Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 193184, 7 pages
doi:10.1155/2011/193184

Research Article

Integral Means and Arc length of Starlike Log-harmonic Mappings

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Received 13 January 2011; Accepted 17 February 2011

Academic Editor: Gaston Mandata N'Guerekata

We use star functions to determine the integral means for starlike log-harmonic mappings. Moreover, we include the upper bound for the arc length of starlike log-harmonic mappings.

1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined on the unit disk $U = \{z : |z| < 1\}$. A Log-harmonic mapping is a solution to the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = a\frac{f_z}{f}, \tag{1.1}$$

where the second dilatation function $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$. It has been shown that if f is a nonvanishing Log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)}, \tag{1.2}$$

where h and g are analytic functions in U. On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the following representation:

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}, \tag{1.3}$$

where Re $\beta > -1/2$, and h and g are analytic functions in U, $g(0) = 1$, and $h(0) \neq 0$ (see [1]). Univalent Log-harmonic mappings have been studied extensively (for details see [1–5]).

Let $f = z|z|^{2\beta}h\overline{g}$ be a univalent Log-harmonic mapping. We say that f is starlike Log-harmonic mapping if

$$\frac{\partial \arg f\left(re^{i\theta}\right)}{\partial\theta} = \operatorname{Re}\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0, \tag{1.4}$$

for all $z \in U$. Denote by $\mathrm{ST}^*_{\mathrm{Lh}}$ the set of all starlike Log-harmonic mappings, and by S^* the set of all starlike analytic mappings. It was shown in [4] that $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in \mathrm{ST}^*_{\mathrm{Lh}}$ if and only if $\varphi(z) = zh(z)/g(z) \in S^*$.

In Section 2, using star functions we determine the integral means for starlike Log-harmonic mappings. Moreover, we include the upper bound for the arc length of starlike Log-harmonic mappings.

2. Main Results

If f is univalent normalized starlike Log-harmonic mapping, then it was shown in [4] that

$$f(z) = zh(z)\overline{g(z)} = H(z)\overline{g(z)} = \varphi(z)\exp\left(2\operatorname{Re}\int_0^z \frac{a(s)}{1-a(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right), \tag{2.1}$$

where $\varphi(z) = H(z)/g(z)$ is starlike and a is analytic with $a(0) = 0$ and $|a(z)| < 1$ for z in the unit disk,

$$H(z) = \varphi(z)\exp\left(\int_0^z \frac{a(s)}{1-a(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right), \tag{2.2}$$

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{1-a(s)}\frac{\varphi'(s)}{\varphi(s)}ds\right). \tag{2.3}$$

Theorem 2.2 of this section is an application of the Baerstein star functions to starlike Log-harmonic mapping. Star function was first introduced and properties were derived by Baerstein [6], [7, Chapter 7]. The first application was the remarkable result: if $f \in S$, then

$$\int \left|f\left(re^{it}\right)\right|^p dt \le \int \left|k\left(re^{it}\right)\right|^p dt, \tag{2.4}$$

where $k(z) = z/(1-z)^2$, $0 < r < 1$, and $p > 0$.

If $u(z)$ is a real L^1 function in an annulus $0 < R_1 < |z| < R_2$, then the definition of the star function of u, u^* is

$$u^*\left(re^{i\theta}\right) = \sup_{|E|=2\theta}\int_E u\left(re^{it}\right)dt, \quad \text{for } R_1 < r < R_2. \tag{2.5}$$

One important property is that when u is a symmetric (even) rearrangement, then

$$u^*\left(re^{i\theta}\right) = \int_{-\theta}^{\theta} u\left(re^{it}\right)dt. \tag{2.6}$$

Other properties [6], [7, Chapter 7] are that the star function is subadditive and the star respects subordination. Respect means that the star of the subordinate function is less than or equal to the star of the function. In addition, it was also shown that star function is additive when functions are symmetric rearrangements. Here is a lemma, quoted in [6], [7, Chapter 7], which we will be used later.

Lemma 2.1. *For g, h real and L^1 on $[-a, a]$, the following are equivalent:*

(a) *for every convex nondecreasing function $\Phi : R \to R$,*

$$\int_{-a}^{a} \Phi(g(x))dx \leq \int_{-a}^{a} \Phi(h(x))dx; \tag{2.7}$$

(b) *for every $t \in R$,*

$$\int_{-a}^{a} (g(x) - t)^+ dt \leq \int_{-a}^{a} (h(x) - t)^+ dt; \tag{2.8}$$

(c) *for every $x \in [0, a]$,*

$$g^*(x) \leq g^*(x). \tag{2.9}$$

Here is the main result of the section.

Theorem 2.2. *If $f(z) = zh(z)\overline{g(z)} = H(z)\overline{g(z)}$ is in ST^*_{Lh}, then for each fixed r, $0 < r < 1$, and as a function of θ*

(a)

$$(\log|H(z)|)^* \leq \left[\log\left[\left|\frac{z}{(1-z)}\right| \exp\left(\operatorname{Re}\frac{2z}{1-z}\right)\right]\right]^*, \tag{2.10}$$

(b)

$$(\log|g(z)|)^* \leq \left[\log\left[|1-\overline{z}| \exp\left(\operatorname{Re}\frac{2z}{1-z}\right)\right]\right]^*, \tag{2.11}$$

(c)

$$(\log|f(z)|)^* \leq \left[\log\left[|z| \exp\left(\operatorname{Re}\frac{4z}{1-z}\right)\right]\right]^*, \tag{2.12}$$

the three results are sharp by the functions

$$
\begin{aligned}
f(z) &= \frac{z(1-\overline{z})}{(1-z)} \exp\left(\operatorname{Re} \frac{4z}{1-z}\right), \\
zh(z) &= \frac{z}{(1-z)} \exp\left(\frac{2z}{1-z}\right), \\
g(z) &= (1-z) \exp\left(\frac{2z}{1-z}\right).
\end{aligned}
\tag{2.13}
$$

Proof. The proofs of the three parts are similar. We will emphasise the proof of part (a).

By (2.2),

$$
H(z) = \Phi(z) \exp\left(\int_0^1 \frac{a(\rho z)}{1-a(\rho z)} \frac{z\Phi'(\rho z)}{\Phi(\rho z)} d\rho\right). \tag{2.14}
$$

Then

$$
\log|H(z)| = \log|\Phi(z)| + \operatorname{Re}\left(\int_0^1 \frac{a(\rho z)}{1-a(\rho z)} \frac{z\Phi'(\rho z)}{\Phi(\rho z)} d\rho\right), \tag{2.15}
$$

where $z = re^{i\theta}$. Write $(a(\rho z)/(1-a(\rho z)))((\rho z)\Phi'(\rho z)/\Phi(\rho z)) = (a(\rho z)/(1-a(\rho z)))((1+\omega(\rho z))/\rho(1-\omega(\rho z)))$, where ω is analytic, $|\omega| < 1$, and $\omega(0) = 0$, (see [7]). Then, as $(a(z)/(1-a(z)))(z\Phi'(z)/\Phi(z))$ is subordinate to $(z(1+z)/(1-z)^2)$ [7],

$$
\frac{a(z)}{1-a(z)} \frac{(z)\Phi'(z)}{\Phi(z)} = \frac{\psi(z)(1+\psi(z))}{(1-\psi(z))^2}, \tag{2.16}
$$

for ψ analytic, $|\psi| < 1$ and $\psi(0) = 0$.

Then (2.15) becomes

$$
\log|H(z)| = \log|\Phi(z)| + \operatorname{Re}\left(\int_0^1 \frac{1}{(\rho)} \frac{\psi(\rho z)(1+\psi(\rho z))}{(1-\psi(\rho z))^2} d\rho\right). \tag{2.17}
$$

As the star function is subadditive,

$$
(\log|H(z)|)^* \leq (\log|\Phi(z)|)^* + \int_0^1 \left(\operatorname{Re} \frac{1}{\rho} \frac{\psi(\rho z)1+\psi(\rho z)}{(1-\psi(\rho z))^2}\right)^* d\rho. \tag{2.18}
$$

Consequently, by (2.4) and the fact that star functions respect subordination,

$$
(\log|H(z)|)^* \leq \left(\log\left|\frac{z}{(1-z)^2}\right|\right)^* + \int_0^1 \frac{1}{\rho}\left(\operatorname{Re} \frac{\rho z(1+\rho z)}{(1-\rho z)^2}\right)^* d\rho. \tag{2.19}
$$

Hence, as star functions are additive when functions are symmetric re-arrangements,

$$\begin{aligned}(\log|H(z)|)^{*} &\leq \left(\log\left|\frac{z}{(1-z)^{2}}\right|\right)^{*}+\int_{0}^{1}\frac{1}{\rho}\left(\operatorname{Re}\frac{\rho z(1+\rho z)}{(1-\rho z)^{2}}\right)^{*}d\rho\\ &=\left[\log\left|\frac{z}{(1-z)^{2}}\right|+\int_{0}^{1}\frac{1}{\rho}\left(\operatorname{Re}\frac{\rho z(1+\rho z)}{(1-\rho z)^{2}}\right)d\rho\right]^{*}\\ &=\left[\log\left(\left|\frac{z}{(1-z)^{2}}\right|\exp\left(\int_{0}^{1}\frac{1}{\rho}\left(\operatorname{Re}\frac{\rho z(1+\rho z)}{(1-\rho z)^{2}}\right)d\rho\right)\right)\right]^{*}\\ &=\left[\log\left[\left|\frac{z}{(1-z)}\right|\exp\left(\operatorname{Re}\frac{2z}{1-z}\right)\right]\right]^{*}.\end{aligned}\tag{2.20}$$

Therefore,

$$(\log|H(z)|)^{*}\leq\left[\log\left[\left|\frac{z}{(1-z)}\right|\exp\left(\operatorname{Re}\frac{2z}{1-z}\right)\right]\right]^{*},\tag{2.21}$$

which is part (a).

By (2.4), (2.15) and in similar fashion to the upper part,

$$\begin{aligned}(\log|g(z)|)^{*} &\leq\int_{0}^{1}\frac{1}{(\rho)}\left(\operatorname{Re}\frac{\rho z(1+\rho z)}{(1-\rho z)^{2}}\right)^{*}d\rho\\ &=\left[\log\left[|1-\overline{z}|\exp\left(\operatorname{Re}\frac{2z}{1-z}\right)\right]\right]^{*}.\end{aligned}\tag{2.22}$$

(2.21) and (2.22) give part (c). □

Now by using Lemma 2.1, we have the following corollary.

Corollary 2.3. *If $f(z)=zh(z)\overline{g(z)}$ is starlike Log-harmonic, then, for $z=e^{i\theta}$,*

$$\begin{aligned}&\int|f(z)|^{p}dt\leq\int\left||z|\exp\left(\operatorname{Re}\frac{4z}{1-z}\right)\right|^{p}d\theta,\quad\forall p>0,\\ &\int(\log^{+}|f(z)|)dt\leq\int\log^{+}|z|d\theta+\int\operatorname{Re}\frac{4z}{1-z}d\theta\leq C<\infty,\end{aligned}\tag{2.23}$$

the later implies that $f\in N^{+}$ hence has radial limits.

Proof. If we choose $\Phi(x)=\exp(px)$ which is nondecreasing convex, then part (a) of Lemma 2.1 and part (c) of Theorem 2.2 give the first integral mean. The choice $\Phi(x)=\log^{+}(x)$ gives the second integral mean. □

In the next theorem, we establish an upper for the arc length of starlike Log-harmonic mappings.

Theorem 2.4. *If* $f(z) = zh(z)\overline{g(z)} \in ST^{*}_{Lh}$ *and for* r, $0 < r < 1$, $|f(re^{i\theta})| \leq M(r)$, *then* $L(r) \leq 4\pi M(r)/(1-r^2)$.

Proof. Let C_r denote the closed curve which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$. Then

$$\begin{aligned} L(r) &= \int_{C_r} |df| = \int_0^{2\pi} \left|zf_z - \overline{z}f_{\overline{z}}\right| d\theta \\ &= \int_0^{2\pi} |f| \left|\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right| d\theta \\ &\leq M(r) \int_0^{2\pi} \left|\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right| d\theta. \end{aligned} \tag{2.24}$$

Now using (2.1), we have

$$\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = \operatorname{Re}\frac{z\varphi'}{\varphi} + i\operatorname{Im}\left[\left(\frac{1+a}{1-a}\right)\left(\frac{z\varphi'}{\varphi}\right)\right]. \tag{2.25}$$

Therefore,

$$\begin{aligned} L(r) &\leq M(r)\int_0^{2\pi} \operatorname{Re}\left(\frac{z\varphi'}{\varphi}\right) d\theta + M(r)\int_0^{2\pi} \left|\operatorname{Im}\left(\frac{1+a}{1-a}\right)\left(\frac{z\varphi'}{\varphi}\right)\right| d\theta \\ &= M(r)I_1 + M(r)I_2. \end{aligned} \tag{2.26}$$

Since $\operatorname{Re}(z\varphi'/\varphi)$ is harmonic, and by the mean value theorem for harmonic functions, $I_1 = 2\pi$. Moreover, $((1+a)/(1-a))(z\varphi'/\varphi)$ is subordinate to $((1+z)/(1-z))^2$; therefore, we have

$$I_2 \leq \int_0^{2\pi} \left|\frac{1+z}{1-z}\right|^2 d\theta = 2\pi\left[1 + 2\sum_{n=1}^{\infty} r^{2n}\right] = 2\pi\left(\frac{1+r^2}{1-r^2}\right). \tag{2.27}$$

Substituting the bounds for I_1 and I_2 in (2.26), we get

$$L(r) \leq 2\pi M(r) + 2\pi M(r)\left(\frac{1+r^2}{1-r^2}\right) \leq \frac{4\pi M(r)}{1-r^2}. \tag{2.28}$$

□

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