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Research Article

Hybrid Projection Algorithms for Asymptotically Strict Quasi-ϕ-Pseudocontractions

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A new nonlinear mapping is introduced. Hybrid projection algorithms are considered for the class of new nonlinear mappings. Strong convergence theorems are established in a real Banach space.

1. Introduction

Let E be a real Banach space, C a nonempty subset of E, and $T : C \to C$ a nonlinear mapping. Denote by $F(T)$ the set of fixed points of T. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

We remark that the mapping T is said to be quasinonexpansive if $F(T) \neq \emptyset$ and (1.1) holds for all $x \in C$ and $y \in F(T)$. T is said to be asymptotically nonexpansive if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

We remark that the mapping T is said to be asymptotically quasinonexpansive if $F(T) \neq \emptyset$ and (1.2) holds for all $x \in C$ and $y \in F(T)$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically

nonexpansive selfmapping T has a fixed point in C. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, many authors have studied the weak and strong convergence problems of iterative algorithms for the class of mappings.

Recall that T is said to be a strict pseudocontraction if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

We remark that the mapping T is said to be a strict quasipseudocontraction if $F(T) \neq \emptyset$ and (1.3) holds for all $x \in C$ and $y \in F(T)$.

The class of strict pseudocontractions was introduced by Browder and Petryshyn [2]. In 2007, Marino and Xu [3] proved that the fixed point set of strict pseudocontractions is closed and convex. They also proved that $I - T$ is demiclosed at the origin in real Hilbert spaces. A strong convergence theorem of hybrid projection algorithms for strict pseudocontractions was established; see [3] for more details.

Recall that T is said to be an asymptotically strict pseudocontraction if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + \mu_n)\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

We remark that the mapping T is said to be an asymptotically strict quasipseudocontraction if $F(T) \neq \emptyset$ and (1.4) holds for all $x \in C$ and $y \in F(T)$.

The class of asymptotically strict pseudocontractions was introduced by Qihou [4] in 1996. Kim and Xu [5] proved that the fixed-point set of asymptotically strict pseudocontractions is closed and convex. They also obtained a strong convergence theorem for the class of asymptotically strict pseudocontractions by hybrid projection algorithms. To be more precise, they proved the following theorem.

Theorem KX. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be an asymptotically κ-strict pseudocontraction for some $0 \leq \kappa < 1$. Assume that the fixed-point set $F(T)$ of T is nonempty and bounded. Let $\{x_n\}$ be the sequence generated by the following (CQ) algorithm:*

$$\begin{gathered} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [\kappa - \alpha_n(1 - \alpha_n)]\|x_n - Tx_n\|^2 + \theta_n \right\}, \\ Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{gathered} \tag{1.5}$$

where

$$\theta_n = \Delta_n^2(1 - \alpha_n)\mu_n \longrightarrow \textit{ or } n \longrightarrow \infty, \qquad \Delta_n = \sup\{\|x_n - z\| : p \in F(T)\}. \tag{1.6}$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\limsup_{n\to\infty}\alpha_n < 1 - \kappa$ then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

It is well known that, in an infinite dimensional Hilbert space, the normal Mann iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Hybrid projection algorithms are popular tool to prove strong convergence of iterative sequences without compactness assumptions. Recently, hybrid projection algorithms have received rapid developments; see, for example, [3, 5–24]. In this paper, we will introduce a new mapping, asymptotically strict quasi-ϕ-pseudocontractions, and give a strong convergence theorem by a simple hybrid projection algorithm in a real Banach space.

2. Preliminaries

Let E be a Banach space with the dual space E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad \forall x \in E, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E^*; see [25]. It is well known that if E^* is strictly convex, then J is single valued, and if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

It is also well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces, and consequently, it is not available in more general Banach spaces. In this connection, Alber [26] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Recall that a Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E enjoys Kadec-Klee property; see [25, 27] for more details. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.3}$$

Observe that, in a Hilbert space H, (2.3) is reduced to $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the following minimization problem:

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y, x). \tag{2.4}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and the strict monotonicity of the mapping J; see, for example, [26–29]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y,x) \leq (\|y\| + \|x\|)^2, \quad \forall x,y \in E, \tag{2.5}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,y,z \in E. \tag{2.6}$$

Remark 2.1. If E is a reflexive, strictly convex, and smooth Banach space, then, for all $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (2.5), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we see that $Jx = Jy$. It follows that $x = y$; see [25, 27] for more details.

Now, we give some definitions for our main results in this paper.

Let C be a closed convex subset of a real Banach space E and $T : C \to C$ a mapping.

(1) A point p in C is said to be an asymptotic fixed point of T [30] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

(2) T is said to be relatively nonexpansive [15, 31, 32] if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p,Tx) \leq \phi(p,x), \quad \forall x \in C, p \in F(T). \tag{2.7}$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [30–32].

(3) T is said to be relatively asymptotically nonexpansive [6, 11] if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p,T^n x) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C, p \in F(T), \tag{2.8}$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 1$ as $n \to \infty$.

(4) T is said to be ϕ-nonexpansive [14, 16, 17] if

$$\phi(Tx,Ty) \leq \phi(x,y), \quad \forall x,y \in C. \tag{2.9}$$

(5) T is said to be quasi-ϕ-nonexpansive [14, 16, 17] if

$$F(T) \neq \emptyset, \quad \phi(p,Tx) \leq \phi(p,x), \quad \forall x \in C, p \in F(T). \tag{2.10}$$

(6) T is said to be asymptotically ϕ-nonexpansive [14] if there exists a real sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \leq (1+\mu_n)\phi(x,y), \quad \forall x,y \in C. \tag{2.11}$$

(7) T is said to be asymptotically quasi-ϕ-nonexpansive [14] if there exists a real sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \quad \phi(p,T^n x) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C,\ p \in F(T). \tag{2.12}$$

(8) T is said to be a strict quasi-ϕ-pseudocontraction if $F(T) \neq \emptyset$, and there exists a constant $\kappa \in [0,1)$ such that

$$\phi(p, Tx) \leq \phi(p, x) + \kappa\phi(x, Tx), \quad \forall x \in C,\ p \in F(T). \tag{2.13}$$

We remark that T is said to be a quasistrict pseudocontraction in [13].

(9) T is said to be asymptotically regular on C if, for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \left\{ \left\| T^{n+1}x - T^n x \right\| \right\} = 0. \tag{2.14}$$

Remark 2.2. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require $F(T) = \widetilde{F}(T)$, where $\widetilde{F}(T)$ denotes the asymptotic fixed-point set of T.

Remark 2.3. In the framework of Hilbert spaces, quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings are reduced to quasinonexpansive mappings and asymptotically quasinonexpansive mappings.

In this paper, we introduce a new nonlinear mapping: asymptotically strict quasi-ϕ-pseudocontractions.

Definition 2.4. Recall that a mapping $T : C \to C$ is said to be an asymptotically strict quasi-ϕ-pseudocontraction if $F(T) \neq \emptyset$, and there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ and a constant $\kappa \in [0,1)$ such that

$$\phi(p, T^n x) \leq (1 + \mu_n)\phi(p, x) + \kappa\phi(x, T^n x), \quad \forall x \in C,\ p \in F(T). \tag{2.15}$$

Remark 2.5. In the framework of Hilbert spaces, asymptotically strict quasi-ϕ-pseudocontractions are asymptotically strict quasipseudocontractions.

Next, we give an example which is an asymptotically strict quasi-ϕ-pseudocontraction.

Let $E = l_2 := \{x = \{x_1, x_2, \ldots\} : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$, and let B_E be the closed unit ball in E. Define a mapping $T : B_E \to B_E$ by

$$T(x_1, x_2, \ldots) = \left(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\right), \tag{2.16}$$

where $\{a_i\}$ is a sequence of real numbers such that $a_2 > 0$, $0 < a_j < 1$, where $i \neq 2$, and $\Pi_{i=2}^{\infty} a_j = 1/2$. Then

$$\begin{aligned} \phi(p, T^n x) &= \left\| p - T^n x \right\|^2 \\ &\leq 2\left(\Pi_{i=2}^{n} a_j\right) \left\| p - x \right\|^2 + \kappa \| x - T^n x \|^2 \\ &= 2\left(\Pi_{i=2}^{n} a_j\right) \phi(p, x) + \kappa\phi(x, T^n x), \quad \forall x \in B_E,\ n \geq 2, \end{aligned} \tag{2.17}$$

where $p = (0, 0, \ldots)$ is a fixed point of T and $\kappa \in [0, 1)$ is a real number. In view of $\lim_{n\to\infty} 2(\Pi_{i=2}^{n} a_j) = 1$, we see that T is an asymptotically strict quasi-ϕ-pseudocontraction.

In order to prove our main results, we also need the following lemmas.

Lemma 2.6 (see [29]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 2.7 (see [26]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$ then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C. \tag{2.18}$$

Lemma 2.8 (see [26]). *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty closed convex subset of E, and $x \in E$ then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.19}$$

3. Main Results

Theorem 3.1. *Let C be a nonempty closed and convex subset of a uniformly convex and smooth Banach space E. Let $T : C \to C$ be a closed and asymptotically strict quasi-ϕ-pseudocontraction with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that T is uniformly asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ C_{n+1} = \left\{ u \in C_n : \phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \left\langle x_n - u, Jx_n - JT^n x_n \right\rangle + \mu_n \frac{M_n}{1-\kappa} \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{gathered} \tag{Υ}$$

where $M_n = \sup\{\phi(p, x_n) : p \in F(T)\}$ then the sequence $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{F(T)} x_0$.

Proof. The proof is split into five steps.

Step 1. Show that $F(T)$ is closed and convex.

Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. We see that $p \in F(T)$. Indeed, we obtain from the definition of T that

$$\phi(p_n, T^n p) \leq (1 + \mu_n)\phi(p_n, p) + \kappa\phi(p, T^n p). \tag{3.1}$$

In view of (2.6), we see that

$$\phi(p_n, T^n p) = \phi(p_n, p) + \phi(p, T^n p) + 2\langle p_n - p, Jp - JT^n p \rangle. \tag{3.2}$$

It follows that

$$\phi(p_n, p) + \phi(p, T^n p) + 2\langle p_n - p, Jp - JT^n p\rangle \leq (1 + \mu_n)\phi(p_n, p) + \kappa\phi(p, T^n p), \tag{3.3}$$

which implies that

$$\phi(p, T^n p) \leq \frac{\mu_n}{1-\kappa}\phi(p_n, p) + \frac{2}{1-\kappa}\langle p - p_n, Jp - JT^n p\rangle, \tag{3.4}$$

from which it follows that

$$\lim_{n\to\infty} \phi(p, T^n p) = 0. \tag{3.5}$$

From Lemma 2.6, we see that $T^n p \to p$ as $n \to \infty$. This implies that $TT^n p = T^{n+1} p \to p$ as $n \to \infty$. From the closedness of T, we obtain that $p \in F(T)$. This proves the closedness of $F(T)$.

Next, we show the convexness of $F(T)$. Let $p_1, p_2 \in F(T)$ and $p_t = tp_1 + (1-t)p_2$, where $t \in (0,1)$. We see that $p_t = Tp_t$. Indeed, we have from the definition of T that

$$\begin{aligned} \phi(p_1, T^n p_t) &\leq (1 + \mu_n)\phi(p_1, p_t) + \kappa\phi(p_t, T^n p_t), \\ \phi(p_2, T^n p_t) &\leq (1 + \mu_n)\phi(p_2, p_t) + \kappa\phi(p_t, T^n p_t). \end{aligned} \tag{3.6}$$

By virtue of (2.6), we obtain that

$$\phi(p_t, T^n p_t) \leq \frac{\mu_n}{1-\kappa}\phi(p_1, p_t) + \frac{2}{1-\kappa}\langle p_t - p_1, Jp_t - JT^n p_t\rangle, \tag{3.7}$$

$$\phi(p_t, T^n p_t) \leq \frac{\mu_n}{1-\kappa}\phi(p_2, p_t) + \frac{2}{1-\kappa}\langle p_t - p_2, Jp_t - JT^n p_t\rangle. \tag{3.8}$$

Multiplying t and $(1-t)$ on both the sides of (3.7) and (3.8), respectively, yields that

$$\phi(p_t, T^n p_t) \leq \frac{t\mu_n}{1-\kappa}\phi(p_1, p_t) + \frac{(1-t)\mu_n}{1-\kappa}\phi(p_2, p_t). \tag{3.9}$$

It follows that

$$\lim_{n\to\infty} \phi(p_t, T^n p_t) = 0. \tag{3.10}$$

In view of Lemma 2.6, we see that $T^n p_t \to p_t$ as $n \to \infty$. This implies that $TTp_t = T^{n+1} p_t \to p_t$ as $n \to \infty$. From the closedness of T, we obtain that $p_t \in F(T)$. This proves that $F(T)$ is convex. This completes Step 1.

Step 2. Show that C_n is closed and convex for each $n \geq 1$.

It is not hard to see that C_n is closed for each $n \geq 1$. Therefore, we only show that C_n is convex for each $n \geq 1$. It is obvious that $C_1 = C$ is convex. Suppose that C_h is convex for some $h \in \mathbb{N}$. Next, we show that C_{h+1} is also convex for the same h. Let $a, b \in C_{h+1}$ and $c = ta + (1-t)b$, where $t \in (0,1)$. It follows that

$$\begin{aligned} \phi\left(x_h, T^h x_h\right) &\leq \frac{2}{1-\kappa}\left\langle x_h - a, Jx_h - JT^h x_h\right\rangle + \mu_h \frac{M_h}{1-\kappa}, \\ \phi\left(x_h, T^h x_h\right) &\leq \frac{2}{1-\kappa}\left\langle x_h - b, Jx_h - JT^h x_h\right\rangle + \mu_h \frac{M_h}{1-\kappa}, \end{aligned} \tag{3.11}$$

where $a, b \in C_h$. From the above two inequalities, we can get that

$$\phi\left(x_h, T^h x_h\right) \leq \frac{2}{1-\kappa}\left\langle x_h - c, Jx_h - JT^h x_h\right\rangle + \mu_h \frac{M_h}{1-\kappa}, \tag{3.12}$$

where $c \in C_h$. It follows that C_{h+1} is closed and convex. This completes Step 2.

Step 3. Show that $F(T) \subset C_n$ for each $n \geq 1$.

It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_h$ for some $h \in \mathbb{N}$. For any $z \in F(T) \subset C_h$, we see that

$$\phi\left(z, T^h x_h\right) \leq (1+\mu_h)\phi(z, x_h) + \kappa\phi\left(x_h, T^h x_h\right). \tag{3.13}$$

On the other hand, we obtain from (2.6) that

$$\phi\left(z, T^h x_h\right) = \phi(z, x_h) + \phi\left(x_h, T^h x_h\right) + 2\left\langle z - x_h, Jx_h - JT^h x_h\right\rangle. \tag{3.14}$$

Combining (3.13) with (3.14), we arrive at

$$\begin{aligned} \phi\left(x_h, T^h x_h\right) &\leq \frac{\mu_h}{1-\kappa}\phi(z, x_h) + \frac{2}{1-\kappa}\left\langle x_h - z, Jx_h - JT^h x_h\right\rangle \\ &\leq \mu_h \frac{M_h}{1-\kappa} + \frac{2}{1-\kappa}\left\langle x_h - z, Jx_h - JT^h x_h\right\rangle, \end{aligned} \tag{3.15}$$

which implies that $z \in C_{h+1}$. This shows that $F(T) \subset C_{h+1}$. This completes Step 3.

Step 4. Show that the sequence $\{x_n\}$ is bounded.

In view of $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \tag{3.16}$$

In view of $F(T) \subset C_n$, we arrive at

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in F(T). \tag{3.17}$$

It follows from Lemma 2.8 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(\Pi_{F(T)} x_0, x_0) - \phi(\Pi_{F(T)} x_0, x_n) \leq \phi(\Pi_{F(T)} x_0, x_0). \tag{3.18}$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is bounded. It follows from (2.5) that the sequence $\{x_n\}$ is also bounded. This completes Step 4.

Step 5. Show that $x_n \to \overline{x}$, where $\overline{x} = \Pi_{F(T)} x_0$, as $n \to \infty$.

Since $\{x_n\}$ is bounded and the space is reflexive, we may assume that $x_n \to \overline{x}$ weakly. Since C_n is closed and convex, we see that $\overline{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}
\phi(\overline{x}, x_0) &= \|\overline{x}\|^2 - 2\langle \overline{x}, Jx_0 \rangle + \|x_0\|^2 \\
&\leq \liminf_{n\to\infty} \left(\|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 \right) \\
&= \liminf_{n\to\infty} \phi(x_n, x_0) \\
&\leq \limsup_{n\to\infty} \phi(x_n, x_0) \\
&\leq \phi(\overline{x}, x_0),
\end{aligned} \tag{3.19}$$

which implies that $\phi(x_n, x_0) \to \phi(\overline{x}, x_0)$ as $n \to \infty$. Hence, $\|x_n\| \to \|\overline{x}\|$ as $n \to \infty$. In view of Kadec-Klee property of E, we see that $x_n \to \overline{x}$ as $n \to \infty$.

Now, we are in a position to show that $\overline{x} \in F(T)$. Notice that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. On the other hand, we see from $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$ that

$$\phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa} \langle x_n - x_{n+1}, Jx_n - JT^n x_n \rangle + \mu_n \frac{M_n}{1-\kappa}, \tag{3.20}$$

from which it follows that $\phi(x_n, T^n x_n) \to 0$ as $n \to \infty$. In view of Lemma 2.6, we arrive at

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.21}$$

Note that $x_n \to \overline{x}$ as $n \to \infty$ in view of

$$\|T^n x_n - \overline{x}\| \leq \|T^n x_n - x_n\| + \|x_n - \overline{x}\|. \tag{3.22}$$

It follows from (3.21) that

$$T^n x_n \longrightarrow \overline{x} \quad \text{as } n \longrightarrow \infty. \tag{3.23}$$

On the other hand, we have

$$\left\| T^{n+1} x_n - \overline{x} \right\| \leq \left\| T^{n+1} x_n - T^n x_n \right\| + \left\| T^n x_n - \overline{x} \right\|. \tag{3.24}$$

It follows from the uniformly asymptotic regularity of T and (3.23) that

$$T^{n+1}x_n \longrightarrow \overline{x} \quad \text{as } n \longrightarrow \infty \tag{3.25}$$

that is, $TT^n x_n \to \overline{x}$. From the closedness of T, we obtain that $\overline{x} = T\overline{x}$.

Finally, we show that $\overline{x} = \Pi_{F(T)}x_0$ which completes the proof. Indeed, we obtain from $x_n = \Pi_{C_n}x_0$ that

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in C_n. \tag{3.26}$$

In particular, we have

$$\langle x_n - w', Jx_0 - Jx_n \rangle \geq 0, \quad \forall w' \in F(T). \tag{3.27}$$

Taking the limit as $n \to \infty$ in (3.27), we obtain that

$$\langle \overline{x} - w', Jx_0 - J\overline{x} \rangle \geq 0, \quad \forall w' \in F(T). \tag{3.28}$$

Hence, we obtain from Lemma 2.7 that $\overline{x} = \Pi_{F(T)}x_0$. This completes the proof. □

As applications of Theorem 3.1, we have the following.

Corollary 3.2. *Let C be a nonempty closed and convex subset of a uniformly convex and smooth Banach space E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-snonexpansive mapping with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that T is uniformly asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}x_0, \\ C_{n+1} = \left\{ u \in C_n : \phi\left(x_n, T^n x_n\right) \leq 2\left\langle x_n - u, Jx_n - JT^n x_n \right\rangle + \mu_n M_n \right\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \quad \forall n \geq 0, \end{gathered} \tag{3.29}$$

where $M = \{\phi(p, x_n) : p \in F(T)\}$ then the sequence $\{x_n\}$ converges strongly to $\overline{x} = \Pi_{F(T)}x_0$.

Proof. Putting $\kappa = 0$ in Theorem 3.1, we can conclude the desired conclusion easily. □

Next, we give two theorems in the framework of real Hilbert spaces.

Theorem 3.3. *Let C a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a closed and asymptotically strict quasipseudocontraction with a sequence $\{\mu_n\} \subset [0, \infty)$ such that*

$\mu_n \to 0$ as $n \to \infty$. Assume that T is uniformly asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:

$$\begin{gathered} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ C_{n+1} = \left\{ u \in C_n : \|x_n - T^n x_n\|^2 \le \frac{2}{1-\kappa} \left\langle x_n - u, x_n - T^n x_n \right\rangle + \mu_n \frac{M_n}{1-\kappa} \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{gathered} \tag{3.30}$$

where $M_n = \sup\{\|p - x_n\|^2 : p \in F(T)\}$ then the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{F(T)} x_0$.

Theorem 3.4. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a closed and asymptotically quasinonexpansive mapping with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that T is uniformly asymptotically regular on C and $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{gathered} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ C_{n+1} = \left\{ u \in C_n : \|x_n - T^n x_n\|^2 \le 2 \left\langle x_n - u, x_n - T^n x_n \right\rangle + \mu_n M_n \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{gathered} \tag{3.31}$$

where $M_n = \{\|p - x_n\|^2 : p \in F(T)\}$ then the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{F(T)} x_0$.

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