Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 976493, 10 pages
doi:10.1155/2010/976493

Research Article

Necessary and Sufficient Conditions for the Boundedness of Dunkl-Type Fractional Maximal Operator in the Dunkl-Type Morrey Spaces

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Received 15 April 2010; Revised 13 June 2010; Accepted 23 June 2010

Academic Editor: Dumitru Baleanu

We consider the generalized shift operator, associated with the Dunkl operator $\Lambda_\alpha(f)(x) = (d/dx)f(x) + ((2\alpha+1)/x)((f(x)-f(-x))/2)$, $\alpha > -1/2$. We study the boundedness of the Dunkl-type fractional maximal operator M_β in the Dunkl-type Morrey space $L_{p,\lambda,\alpha}(\mathbb{R})$, $0 \le \lambda < 2\alpha+2$. We obtain necessary and sufficient conditions on the parameters for the boundedness M_β, $0 \le \beta < 2\alpha+2$ from the spaces $L_{p,\lambda,\alpha}(\mathbb{R})$ to the spaces $L_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p \le q < \infty$, and from the spaces $L_{1,\lambda,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\lambda,\alpha}(\mathbb{R})$, $1 < q < \infty$. As an application of this result, we get the boundedness of M_β from the Dunkl-type Besov-Morrey spaces $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ to the spaces $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$, $1 < p \le q < \infty$, $0 \le \lambda < 2\alpha+2$, $1/p - 1/q = \beta/(2\alpha+2-\lambda)$, $1 \le \theta \le \infty$, and $0 < s < 1$.

1. Introduction

On the real line, the Dunkl operators Λ_α are differential-difference operators introduced in 1989 by Dunkl [1]. For a real parameter $\alpha > -1/2$, we consider the Dunkl operator, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha(f)(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right). \tag{1.1}$$

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [2]).

The Hardy-Littlewood maximal function, fractional maximal function, and fractional integrals are important technical tools in harmonic analysis, theory of functions, and partial differential equations. In the works [3–5], the maximal operator and in [6, 7] the fractional maximal operator associated with the Dunkl operator on $\mathbb{R}$ were studied. In this work, we study the boundedness of the fractional maximal operator M_β (Dunkl-type fractional maximal operator) in Morrey spaces $L_{p,\lambda,\alpha}(\mathbb{R})$ (Dunkl-type Morrey spaces) associated with the Dunkl operator on $\mathbb{R}$. We obtain the necessary and sufficient conditions for the boundedness of the operator M_β from the spaces $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p \le q < \infty$, and from the spaces $L_{1,\lambda,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\lambda,\alpha}(\mathbb{R})$, $1 < q < \infty$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give our main result on the boundedness of the operator M_β in $L_{p,\lambda,\alpha}(\mathbb{R})$. We obtain necessary and sufficient conditions on the parameters for the boundedness of the operator M_β from the spaces $L_{p,\lambda,\alpha}(\mathbb{R})$ to the spaces $L_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p \le q < \infty$, and from the spaces $L_{1,\lambda,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\lambda,\alpha}(\mathbb{R})$, $1 < q < \infty$. As an application of this result, in Section 4 we prove the boundedness of the operator M_β from the Dunkl-type Besov-Morrey spaces $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ to the spaces $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$, $1 < p \le q < \infty$, $0 \le \lambda < 2\alpha + 2$, $1/p - 1/q = \beta/(2\alpha + 2 - \lambda)$, $1 \le \theta \le \infty$, and $0 < s < 1$.

Finally, we mention that, C will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

2. Preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}$, given by

$$d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1} dx. \tag{2.1}$$

For every $1 \le p \le \infty$, we denote by $L_{p,\alpha}(\mathbb{R}) = L_p(d\mu_\alpha)(\mathbb{R})$ the spaces of complex-valued functions f, measurable on $\mathbb{R}$ such that

$$\begin{gathered} \|f\|_{p,\alpha} := \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty \quad \text{if } p \in [1,\infty), \\ \|f\|_{\infty,\alpha} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| \quad \text{if } p = \infty. \end{gathered} \tag{2.2}$$

For $1 \le p < \infty$ we denote by $WL_{p,\alpha}(\mathbb{R})$, the weak $L_{p,\alpha}(\mathbb{R})$ spaces defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with the finite norm

$$\|f\|_{WL_{p,\alpha}} := \sup_{r>0} r\left(\mu_\alpha\{x \in \mathbb{R} : |f(x)| > r\}\right)^{1/p}. \tag{2.3}$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha}, \qquad \|f\|_{WL_{p,\alpha}} \le \|f\|_{p,\alpha} \quad \forall f \in L_{p,\alpha}(\mathbb{R}). \tag{2.4}$$

For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x, y, z) := (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x, y, z), \tag{2.5}$$

where

$$\sigma_{x,y,z} := \begin{cases} \dfrac{x^2 + y^2 - z^2}{2xy} & \text{if } x, y \in \mathbb{R} \setminus 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x, y, z) := \begin{cases} d_\alpha \dfrac{\left(\left[(|x| + |y|)^2 - z^2\right]\left[z^2 - (|x| - |y|)^2\right]\right)^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } |z| \in A_{x,y}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

where $d_\alpha = (\Gamma(\alpha+1))^2/(2^{\alpha-1}\sqrt{\pi}\,\Gamma(\alpha+1/2))$ and $A_{x,y} = [||x| - |y||, |x| + |y|]$.

In the sequel we consider the signed measure $\nu_{x,y}$, on $\mathbb{R}$, given by

$$\nu_{x,y} := \begin{cases} W_\alpha(x, y, z)d\mu_\alpha(z) & \text{if } x, y \in \mathbb{R} \setminus 0, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases} \tag{2.8}$$

For $x, y \in \mathbb{R}$ and f being a continuous function on $\mathbb{R}$, the Dunkl translation operator τ_x is given by

$$\tau_x f(y) := \int_{\mathbb{R}} f(z)d\nu_{x,y}(z). \tag{2.9}$$

Using the change of variable $z = \Psi(x, y, \theta) = \sqrt{x^2 + y^2 - 2xy\cos\theta}$, we have also (see [8])

$$\tau_x f(y) = C_\alpha \int_0^\pi \left[f(\Psi) + f(-\Psi) + \frac{x+y}{\Psi}(f(\Psi) - f(-\Psi))\right] d\nu_\alpha(\theta), \tag{2.10}$$

where $d\nu_\alpha(\theta) = (1 - \cos\theta)\sin^{2\alpha}\theta\, d\theta$ and $C_\alpha = \Gamma(\alpha+1)/2\sqrt{\pi}\,\Gamma(\alpha+1/2)$.

Proposition 2.1 (see Soltani [9]). *For all $x \in \mathbb{R}$ the operator τ_x extends to $L_{p,\alpha}(\mathbb{R})$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(\mathbb{R})$,*

$$\|\tau_x f\|_{L_{p,\alpha}} \leq 4\|f\|_{L_{p,\alpha}}. \tag{2.11}$$

Let $B(x,r) = \{y \in \mathbb{R} : |y| \in]\max\{0,|x|-r\}, |x|+r[\}, r > 0$, and $b_\alpha = [2^{\alpha+1}\ (\alpha+1)\ \Gamma(\alpha+1)]^{-1}$. Then $B(0,r) =]-r,r[$ and $\mu_\alpha B(0,r) = b_\alpha r^{2\alpha+2}$.

Now we define the Dunkl-type fractional maximal function (see [3–5]) by

$$M_\beta f(x) = \sup_{r>0}\left(\mu_\alpha B(0,r)\right)^{-1+\beta/(2\alpha+2)} \int_{B(0,r)} \tau_x|f|(y)\,d\mu_\alpha(y), \quad 0 \le \beta < 2\alpha+2. \tag{2.12}$$

If $\beta = 0$, then $M = M_0$ is the Dunkl-type maximal operator.

In [3–5] was proved the following theorem (see also [10]).

Theorem 2.2. (1) *If $f \in L_{1,\alpha}(\mathbb{R})$, then for every $\beta > 0$*

$$\mu_\alpha\{x \in \mathbb{R} : Mf(x) > \beta\} \le \frac{C}{\beta}\|f\|_{L_{1,\alpha}}, \tag{2.13}$$

where $C > 0$ is independent of f.

(2) *If $f \in L_{p,\alpha}(\mathbb{R}),\ 1 < p \le \infty$, then $Mf \in L_{p,\alpha}(\mathbb{R})$ and*

$$\|Mf\|_{L_{p,\alpha}} \le C_p\|f\|_{L_{p,\alpha}}, \tag{2.14}$$

where $C_p > 0$ is independent of f.

Definition 2.3. Let $1 \le p < \infty$, $0 \le \lambda \le 2\alpha+2$. We denote by $L_{p,\lambda,\alpha}(\mathbb{R})$ Morrey space ($\equiv$ Dunkl-type Morrey space), associated with the Dunkl operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$, with the finite norm

$$\|f\|_{p,\lambda,\alpha} = \sup_{x\in\mathbb{R}, r>0}\left(r^{-\lambda}\int_{B(0,r)} \tau_x|f(y)|^p d\mu_\alpha(y)\right)^{1/p}. \tag{2.15}$$

Note that $L_{p,0,\alpha}(\mathbb{R}) = L_{p,\alpha}(\mathbb{R})$, and if $\lambda < 0$ or $\lambda > 2\alpha+2$, then $L_{p,\lambda,\alpha}(\mathbb{R}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}$ (see also [7]).

Definition 2.4. Let $1 \le p < \infty$ and $0 \le \lambda \le 2\alpha+2$. We denote by $WL_{p,\lambda,\alpha}(\mathbb{R})$ a weak Dunkl-type Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}$ with finite norm

$$\|f\|_{WL_{p,\lambda,\alpha}} = \sup_{t>0} t \sup_{x\in\mathbb{R}, r>0}\left(r^{-\lambda}\int_{\{y\in B(0,r):\ \tau_x|f(y)|>t\}} d\mu_\alpha(y)\right)^{1/p}. \tag{2.16}$$

We note that

$$L_{p,\lambda,\alpha}(\mathbb{R}) \subset WL_{p,\lambda,\alpha}(\mathbb{R}), \qquad \|f\|_{WL_{p,\lambda,\alpha}} \le \|f\|_{p,\lambda,\alpha}. \tag{2.17}$$

3. Main Results

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the Dunkl-type fractional maximal operator M_β to be bounded from the spaces $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$, $1 < p < q < \infty$ and from the spaces $L_{1,\lambda,\alpha}(\mathbb{R})$ to the weak spaces $WL_{q,\lambda,\alpha}(\mathbb{R})$, $1 < q < \infty$.

Theorem 3.1. *Let* $0 \le \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2$, *and* $1 \le p \le (2\alpha + 2 - \lambda)/\beta$.

(1) *If* $p = 1$, *then the condition* $1 - 1/q = \beta/(2\alpha + 2 - \lambda)$ *is necessary and sufficient for the boundedness of* M_β *from* $L_{1,\lambda,\alpha}(\mathbb{R})$ *to* $WL_{q,\lambda,\alpha}(\mathbb{R})$.

(2) *If* $1 < p < (2\alpha + 2 - \lambda)/\beta$, *then the condition* $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$ *is necessary and sufficient for the boundedness of* M_β *from* $L_{p,\lambda,\alpha}(\mathbb{R})$ *to* $L_{q,\lambda,\alpha}(\mathbb{R})$.

(3) *If* $p = (2\alpha + 2 - \lambda)/\beta$, *then* M_β *is bounded from* $L_{p,\lambda,\alpha}(\mathbb{R})$ *to* $L_\infty(\mathbb{R})$.

For $1 \le p,\ \theta \le \infty$, $0 \le \lambda < 2\alpha + 2$, and $0 < s < 2$, the Dunkl-type Besov-Morrey $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ consists of all functions f in $L_{p,\lambda,\alpha}(\mathbb{R})$ so that

$$\|f\|_{B^s_{p\theta,\lambda,\alpha}} = \|f\|_{L_{p,\lambda,\alpha}} + \left(\int_{\mathbb{R}} \frac{\|\tau_x f(\cdot) - f(\cdot)\|^\theta_{L_{p,\lambda,\alpha}}}{|x|^{2\alpha+2+s\theta}} d\mu_\alpha(x)\right)^{1/\theta} < \infty. \tag{3.1}$$

Besov spaces in the setting of the Dunkl operators were studied by Abdelkefi and Sifi [11], Bouguila et al. [12], Guliyev and Mammadov [10], and Kamoun [13]. In the following theorem, we prove the boundedness of the Dunkl-type fractional maximal operator in the Dunkl-type Besov-Morrey spaces.

Theorem 3.2. *For* $1 < p \le q < \infty$, $0 \le \lambda < 2\alpha + 2$, $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$, $1 \le \theta \le \infty$, *and* $0 < s < 1$, *the Dunkl-type fractional maximal operator* M_β *is bounded from* $B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$ *to* $B^s_{q\theta,\lambda,\alpha}(\mathbb{R})$. *More precisely, there is a constant* $C > 0$ *such that*

$$\|M_\beta f\|_{B^s_{q\theta,\lambda,\alpha}} \le C\|f\|_{B^s_{p\theta,\lambda,\alpha}} \tag{3.2}$$

hold for all $f \in B^s_{p\theta,\lambda,\alpha}(\mathbb{R})$.

Remark 3.3. Note that Theorem 3.2 in the case $\lambda = 0$ was proved in [10].

4. Boundedness of the Dunkl-Type Fractional Maximal Operator in the Dunkl-Type Morrey Spaces

In the following theorem, we obtain the boundedness of the Dunkl-type fractional maximal operator M_β in the Dunkl-type Morrey spaces $L_{p,\lambda,\alpha}(\mathbb{R})$.

Theorem 4.1. *Let* $0 \le \beta < 2\alpha + 2$, $0 \le \lambda < 2\alpha + 2$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$, *and* $1 \le p \le (2\alpha + 2 - \lambda)/\beta$.

(1) *If* $p = 1$ *and* $1 - 1/q = \beta/(2\alpha + 2 - \lambda)$, *then* $M_\beta f \in WL_{q,\lambda,\alpha}(\mathbb{R})$ *and*

$$\|M_\beta f\|_{WL_{q,\lambda,\alpha}} \le C\|f\|_{1,\lambda,\alpha}, \tag{4.1}$$

where $C > 0$ *is independent of* f.

(2) *If* $1 < p < (2\alpha + 2 - \lambda)/\beta$ *and* $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$, *then* $M_\beta f \in L_{q,\lambda,\alpha}(\mathbb{R})$ *and*

$$\|M_\beta f\|_{q,\lambda,\alpha} \le C\|f\|_{p,\lambda,\alpha}, \tag{4.2}$$

where $C > 0$ *is independent of* f.

(3) *If* $p = (2\alpha + 2 - \lambda)/\beta$ *and* $q = \infty$, *then* $M_\beta f \in L_\infty(\mathbb{R})$ *and*

$$\|M_\beta f\|_\infty \le b_\alpha^{-1/p(2\alpha+2)}\|f\|_{p,\lambda,\alpha}. \tag{4.3}$$

Proof. The maximal function $Mf(x)$ may be interpreted as a maximal function defined on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying

$$\mu E(x, 2r) \le C_0 \mu E(x, r) \tag{4.4}$$

with a constant C_0 being independent of x and $r > 0$. Here $E(x, r) = \{y \in X : \rho(x, y) < r\}$, $\rho(x, y) = |x - y|$. Let (X, ρ, μ) be a space of homogeneous type, where $X = \mathbb{R}$, $\rho(x, y) = |x - y|$, and $d\mu(x) = d\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (4.4). Define

$$M_\mu f(x) = \sup_{r>0} (\mu E(x, r))^{-1} \int_{E(x,r)} |f(y)| d\mu(y). \tag{4.5}$$

It is well known that the maximal operator M_μ is bounded from $L_{1,\lambda}(X, \mu)$ to $WL_{1,\lambda}(X, \mu)$ and is bounded on $L_{p,\lambda}(X, \mu)$ for $1 < p < \infty$, $0 \le \lambda < 2\alpha + 2$ (see [14, 15]).

The following inequality was proved in [6]

$$Mf(x) \le CM_\mu f(x), \tag{4.6}$$

where $C > 0$ is independent of f.

Then from (4.6) we get the boundedness of the operator M from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{1,\lambda,\alpha}(\mathbb{R})$ and on $L_{p,\lambda,\alpha}(\mathbb{R})$, $1 < p < \infty$. Thus in the case $\beta = 0$ we complete the proof of (1) and (2).

Let $t > 0$, $0 < \beta < 2\alpha + 2$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$, $1 \le p \le (2\alpha + 2 - \lambda)/\beta$ and $(1/p) - (1/q) = \beta/(2\alpha + 2 - \lambda)$. Applying the Hölders inequality we have

$$\begin{aligned} M_\beta f(x) = \max\Bigg\{ &\sup_{r \ge t} (\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1} \int_{B(0,r)} \tau_x |f(y)| d\mu_\alpha(y), \\ &\sup_{r<t} (\mu_\alpha B(0,r))^{\beta/(2\alpha+2)-1} \int_{B(0,r)} \tau_x |f(y)| d\mu_\alpha(y) \Bigg\} \\ \le b_\alpha^{\beta/(2\alpha+2)} &\max\Big\{ b_\alpha^{-1/p}\ t^{\beta-(2\alpha+2-\lambda)/p} \|f\|_{p,\lambda,\alpha}, t^\beta Mf(x) \Big\}. \end{aligned} \tag{4.7}$$

Therefore, for all $t > 0$, we get

$$M_\beta f(x) \le b_\alpha^{\beta/(2\alpha+2)} \Big(b_\alpha^{-1/p} t^{\beta-(2\alpha+2-\lambda)/p} + \|f\|_{p,\lambda,\alpha}, t^\beta Mf(x) \Big). \tag{4.8}$$

The minimum value of the right-hand side (4.8) is attained at

$$t = \left(\frac{2\alpha + 2 - \lambda}{p} b_\alpha^{-1/p} \frac{\|f\|_{p,\lambda,\alpha}}{Mf(x)} \right)^{p/(2\alpha+2-\lambda)} \tag{4.9}$$

and hence

$$M_\beta f(x) \le b_\alpha^{\beta/(2\alpha+2)-\beta/(2\alpha+2-\lambda)} \|f\|_{p,\lambda,\alpha}^{1-p/q} (Mf(x))^{p/q}. \tag{4.10}$$

Then for $1 < p \le (2\alpha + 2 - \lambda)/\beta$ from (4.10), we have

$$\begin{aligned} \|M_\beta f\|_{q,\lambda,\alpha} &= \sup_{r>0} \left(r^{-\lambda} \int_{B(0,r)} \tau_x (M_\beta f(y))^q d\mu_\alpha(y) \right)^{1/q} \\ &\le b_\alpha^{\beta/(2\alpha+2)-\beta/(2\alpha+2-\lambda)} \|f\|_{p,\lambda,\alpha}^{1-p/q} \left(r^{-\lambda} \int_{B(0,r)} \tau_x (Mf(y))^p d\mu_\alpha(y) \right)^{1/q} \\ &\le b_\alpha^{\beta/(2\alpha+2)-\beta/(2\alpha+2-\lambda)} \|f\|_{p,\lambda,\alpha}^{1-p/q} \|Mf\|_{p,\lambda,\alpha}^{p/q} \\ &\le C \|f\|_{p,\lambda,\alpha}, \end{aligned} \tag{4.11}$$

where $C > 0$ is independent of f.

Also for $p = 1$ from (4.10) we have

$$\begin{aligned}\|M_\beta f\|_{WL_{q,\lambda,\alpha}} &= \sup_{t>0} t \sup_{x\in\mathbb{R},r>0}\left(r^{-\lambda}\int_{\{y\in B(0,r)\,:\,\tau_x M_\beta f(y)>t\}} d\mu_\alpha(y)\right)^{1/q} \\ &\leq \sup_{t>0} t \sup_{x\in\mathbb{R},r>0}\left(r^{-\lambda}\int_{\left\{y\in B(0,r):\,\tau_x Mf(y)>b_\alpha^{-\beta q/(2\alpha+2-\lambda)+\beta q/(2\alpha+2)}\|f\|_{1,\lambda,\alpha}^{1-q}t^q\right\}} d\mu_\alpha(y)\right)^{1/q} \\ &\leq b_\alpha^{\beta/(2\alpha+2-\lambda)-\beta/(2\alpha+2)}\|f\|_{1,\lambda,\alpha}^{1-1/q}\|Mf\|_{WL_{1,\lambda,\alpha}}^{1/q} \\ &\leq C\|f\|_{1,\lambda,\alpha},\end{aligned} \tag{4.12}$$

where $C > 0$ is independent of f.

Therefore, the case $\beta > 0$ complete the proof of (1) and (2).

(3) Let $p = (2\alpha + 2 - \lambda)/\beta$, $f \in L_{p,\lambda,\alpha}(\mathbb{R})$; then applying Hölders inequality, we obtain

$$\begin{aligned}&(\mu_\alpha B(0,r))^{-1+\beta/(2\alpha+2)}\int_{B(0,r)}\tau_x|f|(y)\,d\mu_\alpha(y) \\ &\quad\leq (\mu_\alpha B(0,r))^{-1+\beta/(2\alpha+2)+1/p}\left(\int_{B(0,r)}\tau_x|f(y)|^p d\mu_\alpha(y)\right)^{1/p} \\ &\quad= b_\alpha^{-\lambda/p(2\alpha+2)}\left(r^{-\lambda}\int_{B(0,r)}\tau_x|f(y)|^p d\mu_\alpha(y)\right)^{1/p} \\ &\quad\leq b_\alpha^{-\lambda/p(2\alpha+2)}\|f\|_{p,\lambda,\alpha}.\end{aligned} \tag{4.13}$$

Thus the case $\beta > 0$ completes the proof of (3).

Theorem 4.1 has been proved. □

Proof of Theorem 3.1. Sufficiency part of the proof follows from Theorem 4.1.

Necessity. (1) Let $1 < p \leq (2\alpha + 2 - \lambda)/\alpha$ and M_β be bounded from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$.

Define $f_t(x) := f(tx)$, $t > 0$. Then

$$\begin{aligned}\|f_t\|_{p,\lambda,\alpha} &= t^{-(2\alpha+2)/p}\sup_{x\in\mathbb{R},r>0}\left(r^{-\lambda}\int_{B(0,tr)}\tau_{tx}|f(y)|^p d\mu_\alpha(y)\right)^{1/p} \\ &= t^{-(2\alpha+2-\lambda)/p}\ \|f\|_{p,\lambda,\alpha}\end{aligned} \tag{4.14}$$

and $M_\beta f_t(x) = t^{-\beta} M_\beta f(tx)$,

$$\begin{aligned}
\|M_\beta f_t\|_{L_{q,\lambda,\alpha}} &= t^{-\beta} \sup_{x\in\mathbb{R}, r>0} \left(r^{-\lambda} \int_{B(0,r)} \tau_{tx} |M_\beta f(y)|^q \, d\mu_\alpha(y) \right)^{1/q} \\
&= t^{-\beta-(2\alpha+2)/q} \sup_{x\in\mathbb{R}, r>0} \left(r^{-\lambda} \int_{B(0,tr)} \tau_{x} |M_\beta f(y)|^q \, d\mu_\alpha(y) \right)^{1/q} \\
&= t^{-\beta-(2\alpha+2-\lambda)/q} \|M_\beta f\|_{L_{q,\lambda,\alpha}}.
\end{aligned} \tag{4.15}$$

By the boundedness of M_β from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$,

$$\begin{aligned}
\|M_\beta f\|_{L_{q,\lambda,\alpha}} &= r^{\beta+(2\alpha+2-\lambda)/q} \|M_\beta f_r\|_{L_{q,\lambda,\alpha}} \\
&\le C r^{\beta+(2\alpha+2-\lambda)/q} \|f_r\|_{p,\lambda,\alpha} \\
&= C r^{\beta+(2\alpha+2-\lambda)/q-(2\alpha+2-\lambda)/p} \|f\|_{p,\lambda,\alpha},
\end{aligned} \tag{4.16}$$

where C depends only on p, β, λ, and α.

If $1/p > 1/q + \beta/(2\alpha+2-\lambda)$, then for all $f \in L_{p,\lambda,\alpha}(\mathbb{R})$ we have $\|M_\beta f\|_{q,\lambda,\alpha} = 0$ as $r \to 0$, which is impossible. Similarly, if $1/p < 1/q + \beta/(2\alpha+2-\lambda)$, then for all $f \in L_{p,\lambda,\alpha}(\mathbb{R})$ we obtain $\|M_\beta f\|_{q,\lambda,\alpha} = 0$ as $r \to \infty$, which is also impossible.

Therefore, we get $1/p = 1/q + \beta/(2\alpha+2-\lambda)$.

Necessity. Let M_β be bounded from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{q,\lambda,\alpha}(\mathbb{R})$. We have

$$\|M_\beta f_r\|_{WL_{q,\lambda,\alpha}} = r^{-\beta-(2\alpha+2-\lambda)/q} \|M_\beta f\|_{WL_{q,\lambda,\alpha}}. \tag{4.17}$$

By the boundedness of M_β from $L_{1,\lambda,\alpha}(\mathbb{R})$ to $WL_{q,\lambda,\alpha}(\mathbb{R})$ it follows that

$$\begin{aligned}
\|M_\beta f\|_{WL_{q,\lambda,\alpha}} &= r^{\beta+(2\alpha+2-\lambda)/q} \|M_\beta f_r\|_{WL_{q,\lambda,\alpha}} \\
&\le C r^{\beta+(2\alpha+2-\lambda)/q} \|f_r\|_{1,\lambda,\alpha} \\
&= C r^{\beta+(2\alpha+2-\lambda)/q-(2\alpha+2)} \|f\|_{1,\lambda,\alpha},
\end{aligned} \tag{4.18}$$

where C depends only on β, λ, and α.

If $1 < 1/q + \beta/(2\alpha+2-\lambda)$, then for all $f \in L_{1,\lambda,\alpha}(\mathbb{R})$ we have $\|M_\beta f\|_{WL_{q,\lambda,\alpha}} = 0$ as $r \to 0$. Similarly, if $1 > 1/q + \beta/(2\alpha+2-\lambda)$, then for all $f \in L_{1,\lambda,\alpha}(\mathbb{R})$ we obtain $\|M_\beta f\|_{WL_{q,\lambda,\alpha}} = 0$ as $r \to \infty$.

Hence we get $1 = 1/q + \beta/(2\alpha+2-\lambda)$. Thus the proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2. For $x \in \mathbb{R}$, let τ_x be the generalized translation by x. By definition of the Besov spaces, it suffices to show that

$$\left\|\tau_x M_\beta f - M_\beta f\right\|_{L_{q,\lambda,\alpha}} \leq C_2 \left\|\tau_x f - f\right\|_{L_{p,\lambda,\alpha}}. \tag{4.19}$$

It is easy to see that τ_x commutes with M_β, that is, $\tau_x M_\beta f = M_\beta(\tau_x f)$. Hence we have

$$\left|\tau_x M_\beta f - M_\beta f\right| = \left|M_\beta(\tau_x f) - M_\beta f\right| \leq M_\beta\left(\left|\tau_x f - f\right|\right). \tag{4.20}$$

Taking $L_{p,\lambda,\alpha}(\mathbb{R})$ norm on both ends of the above inequality, by the boundedness of M_β from $L_{p,\lambda,\alpha}(\mathbb{R})$ to $L_{q,\lambda,\alpha}(\mathbb{R})$, we obtain the desired result. Theorem 3.2 is proved. □

Acknowledgment

The authors express their thanks to the referee for careful reading, and helpful comments and suggestions on the manuscript of this paper.

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