Ker}(\lambda I - \mathcal{A}) \neq \{0\}$ if and only if $\operatorname{Ker}\Delta(\lambda) \neq \{0\}$. This concludes, by definition, $\sigma_P(\mathcal{A}) = \sigma_P(\Delta)$. It is easy to see that $\phi = (\phi(0), \phi) \in \operatorname{Ker}(\lambda I - \mathcal{A})$ if and only if $\phi(0) \in \operatorname{Ker}\Delta(\lambda)$ and $\phi(\theta) = e^{\lambda\theta}\phi_1(0)$. By this equivalence it is easily seen that $\dim \operatorname{Ker} \Delta(\lambda) = \dim \operatorname{Ker}(\lambda I - \mathcal{A})$. This shows $\sigma_d(\mathcal{A}) = \sigma_d(\Delta)$. The other equalities $\sigma_P(\mathcal{A}^*) = \sigma_P(\Delta^*)$ and $\sigma_d(\mathcal{A}^*) = \sigma_d(\Delta^*)$ can be proved similarly. □

In what follows we omit the symbol I for the identity operator; for example, $\lambda - \mathcal{A}$ denotes $\lambda I - \mathcal{A}$. For each isolated point $\lambda \in \sigma(\mathcal{A})$, the spectral projection P_λ and the quasinilpotent operator Q_λ are defined, respectively, by

$$P_\lambda = \frac{1}{2\pi i}\int_{\gamma_\lambda} (z - \mathcal{A})^{-1} dz, \qquad Q_\lambda = \frac{1}{2\pi i}\int_{\gamma_\lambda} (z - \lambda)(z - \mathcal{A})^{-1} dz, \tag{4.3}$$

where γ_λ is a small circle with center λ such that its interior and γ_λ contain no points of $\sigma(\mathcal{A})$. Let $\mathcal{N}_\lambda = P_\lambda \mathcal{X}$ be the generalized eigenspace corresponding to the eigenvalue λ of A. It is obvious that

$$Q_\lambda^j = \frac{1}{2\pi i}\int_{\gamma_\lambda}(z-\lambda)^j(z-\mathcal{A})^{-1}dz, \quad j = 1,2,\ldots. \tag{4.4}$$

Further, if λ is a pole of $(z-\mathcal{A})^{-1}$ of order k_λ, then we have

$$Q_\lambda^{k_\lambda} = \mathrm{O}, \qquad \operatorname{Im} Q_\lambda \subset \mathcal{N}_\lambda, \tag{4.5}$$

$$\operatorname{Ker}(\lambda - \mathcal{A}) = \mathcal{N}_\lambda \cap \operatorname{Ker} Q_\lambda, \tag{4.6}$$

where O is the null operator, and the direct sum decompositions of $\mathcal{X}$

$$\mathcal{N}_\lambda = \operatorname{Ker}(\lambda - \mathcal{A})^{k_\lambda}, \qquad \mathcal{X} = \mathcal{N}_\lambda \oplus \operatorname{Im}(\lambda - \mathcal{A})^{k_\lambda} \tag{4.7}$$

hold (cf. Chapter 3 in Kato [12], Chapter 8 in Tanabe [13]). The relations (4.3)–(4.7) hold for $\mathcal{A}^*$ and each isolated $\lambda \in \sigma(\mathcal{A}^*)$. In view of Proposition 2.3, we see that

$$\operatorname{Ker}(\lambda - \mathcal{A}) = \left\{\left(\phi_0, e^{\lambda\cdot}\phi_0\right) : \Delta(\lambda)\phi_0 = 0\right\}. \tag{4.8}$$

Note that λ is a pole and each $\operatorname{Ker}(\lambda - \mathcal{A})$ and $\mathcal{N}_\lambda$ are finite dimensional if $\lambda \in \sigma_d(\Delta)$. Let $\Lambda \subset \sigma_d(\mathcal{A})$ be a finite set of isolated spectrum. Suppose that there exists a rectifiable Jordan curve γ_Λ which surrounds Λ and separates Λ and $\mathbb{C}^1 \setminus \Lambda$. We define the projection P_Λ on Λ by

$$P_\Lambda = \frac{1}{2\pi i}\int_{\gamma_\Lambda}(z-\mathcal{A})^{-1}dz. \tag{4.9}$$

Then the following decomposition of $\mathcal{X}$ holds:

$$\mathcal{X} = \mathcal{N}_\Lambda \oplus \mathcal{R}_\Lambda, \tag{4.10}$$

where

$$\mathcal{N}_\Lambda = P_\Lambda \mathcal{X}, \qquad \mathcal{R}_\Lambda = (I - P_\Lambda)\mathcal{X}, \tag{4.11}$$

and I is the identity operator on $\mathcal{X}$.

Now we introduce the bounded operator $F_\lambda : X^* \to L^2([-r,0];X^*)$ defined by

$$[F_\lambda \phi_0^*](\theta) = \left[-B_0^*(\theta) + \int_{-r}^{\theta} e^{\lambda(\tau-\theta)}d\eta^*(\tau) + \int_{-r}^{\theta}\lambda e^{\lambda(\tau-\theta)}B_0^*(\tau)d\tau\right]\phi_0^*, \quad \text{a.e. } \theta \in [-r,0] \tag{4.12}$$

for any $\lambda \in \mathbb{C}^1$. Then by applying the adjoint version of Proposition 3.1, we have

$$\operatorname{Ker}\left(\overline{\lambda} - \mathcal{A}^*\right) = \left\{(\phi_0^*, F_{\overline{\lambda}}\phi_0^*) : \Delta^*\left(\overline{\lambda}\right)\phi_0^* = 0\right\}, \tag{4.13}$$

where $\Delta^*(\overline{\lambda}) = \Delta(\overline{\lambda}, A^*, \eta^*, \zeta^*)$ and, if $\overline{\lambda} \in \sigma_d(\mathcal{A}^*) = \sigma_d(\Delta^*)$, then

$$\dim \operatorname{Ker}\left(\overline{\lambda} - \mathcal{A}^*\right) = \dim \operatorname{Ker} \Delta^*\left(\overline{\lambda}\right) < \infty. \tag{4.14}$$

Let $\lambda \in \sigma(\mathcal{A})$ be an isolated point. Then, by Kato [12], $\overline{\lambda} \in \sigma(\mathcal{A}^*)$ is also isolated and the eigenspace $\operatorname{Ker}(\overline{\lambda} - \mathcal{A}^*)$ and the generalized eigenspace $\mathcal{N}_{\overline{\lambda}}^* = (P_\lambda)^*\mathcal{X}^* = P_{\overline{\lambda}}^*\mathcal{X}^*$ are well defined, and

$$\dim \operatorname{Ker}(\lambda - \mathcal{A}) = \dim \operatorname{Ker}\left(\overline{\lambda} - \mathcal{A}^*\right) \leq \infty, \qquad \dim \mathcal{N}_\lambda = \dim \mathcal{N}_{\overline{\lambda}}^* \leq \infty, \tag{4.15}$$

where $P_{\overline{\lambda}}^*$ is the projection

$$P_{\overline{\lambda}}^* = \frac{1}{2\pi i}\int_{\gamma_\lambda^\vee} (z - \mathcal{A}^*)^{-1} dz \tag{4.16}$$

and $\gamma_\lambda^\vee$ is the mirror image of γ_λ in (4.3). Hence, we have the following result.

Proposition 4.2. *For each $\lambda \in \sigma_d(\Delta)$, one has $\overline{\lambda} \in \sigma_d(\Delta^*) = \sigma_d(\mathcal{A}^*)$ and*

$$\begin{aligned} \operatorname{Ker}\left(\overline{\lambda} - \mathcal{A}^*\right) &= \left\{(\phi_0^*, F_{\overline{\lambda}}\phi_0^*) : \phi_0^* \in \operatorname{Ker} \Delta^*\left(\overline{\lambda}\right)\right\}, \\ \dim \operatorname{Ker}\left(\overline{\lambda} - \mathcal{A}^*\right) &= \dim \operatorname{Ker} \Delta^*\left(\overline{\lambda}\right) < \infty, \\ \dim \mathcal{N}_{\overline{\lambda}}^* &= \dim \mathcal{N}_\lambda < \infty. \end{aligned} \tag{4.17}$$

5. Pole Assignment

In general, for system (2.1) we have no ideas whether or not the associated generator $\mathcal{A}$ in Theorem 2.2 satisfies the spectral determined growth condition

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{A})\} = \lim_{t\to\infty} \ln \frac{\|e^{t\mathcal{A}}\|}{t}. \tag{5.1}$$

Consequently, it is difficult for standard results, for example, those established by Hale [11], to be applied to the mild solution $y(\cdot, \phi)$ of (2.1). Instead of considering the stability and stabilizability problem for the control system (2.4), we will study in this section the finite pole assignment problem for (2.4) on the product space $\mathcal{X}$.

We are concerned about the finite pole assignment problem of the control system (2.4): under what conditions on M can we construct a feedback law such that any finite set in $\sigma_d(\Delta)$ is shifted to any preassigned set in the complex plane?

To this end, first note that, by means of $\mathcal{A}$, we can reformulate system (2.4) into the space $\mathcal{X}$ as a control system without delay:

$$\frac{dY(t)}{dt} = \mathcal{A}Y(t) + \mathcal{M}u(t), \quad t \geq 0, Y(0) = \phi = (\phi_0, \phi_1) \in \mathcal{X},\ u \in L^2([0,\infty);U), \tag{5.2}$$

where $\mathcal{M} : U \to \mathcal{X}$ is defined by $\mathcal{M}u = (Mu, 0)$, $u \in U$.

Definition 5.1. Let $\Lambda_0 = \{\lambda_1, \ldots, \lambda_l\} \subset \sigma_d(\Delta)$ and $\Lambda_1 = \{\mu_1, \ldots, \mu_l\}$ be finite sets in the complex plane. The control system (5.2) is said to be *pole assignable* with respect to (Λ_0, Λ_1) if and only if there exists a bounded linear operator $K \in \mathcal{L}(\mathcal{X}, U)$ such that

$$\sigma(\mathcal{A} + \mathcal{M}K) = (\sigma(\mathcal{A}) \setminus \Lambda_0) \cup \Lambda_1. \tag{5.3}$$

We remark that the operator K has the form

$$K\phi = K_0\phi_0 + \int_{-r}^{0} K_1(\theta)\phi_1(\theta)d\theta, \quad \phi = (\phi_0, \phi_1) \in \mathcal{X}, \tag{5.4}$$

where $K_0 \in \mathcal{L}(X, U)$ and $K_1 \in L^2([-r, 0]; \mathcal{L}(X, U))$.

We will show three results which are important in the subsequent finite pole assignability.

Proposition 5.2. *For arbitrary $\lambda \in \mathbb{C}^1$, the following relations are equivalent:*

(i) $\operatorname{Im}(\lambda - \mathcal{A}) + \operatorname{Im}\mathcal{M} = \mathcal{X}$;

(ii) $\operatorname{Im}\Delta(\lambda) + \operatorname{Im} M = X$.

Proof. Relation (i) holds if and only if, for any $\psi \in \mathcal{X}$, there exist $\phi = (\phi_0, \phi_1) \in \mathfrak{D}(\mathcal{A})$ with $\phi_0 = \phi_1(0) \in \mathfrak{D}(A)$ and $u \in U$ such that

$$(\lambda - \mathcal{A})\phi + \mathcal{M}u = \psi. \tag{5.5}$$

This is equivalent, in view of Proposition 2.3, to

$$\lambda\phi_1(0) - A\phi_1(0) - \textstyle\int_{-r}^{0} d\zeta(\theta)\phi_1'(\theta) - \int_{-r}^{0} d\eta(\theta)\phi_1(\theta) + Mu = \psi_0, \quad \phi_1(0) \in \mathfrak{D}(A), \tag{5.6}$$

$$\lambda\phi_1(\theta) - \frac{d}{d\theta}\phi_1(\theta) = \psi_1(\theta), \quad \theta \in [-r, 0]. \tag{5.7}$$

We solve the differential equation (5.7) to obtain

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad \theta \in [-r, 0]. \tag{5.8}$$

Substituting (5.8) into (5.6) and using Fubini's theorem, we have

$$\begin{aligned}\Delta(\lambda)\phi_1(0) + Mu = \psi_0 &+ \int_{-r}^0 d\eta(\theta)\int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau + \int_{-r}^0 d\zeta(\theta)\psi_1(\theta)\\ &+ \int_{-r}^0 d\zeta(\theta)\int_\theta^0 \lambda e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad \phi_1(0) \in \mathfrak{D}(A).\end{aligned} \tag{5.9}$$

Also, condition (ii) holds if and only if, for any $\varphi_0 \in X$, there exist $\kappa_0 \in \mathfrak{D}(A)$ and $u \in U$ such that

$$\Delta(\lambda)\kappa_0 + Mu = \varphi_0. \tag{5.10}$$

Assume that (i) holds and let $\varphi_0 \in X$ be an arbitrarily given vector. If we put $\psi = (\psi_0, \psi_1) = (\varphi_0, 0) \in \mathcal{X}$, then by virtue of (5.9) there exist $\phi = (\phi_1(0), \phi_1) \in \mathfrak{D}(\mathcal{A})$ and $u \in U$ such that

$$\Delta(\lambda)\phi_1(0) + Mu = \varphi_0, \quad \phi = \left(\phi_1(0), e^{\lambda\cdot}\phi_1(0)\right). \tag{5.11}$$

By setting $\kappa_0 = \phi_1(0)$, we have (5.10) so that (ii) is valid. Next, we will show the implication (ii) $\Rightarrow$ (i). To this end, assume that (ii) is valid and let $\psi = (\psi_0, \psi_1) \in \mathcal{X}$. If we put

$$\varphi_0 = \psi_0 + \int_{-r}^0 d\eta(\theta)\int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau + \int_{-r}^0 d\zeta(\theta)\int_\theta^0 \lambda e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau + \int_{-r}^0 d\zeta(\theta)\psi_1(\theta), \tag{5.12}$$

then by virtue of (5.10) we have

$$\Delta(\lambda)\kappa_0 + Mu = \varphi_0 \tag{5.13}$$

for some $\kappa_0 \in \mathfrak{D}(A)$ and $u \in U$. For such a vector κ_0, we define $\phi_1(\theta)$ by

$$\phi_1(\theta) = e^{\lambda\theta}\kappa_0 + \int_\theta^0 e^{\lambda(\theta-\tau)}\psi_1(\tau)d\tau, \quad \theta \in [-r, 0]. \tag{5.14}$$

Then the function $\phi_1(\theta)$ satisfies $\phi_1(0) = \kappa_0 \in \mathfrak{D}(A), \phi = (\phi_1(0), \phi_1(\cdot)) \in \mathfrak{D}(\mathcal{A})$ satisfies (5.6) and (5.7), and relation (i) is therefore proved to be valid. $\square$

Proposition 5.3. *For $\lambda \in \mathbb{C}^1$, the following relations are equivalent:*

(i) $\operatorname{Ker}(\overline{\lambda} - \mathcal{A}^*) \cap \operatorname{Ker} \mathcal{M}^* = \{0\}$;

(ii) $\operatorname{Ker} \Delta^*(\overline{\lambda}) \cap \operatorname{Ker} M^* = \{0\}$;

(iii) $\overline{\operatorname{Im}(\lambda - \mathcal{A}) + \operatorname{Im} \mathcal{M}} = \mathcal{X}$, *that is,* $\operatorname{Im}(\lambda - \mathcal{A}) + \operatorname{Im} \mathcal{M}$ *is dense in* $\mathcal{X}$

(iv) $\overline{\operatorname{Im} \Delta(\lambda) + \operatorname{Im} M} = X$, *that is,* $\operatorname{Im} \Delta(\lambda) + \operatorname{Im} M$ *is dense in* X.

Proof. We first show the equivalence of (i) and (ii). By the very definitions of adjoint operators and operator $\mathcal{M}$, we have that, for any $\phi^* = (\phi_0^*, \phi_1^*) \in \mathcal{X}^*$ and $u \in U$,

$$\begin{gathered}\langle u, M^* \phi_0^* \rangle_{U,U^*} = \langle Mu, \phi_0^* \rangle_{X,X^*}, \\ \langle u, \mathcal{M}^* \phi^* \rangle_{U,U^*} = \langle \mathcal{M}u, \phi^* \rangle_{\mathcal{X},\mathcal{X}^*} = \langle (Mu, 0), (\phi_0^*, \phi_1^*) \rangle_{\mathcal{X},\mathcal{X}^*} = \langle Mu, \phi_0^* \rangle_{X,X^*}.\end{gathered} \tag{5.15}$$

Thus, the condition $\phi^* \in \operatorname{Ker} \mathcal{M}^*$ is equivalent to the condition $\phi_0^* \in \operatorname{Ker} M^*$. Now assume that (i) holds and let $\phi_0^* \in \operatorname{Ker} \Delta^*(\overline{\lambda}) \cap \operatorname{Ker} M^*$. If we set

$$\phi^* = (\phi_0^*, \phi_1^*) = \left(\phi_0^*, -B_0^*(\cdot) + \int_{-r}^{\cdot} e^{\overline{\lambda}(\theta - \cdot)} d\eta^*(\theta) \phi_0^* + \int_{-r}^{\cdot} \overline{\lambda} e^{\overline{\lambda}(\theta - \cdot)} B_0^*(\theta) d\theta \phi_0^* \right) \in \mathcal{X}^*, \tag{5.16}$$

then, by virtue of Proposition 3.1, we have $\phi^* \in \operatorname{Ker}(\overline{\lambda} - \mathcal{A}^*) \cap \operatorname{Ker} \mathcal{M}^*$, so that, by (i), $\phi^* = (\phi_0^*, \phi_1^*) = 0$ and thus $\phi_0^* = 0$. This proves the implication (i) $\Rightarrow$ (ii). To show the converse implication, suppose that (ii) is true and let $\phi^* = (\phi_0^*, \phi_1^*) \in \operatorname{Ker}(\overline{\lambda} - \mathcal{A}^*) \cap \operatorname{Ker} \mathcal{M}^*$. Then again by virtue of Proposition 3.1, we have that $\phi_0^* \in \operatorname{Ker} \Delta^*(\overline{\lambda})$,

$$\phi^* = \left(\phi_0^*, -B_0^*(\cdot) + \int_{-r}^{\cdot} e^{\overline{\lambda}(\theta - \cdot)} d\eta^*(\theta) \phi_0^* + \int_{-r}^{\cdot} \overline{\lambda} e^{\overline{\lambda}(\theta - \cdot)} B_0^*(\theta) d\theta \phi_0^* \right), \tag{5.17}$$

and $\phi_0^* \in \operatorname{Ker} M^*$; hence $\phi_0^* = 0$ in view of (ii). Then $\phi^* = 0$, and thus relation (i) is shown to be true.Now we show the equivalence of (i) and (iii). Define the closed operator $[\lambda - \mathcal{A}, \mathcal{M}] : \mathfrak{D}(\mathcal{A}) \times U \subset \mathcal{X} \times U \to \mathcal{X}$ by

$$[\lambda - \mathcal{A}, \mathcal{M}](\phi, u) = (\lambda - \mathcal{A})\phi + \mathcal{M}u, \quad (\phi, u) \in \mathfrak{D}(\mathcal{A}) \times U. \tag{5.18}$$

Here $\mathcal{X} \times U$ is a complex Banach space equipped with the norm $\|(\phi, u)\|_{\mathcal{X} \times U} = \|\phi\|_{\mathcal{X}} + \|u\|_U$ for any $(\phi, u) \in \mathcal{X} \times U$. Then by the duality theorem, condition (iii) is equivalent to $\operatorname{Ker}[\lambda - \mathcal{A}, \mathcal{M}]^* = \{0\}$. By calculating the adjoint operator that involves duality pairings, we can readily verify that the adjoint $[\lambda - \mathcal{A}, \mathcal{M}]^* : \mathcal{X}^* \to \mathcal{X}^* \times U^*$ is given by

$$[\lambda - \mathcal{A}, \mathcal{M}]^* \phi^* = \left(\left(\overline{\lambda} I - \mathcal{A}^* \right) \phi^*, \mathcal{M}^* \phi^* \right), \quad \phi^* \in \mathcal{X}^*. \tag{5.19}$$

It then follows from (5.19) that $\operatorname{Ker}[\lambda - \mathcal{A}, \mathcal{M}]^* = \{0\}$ if and only if

$$\operatorname{Ker}\left(\overline{\lambda} - \mathcal{A}^* \right) \cap \operatorname{Ker} \mathcal{M}^* = \{0\}. \tag{5.20}$$

This proves the desired equivalence of (i) and (iii). We note here that the adjoint operator $\Delta(\lambda)^*$ is given by $\Delta^*(\overline{\lambda})$. Then the equivalence of (ii) and (iv) can be shown as in the proof of the equivalence of (i) and (iii). Hence the proof is complete.Given arbitrarily sets

$$\Lambda_0 = \{\lambda_1, \ldots, \lambda_l\} \subset \sigma_d(\Delta), \qquad \Lambda_1 = \{\mu_1, \ldots, \mu_l\} \subset \mathbb{C}^l, \tag{5.21}$$

let $U = \mathbb{C}^N$ and the controller $M : \mathbb{C}^N \to X$ be defined by

$$Mu = \sum_{i=1}^{N} u_k b_k, \quad u = (u_1, \ldots, u_N) \in \mathbb{C}^N, \; b_k \in X, \; k = 1, \ldots, N. \tag{5.22}$$

For $\lambda \in \sigma_d(\Delta)$, it is clear that $\overline{\lambda} \in \sigma_d(\mathcal{A}^*) = \sigma_d(\Delta^*)$ and we can thus denote the basis of the kernel Ker $\Delta^*(\overline{\lambda})$ by $\{\varphi^*_{\lambda j}\}_{j=1}^{d_\lambda}$, where $d_\lambda =$ dim Ker $\Delta^*(\overline{\lambda})$. □

Proposition 5.4. *Assume that M is given by* (5.22). *For any $\lambda \in \sigma_d(\Delta)$, the following conditions are equivalent:*

(i) Ker $\Delta^*(\overline{\lambda}) \cap$ Ker $M^* = \{0\}$;

(ii) $\mathrm{Rank}(\langle b_k, \varphi^*_{\lambda j}\rangle_{X,X^*} : k = 1, \ldots, N, j = 1, \ldots, d_\lambda) = d_\lambda$.

Proof. First we note that Ker M^* is given by the orthogonal complement

$$\mathrm{Ker}\, M^* = (\mathrm{Im}\, M)^\perp = \{b_k : 1 \le k \le N\}^\perp. \tag{5.23}$$

To show the implication (i) ⇒ (ii), let us suppose contrarily that the rank condition (ii) is not satisfied. Then there exists a nonzero vector $z = (z_1, \ldots, z_{d_\lambda}) \in \mathbb{C}^{d_\lambda}$ such that

$$\sum_{j=1}^{d_\lambda} z_j \langle b_k, \varphi^*_{\lambda j}\rangle_{X,X^*} = 0, \quad k = 1, \ldots, N. \tag{5.24}$$

If we set $\varphi^* = \sum_{j=1}^{d_\lambda} \overline{z}_j \varphi^*_{\lambda j}$, then $\varphi^* \in$ Ker $\Delta^*(\overline{\lambda})$ is nonzero and

$$\langle b_k, \varphi^*\rangle_{X,X^*} = \left\langle b_k, \sum_{j=1}^{d_\lambda} \overline{z}_j \varphi^*_{\lambda j} \right\rangle_{X,X^*} = \sum_{j=1}^{d_\lambda} z_j \langle b_k, \varphi^*_{\lambda j}\rangle_{X,X^*} = 0, \quad k = 1, \ldots, N. \tag{5.25}$$

Thus $\varphi^* \in \{b_k : 1 \le k \le N\}^\perp = \operatorname{Ker} M^*$. This implies that (i) does not hold. Next we will show the converse implication (ii) $\Rightarrow$ (i). Assume that (ii) is valid and let $\varphi^* \in \operatorname{Ker} \Delta^*(\overline{\lambda}) \cap \{b_k : 1 \le k \le N\}^\perp$. Suppose that φ^* is represented as $\varphi^* = \sum_{j=1}^{d_\lambda} z_j \varphi^*_{\lambda j}$, $z_j \in \mathbb{C}^1$, by use of the basis $\{\varphi^*_{\lambda j}\}$ of $\operatorname{Ker} \Delta^*(\overline{\lambda})$, and the condition $\langle b_k, \varphi^* \rangle_{X,X^*} = 0$, $k = 1, \dots, N$, are written as

$$\begin{aligned}(0,\dots,0)^T &= \left(\left\langle b_1, \sum_{j=1}^{d_\lambda} z_j \varphi^*_{\lambda j} \right\rangle_{X,X^*}, \dots, \left\langle b_N, \sum_{j=1}^{d_\lambda} z_j \varphi^*_{\lambda j} \right\rangle_{X,X^*} \right)^T \\ &= \left(\sum_{j=1}^{d_\lambda} \overline{z}_j \left\langle b_1, \varphi^*_{\lambda j} \right\rangle_{X,X^*}, \dots, \sum_{j=1}^{d_\lambda} \overline{z}_j \left\langle b_N, \varphi^*_{\lambda j} \right\rangle_{X,X^*} \right)^T \\ &= B_\lambda (\overline{z}_1, \dots, \overline{z}_{d_\lambda})^T, \quad \text{in } \mathbb{C}^N,\end{aligned} \tag{5.26}$$

where $B_\lambda = (\langle b_k, \varphi^*_{\lambda j} \rangle_{X,X^*} : k = 1, \dots, N, j = 1, \dots, d_\lambda)$. Here $(\cdot)^T$ means the transpose operation of matrices. So the rank condition (ii) implies that $\overline{z}_j = z_j = 0$, $j = 1, \dots, d_\lambda$. Thus $\varphi^* = 0$, which obviously shows (i). $\square$

We can summarize the previous results in the following form.

Theorem 5.5. *Assume that M is given by* (5.22)*. For any $\lambda \in \sigma_d(\Delta)$, the following relations* (i)–(v) *are equivalent:*

(i) $\operatorname{Im}(\lambda - \mathcal{A}) + \operatorname{Im} \mathcal{M} = \mathcal{X}$;

(ii) $\operatorname{Im} \Delta(\lambda) + \operatorname{Im} M = X$;

(iii) $\operatorname{Ker}(\overline{\lambda} - \mathcal{A}^*) \cap \operatorname{Ker} \mathcal{M}^* = \{0\}$;

(iv) $\operatorname{Ker} \Delta^*(\overline{\lambda}) \cap \operatorname{Ker} M^* = \{0\}$;

(v) $\operatorname{Rank}(\langle b_k, \varphi^*_{\lambda j} \rangle_{X,X^*} : k = 1, \dots, N, j = 1, \dots, d_\lambda) = d_\lambda$.

Proof. Since M is given by (5.22), $\operatorname{Im} M$ is finite dimensional. Whereas $\dim P_\lambda \mathcal{X} = \dim \mathcal{M}_\lambda$ is finite by $\lambda \in \sigma_d(\mathcal{A})$, from Theorem 5.28 by Kato in [12], the operator $\lambda - \mathcal{A}$ is Fredholm and hence by Lemma 1.9 by Kato in [12], the sum $\operatorname{Im}(\lambda - \mathcal{A}) + \operatorname{Im} \mathcal{M}$ is closed. It is also clear that $\operatorname{Im} \Delta(\lambda)$ is closed and so is $\operatorname{Im} \Delta(\lambda) + \operatorname{Im} M$ for $\lambda \in \sigma_d(\Delta)$. Then the equivalences (i)–(v) follow from Propositions 5.2, 5.3 and 5.4. $\square$

For a finite set $\Lambda_0 = \{\lambda_1, \dots, \lambda_l\} \subset \sigma_d(\Delta)$, there exists, by assumption, a rectifiable Jordan curve γ_{Λ_0} which surrounds Λ_0 and separates Λ_0 and $\mathbb{C}^1 \setminus \Lambda_0$. If we denote by P_{Λ_0} the projection on Λ_0, then we can decompose the space $\mathcal{X}$ as

$$\mathcal{X} = \mathcal{N}_{\Lambda_0} \oplus \mathcal{R}_{\Lambda_0}, \tag{5.27}$$

where

$$\mathcal{N}_{\Lambda_0} = P_{\Lambda_0} \mathcal{X}, \qquad \mathcal{R}_{\Lambda_0} = (I - P_{\Lambda_0}) \mathcal{X}. \tag{5.28}$$

As $\Lambda_0 \subset \sigma_d(\Delta)$, each $\lambda_i \in \Lambda_0$ is a pole of $(z - \mathcal{A})^{-1}$ and the subspace $\mathcal{N}_{\Lambda_0}$ is finite dimensional by Proposition 4.2. For the mappings $\mathcal{M}$ and $\mathcal{A}$, we define the operators $\mathcal{A}_{\Lambda_0}$ and $\mathcal{M}_{\Lambda_0}$ by

$$\mathcal{A}_{\Lambda_0} = P_{\Lambda_0}\mathcal{A}, \qquad \mathcal{M}_{\Lambda_0} = P_{\Lambda_0}\mathcal{M}. \tag{5.29}$$

Since the operators $\mathcal{A}_{\Lambda_0}$ and $\mathcal{M}_{\Lambda_0}$ are bounded and linear in the finite-dimensional space $\mathcal{N}_{\Lambda_0}$, the exponential operator $e^{t\mathcal{A}_{\Lambda_0}} = \mathcal{S}_{\Lambda_0}(t)$ is well defined on $\mathcal{N}_{\Lambda_0}$. Let $\phi \in \mathcal{X}$ and $u \in L_2(\mathbb{R}_+;\mathbb{C}^N)$. We introduce the following finite-dimensional control system on $\mathcal{M}_{\Lambda_0}$ by

$$\begin{gathered} \frac{dY_0(t)}{dt} = \mathcal{A}_{\Lambda_0}Y_0(t) + \mathcal{M}_{\Lambda_0}u(t), \quad t \geq 0, \\ Y_0(0) = P_{\Lambda_0}\phi \in \mathcal{N}_{\Lambda_0}. \end{gathered} \tag{5.30}$$

In view of the study by Wonham in [14], the dual observed system of (5.30) on the adjoint space $(\mathcal{N}_{\Lambda_0})^* \subset \mathcal{X}^*$ is given by

$$\begin{gathered} \frac{dZ_0(t)}{dt} = \mathcal{A}^*_{\Lambda_0}Z_0(t), \quad t \geq 0, \\ Z_0(0) = (P_{\Lambda_0})^*\phi^* \in (\mathcal{N}_{\Lambda_0})^* = \mathcal{N}^*_{\overline{\Lambda}_0}, \\ Y_0(t) = \mathcal{M}^*_{\Lambda_0}Z_0(t), \quad t \geq 0, \end{gathered} \tag{5.31}$$

where $(P_{\Lambda_0})^* = P^*_{\overline{\Lambda}_0}$, $\overline{\Lambda}_0 = \{\overline{\lambda}_1, \ldots, \overline{\lambda}_l\}$ is given by $P^*_{\overline{\Lambda}_0} = \dfrac{1}{2\pi i}\int_{\gamma^\vee_{\Lambda_0}}(z - \mathcal{A}^*)^{-1}dz$, $\gamma^\vee_{\Lambda_0}$ being mirror image of γ_{Λ_0}, $\mathcal{N}^*_{\overline{\Lambda}_0} = P^*_{\overline{\Lambda}_0}\mathcal{X}^*$, $A^*_{\Lambda_0} = (P_{\Lambda_0}A)^* = P^*_{\overline{\Lambda}_0}\mathcal{A}^*$ is a bounded linear operator on $\mathcal{N}^*_{\overline{\Lambda}_0}$, $\mathcal{M}^*_{\Lambda_0} \in \mathcal{L}(\mathcal{X}^*, \mathbb{C}^N)$ is given by $\mathcal{M}^*_{\Lambda_0}\psi^* = \mathcal{M}^*P^*_{\overline{\Lambda}_0}\psi^*, \psi^* \in \mathcal{X}^*$. And We denote by $\mathcal{S}^*_{\Lambda_0}(t)$ the exponential operator generated by $\mathcal{A}^*_{\Lambda_0}$. It is obvious that $\mathcal{S}^*_{\Lambda_0}(t) = \mathcal{S}^*(t)P^*_{\overline{\Lambda}_0}$.

In finite-dimensional control theory it is well known (cf., Wonham [14]) that the finite-dimensional control system (5.30) is controllable; that is,

$$\bigcup_{t>0}\left\{\int_0^t \mathcal{S}_{\Lambda_0}(\mathrm{t}-s)\mathcal{M}_{\Lambda_0}u(s)ds : u \in L^2\Big([0,t];\mathbb{C}^N\Big)\right\} = \mathcal{N}_{\Lambda_0} \tag{5.32}$$

if and only if the observed system (5.31) is observable; that is,

$$\mathcal{M}^*_{\Lambda_0}\mathcal{S}^*_{\Lambda_0}(t)\psi^* = 0, \quad t \geq 0,\ \psi^* \in \mathcal{N}^*_{\overline{\Lambda}_0} \text{ implies that } \psi^* = 0. \tag{5.33}$$

The space $\mathcal{N}_{\Lambda_0}$ is decomposed as $\mathcal{N}_{\Lambda_0} = \mathcal{N}_{\lambda_1} \oplus \cdots \oplus \mathcal{N}_{\lambda_l}$ (direct sum) so that we have the similar direct sum $\mathcal{N}^*_{\overline{\Lambda}_0} = \mathcal{N}^*_{\overline{\lambda}_1} \oplus \cdots \oplus \mathcal{N}^*_{\overline{\lambda}_l}$. The projection $P^*_{\overline{\Lambda}_0}$ is also decomposed as $P^*_{\overline{\Lambda}_0} = P^*_{\overline{\lambda}_1} + \cdots + P^*_{\overline{\lambda}_l}$, and hence

$$\mathcal{M}^*_{\Lambda_0}\mathcal{S}^*_{\Lambda_0}(t)\psi^* = \mathcal{M}^*\mathcal{S}^*(t)P^*_{\overline{\Lambda}_0}\psi^* = \mathcal{M}^*\mathcal{S}^*(t)P^*_{\overline{\lambda}_1}\psi^* + \cdots + \mathcal{M}^*\mathcal{S}^*(t)P^*_{\overline{\lambda}_l}\psi^*. \tag{5.34}$$

Therefore, condition (5.33) is equivalent to the statement

$$\mathcal{M}^*\mathcal{S}^*(t)\psi^* = 0, \quad t \geq 0,\ \psi^* \in \mathcal{N}^*_{\overline{\lambda}_i} \text{ implies that } \psi^* = 0 \tag{5.35}$$

for each $i = 1, \dots, l$.

Since the eigenvalue $\overline{\lambda}_i$ of $\mathcal{A}^*$ is a pole of $(z - \mathcal{A}^*)^{-1}$ with order k_{λ_i}, we have

$$\begin{aligned}
\mathcal{S}^*(t)P^*_{\overline{\lambda}_i} &= \frac{1}{2\pi i}\int_{\gamma_{\overline{\lambda}_i}} e^{tz}(z-\mathcal{A}^*)^{-1}dz \\
&= e^{t\overline{\lambda}_i}\cdot\frac{1}{2\pi i}\int_{\gamma_{\overline{\lambda}_i}} e^{t(z-\overline{\lambda}_i)}(z-\mathcal{A}^*)^{-1}dz \\
&= e^{t\overline{\lambda}_i}\sum_{j=0}^{k_{\lambda_i}-1}\frac{t^j}{j!}\cdot\frac{1}{2\pi i}\int_{\gamma_{\overline{\lambda}_i}}\left(z-\overline{\lambda}_i\right)^j(z-\mathcal{A}^*)^{-1}dz \\
&= e^{t\overline{\lambda}_i}\left(P^*_{\overline{\lambda}_i} + \sum_{j=1}^{k_{\lambda_i}-1}\frac{t^j}{j!}\left(Q^*_{\overline{\lambda}_i}\right)^j\right).
\end{aligned} \tag{5.36}$$

Hence, for $\psi^* \in \mathcal{N}^*_{\overline{\lambda}_i}$ the equality

$$\mathcal{M}^*\mathcal{S}^*(t)\psi^* = 0, \quad t \geq 0, \tag{5.37}$$

is equivalent to

$$\mathcal{M}^*P^*_{\overline{\lambda}_i}\psi^* = 0, \quad \mathcal{M}^*\left(Q^*_{\overline{\lambda}_i}\right)^j\psi^* = 0, \quad j = 1, \dots, k_{\lambda_i} - 1. \tag{5.38}$$

Recall that, by virtue of Proposition 4.2, we have

$$d_i = \dim\,\mathrm{Ker}(\lambda_i - \mathcal{A}) = \dim\,\mathrm{Ker}\left(\overline{\lambda}_i - \mathcal{A}^*\right) = \dim\,\mathrm{Ker}\,\Delta^*\left(\overline{\lambda}_i\right) < \infty. \tag{5.39}$$

We denote the basis of Ker $\Delta^*(\overline{\lambda}_i)$ by $\{\varphi^*_{i1}, \dots, \varphi^*_{id_i}\} \subset \mathfrak{D}(A^*)$. Then again by Proposition 4.2, the basis of Ker $(\overline{\lambda}_i - \mathcal{A}^*)$ is given by

$$\left\{\left(\varphi^*_{i1}, F_{\overline{\lambda}_i}\varphi^*_{i1}\right), \dots, \left(\varphi^*_{id_i}, F_{\overline{\lambda}_i}\varphi^*_{id_i}\right)\right\} \subset \mathcal{X}^*. \tag{5.40}$$

We set $\Phi^*_{ij} = (\varphi^*_{ij}, F_{\overline{\lambda}_i}\varphi^*_{id_i})$, then

$$\left\langle b_k, \varphi^*_{ij}\right\rangle_{X,X^*} = \langle (b_k, 0), \Phi^*_{ij}\rangle_{\mathcal{X},\mathcal{X}^*} \tag{5.41}$$

holds for each $k = 1, \ldots, N$ and $j = 1, \ldots, d_i$. Indeed, we have

$$\langle (b_k, 0), \Phi_{ij}^* \rangle_{\mathcal{X},\mathcal{X}^*} = \langle (b_k, 0), \left(\varphi_{ij}^*, F_{\overline{\lambda}_i} \varphi_{id_i}^*\right) \rangle_{\mathcal{X},\mathcal{X}^*} = \langle b_k, \varphi_{ij}^* \rangle_{X,X^*}. \tag{5.42}$$

In order to prove the finite pole assignability theorem we need the following proposition on the rank condition.

Proposition 5.6. *Assume that* M *is given by* (5.22). *Then the following two statements are equivalent*

(i) *The equalities*

$$\mathcal{M}^* P_{\overline{\lambda}_i}^* \psi^* = 0, \quad \mathcal{M}^* \left(Q_{\overline{\lambda}_i}^*\right)^j \psi^* = 0, \quad j = 1, \ldots, k_{\lambda_i}, \ \psi^* \in \mathcal{N}_{\overline{\lambda}_i}^*, \tag{5.43}$$

imply that $\psi^* = 0$.

(ii) *The rank condition*

$$\operatorname{Rank}\left(\langle b_k, \varphi_{ij}^* \rangle_{X,X^*} : k = 1, \ldots, N, j = 1, \ldots, d_i\right) \quad = \quad d_i \tag{5.44}$$

holds.

Proof. Since M is given by (5.22), it follows from standard calculations that the adjoint operator $M_0^* \in \mathcal{L}(X^*, \mathbb{C}^N)$ is given by

$$M_0^* \psi_0^* = \left(\overline{\langle b_1, \psi_0^* \rangle}, \ldots, \overline{\langle b_N, \psi_0^* \rangle}\right), \quad \psi_0^* \in X^*, \tag{5.45}$$

and $\mathcal{M}^* \in \mathcal{L}(\mathcal{X}^*, \mathbb{C}^N)$ is given by $\mathcal{M}^* \psi^* = M_0^* \psi_0^*$ for $\psi^* = (\psi_0^*, \psi_1^*) \in \mathcal{X}^*$. Hence, the equalities (5.43) can be rewritten as

$$\left\langle b_k, \left[P_{\overline{\lambda}_i}^* \psi^*\right]_0 \right\rangle_{X,X^*} = \left\langle (b_k, 0), P_{\overline{\lambda}_i}^* \psi^* \right\rangle_{\mathcal{X},\mathcal{X}^*} = 0, \quad k = 1, \ldots, N, \tag{5.46}$$

$$\left\langle b_k, \left[\left(Q_{\overline{\lambda}_i}^*\right)^j \psi^*\right]_0 \right\rangle_{X,X^*} = \left\langle (b_k, 0), (Q_{\overline{\lambda}_i}^*)^j \psi^* \right\rangle_{\mathcal{X},\mathcal{X}^*} = 0, \quad k = 1, \ldots, N, \ j = 1, \ldots, k_{\lambda_i}. \tag{5.47}$$

We first show the implication (i) $\Rightarrow$ (ii). Suppose to the contrary that the rank condition (5.44), or equivalently by the rank condition (5.41),

$$\operatorname{Rank}\left(\langle (b_k, 0), \Phi_{ij}^* \rangle_{\mathcal{X},\mathcal{X}^*} : k = 1, \ldots, N, j = 1, \ldots, d_i\right) = d_i \tag{5.48}$$

is not satisfied. Then there exists a nonzero vector $z = (z_1, \ldots, z_{d_i}) \in \mathbb{C}^N$ such that

$$\sum_{j=1}^{d_i} z_j \left\langle (b_k, 0), \Phi_{ij}^* \right\rangle_{\mathcal{X},\mathcal{X}^*} = \langle (b_k, 0), \quad \sum_{j=1}^{d_i} \overline{z}_j \Phi_{ij}^* \rangle_{\mathcal{X},\mathcal{X}^*} = 0, \quad k = 1, \ldots, N. \tag{5.49}$$

If we set $\Phi^* = \sum_{j=1}^{d_i} \overline{z}_j \Phi^*_{ij}$, then $\Phi^* \in \operatorname{Ker}(\overline{\lambda}_i - \mathcal{A}^*) \subset \mathcal{N}^*_{\overline{\lambda}_i}$ is nonzero and by (5.49)

$$\langle (b_k, 0), \Phi^* \rangle_{\mathcal{X},\mathcal{X}^*} = 0, \quad k = 1, \dots, N. \tag{5.50}$$

Since $P^*_{\overline{\lambda}_i} \Phi^* = \Phi^*$, equalities (5.46) hold owing to (5.50). The relation $\operatorname{Ker}(\overline{\lambda}_i - \mathcal{A}^*) = \mathcal{N}^*_{\overline{\lambda}_i} \cap \operatorname{Ker} Q^*_{\overline{\lambda}_i}$ (cf., (4.6)) yields $Q^*_{\overline{\lambda}_i} \Phi^* = 0$, and hence $(Q^*_{\overline{\lambda}_i})^j \Phi^* = 0$ for each $j = 1, 2, \dots$. This implies equality (5.47). Hence (i) does not hold, which is a contradiction. □

Next we show the converse implication (ii) ⇒ (i). Let $\varphi^* \in \mathcal{N}^*_{\overline{\lambda}_i}$. Assume that the rank condition (5.44), or equivalently (5.48), and equalities (5.46) hold. Since $(Q^*_{\overline{\lambda}_i})^{k_{\lambda_i}} = O$ and $\operatorname{Im} Q^*_{\overline{\lambda}_i} \subset \mathcal{M}^*_{\overline{\lambda}_i}$ by (4.5), then $\varphi_1^* \equiv (Q^*_{\overline{\lambda}_i})^{k_{\lambda_i}-1} \varphi^* \in \operatorname{Ker} Q^*_{\overline{\lambda}_i}$, so that $\varphi_1^* \in \operatorname{Ker}(\overline{\lambda}_i - \mathcal{A}^*)$ by (4.6). Then φ_1^* is written as

$$\varphi_1^* = \sum_{j=1}^{d_i} z_j \Phi^*_{ij}, \quad z_j \in \mathbb{C}^1, \quad j = 1, \dots, d_i. \tag{5.51}$$

Hence, it follows from (5.41), (5.45), and the last equality in (5.47) that

$$\begin{aligned}
\mathcal{M}^* \left(Q^*_{\overline{\lambda}_i} \right)^{k_{\lambda_i}-1} \varphi^* &= M^* \left[\sum_{j=1}^{d_i} z_j \Phi^*_{ij} \right]_0 \\
&= \left(\overline{\left\langle (b_1, 0), \sum_{j=1}^{d_i} z_j \Phi^*_{ij} \right\rangle_{\mathcal{X},\mathcal{X}^*}}, \dots, \overline{\left\langle (b_N, 0), \sum_{j=1}^{d_i} z_j \Phi^*_{ij} \right\rangle_{\mathcal{X},\mathcal{X}^*}} \right) \\
&= \left(\overline{\left\langle b_1, \sum_{j=1}^{d_i} z_j \varphi^*_{ij} \right\rangle_{X,X^*}}, \dots, \overline{\left\langle b_N, \sum_{j=1}^{d_i} z_j \varphi^*_{ij} \right\rangle_{X,X^*}} \right) \\
&= (0, \dots, 0) \in \mathbb{C}^N,
\end{aligned} \tag{5.52}$$

and thus

$$B_i (z_1, \dots, z_{d_i})^T = (0, \dots, 0)^T, \tag{5.53}$$

where $B_i = (\langle b_k, \varphi^*_{ij} \rangle_{X,X^*})$ and $\cdot^T$ denotes the transpose of the vector. Since the rank condition (5.44) is satisfied, (5.53) implies that $z_1 = \cdots = z_{d_i} = 0$. That is, $\varphi_1^* = 0$. Hence $\varphi_2^* \equiv (Q^*_{\overline{\lambda}_i})^{k_{\lambda_i}-2} \varphi^*$

is an element of Ker $Q^*_{\overline{\lambda}_i}$, so that $\varphi^*_2 \in \operatorname{Ker}(\overline{\lambda}_i - A^*)$ by (4.6). We can repeat this procedure via (5.44) to obtain

$$\varphi^*_2 = 0, \quad \left(Q^*_{\overline{\lambda}_i}\right)^{k_{\lambda_i}-3}\psi^* = 0, \dots, \quad Q^*_{\overline{\lambda}_i}\psi^* = 0, \quad P^*_{\overline{\lambda}_i}\psi^* = \psi^* = 0. \tag{5.54}$$

Therefore (i) is shown.Recall notation (5.21), and for each λ_i, $i = 1, \dots, l$, let $\{\varphi^*_{ij}\}_{j=1}^{d_i} \subset X^*$ be the basis of the null space Ker $\Delta^*(\overline{\lambda})$, where $d_i =$ dim Ker $\Delta^*(\overline{\lambda})$. We may further obtain the following result by virtue of Proposition 5.6.

Theorem 5.7. *Assume that M is given by* (5.22) *in system* (5.2)*. Let a finite set $\Lambda_0 = \{\lambda_1, \dots, \lambda_l\} \subset \sigma_d(\Delta)$ be given. For each $\lambda_i \in \Lambda_0$, let B_i, $i = 1, \dots, l$, be $N \times d_i$ matrices given by*

$$B_i = \left(\langle b_k, \varphi^*_{ij}\rangle_{X,X^*} : k = 1, \dots N, j = 1, \dots, d_i\right). \tag{5.55}$$

Then the control system (5.2) *is pole assignable with respect to* (Λ_0, Λ_1) *for any finite set* $\Lambda_1 = \{\mu_1, \dots, \mu_l\}$ *in* $\mathbb{C}^1$ *if and only if the rank conditions*

$$\text{Rank } B_i = d_i, \quad \textit{for each } i = 1, \dots, l, \tag{5.56}$$

are satisfied.

Proof. Given that $\Lambda_1 = \{\mu_1, \dots, \mu_l\} \subset \mathbb{C}^1$, by Theorem 5.5 and Proposition 5.6 we have the equivalences of (5.32), (5.33), (5.35), (5.43), (5.44) and (5.56). Condition (5.32) means that the finite-dimensional control system (5.30) on $\mathcal{N}_{\Lambda_0}$ is controllable. Then by Wonham's pole assignment theorem [14], (5.30) on $\mathcal{N}_{\Lambda_0}$ is controllable if and only if there exists a linear operator $K_0 \in \mathcal{L}(\mathcal{N}_{\Lambda_0}, \mathbb{C}^N)$ such that

$$\sigma(A_{\Lambda_0} + B_{\Lambda_0}K_0) = \Lambda_1, \quad \text{on } \mathcal{N}_{\Lambda_0}. \tag{5.57}$$

Define the operator $K \in \mathcal{L}(\mathcal{X}, \mathbb{C}^N)$ by

$$K\phi = \begin{cases} K_0\phi, & \phi \in \mathcal{N}_{\Lambda_0}, \\ 0, & \phi \in \mathcal{R}_{\Lambda_0}. \end{cases} \tag{5.58}$$

It is clear that $\mathcal{M}K = O$ on $\mathcal{R}_{\Lambda_0}$ and $\mathcal{M}K = \mathcal{M}_{\Lambda_0}K_0$ on $\mathcal{N}_{\Lambda_0}$. Hence $\mathcal{A} + \mathcal{M}K = \mathcal{A}$ on $\mathcal{R}_{\Lambda_0}$ and $\mathcal{A} + \mathcal{M}K = \mathcal{A}_{\Lambda_0} + \mathcal{M}_{\Lambda_0}K_0$ on $\mathcal{M}_{\Lambda_0}$, which implies that $\sigma(\mathcal{A} + \mathcal{M}K) = \sigma(\mathcal{A}) \setminus \Lambda_0$ on $\mathcal{R}_{\Lambda_0}$ and $\sigma(\mathcal{A} + \mathcal{M}K) = \Lambda_1$ on $\mathcal{N}_{\Lambda_0}$ by (5.57). Thus, we obtain the conclusion (5.3) by the direct sum decomposition (4.10). This completes the proof of the theorem. □

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