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Research Article

Superstability of Generalized Higher Derivations

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We define the notion of an approximate generalized higher derivation and investigate the superstability of strong generalized higher derivations.

1. Introduction and Preliminaries

The problem of stability of functional equations was originally raised by Ulam [1, 2] in 1940 concerning the stability of group homomorphisms. Hyers [3] gave an affirmative answer to the question of Ulam. Superstability, the result of Hyers, was generalized by Aoki [4], Bourgin [5], and Rassias [6]. During the last decades, several stability problems for various functional equations have been investigated by several authors. We refer the reader to the monographs [7–10].

Let $(E, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by $\mathfrak{C}_k$ the group of permutations on k symbols.

Definition 1.1. A multi-norm on $\{E^k : k \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, $\|x\|_1 = \|x\|$ for each $x \in E$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

$(M1)$ $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k (\sigma \in \mathfrak{C}_k, x_1, \ldots, x_k \in E)$;

$(M2)$ $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|(x_1, \ldots, x_k)\|_k (\alpha_1, \ldots, \alpha_k \in \mathbb{C},\ x_1, \ldots, x_k \in E)$;

$(M3)$ $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}\ (x_1, \ldots, x_k \in E)$;

$(M4)$ $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}\ (x_1, \ldots, x_k \in E)$.

In this case, we say that $((E^k, \|\cdot\|)k \in \mathbb{N})$ is a multi-normed space.

We recall that the notion of multi-normed space was introduced by Dales and Polyakov in [11]. Motivations for the study of multi-normed spaces and many examples are given in [11].

Suppose that $((E^k, \|\cdot\|_k)k \in \mathbb{N})$ is a multi-normed space, and $k \in \mathbb{N}$. The following properties are almost immediate consequences of the axioms:

(i) $\|(x,\dots,x)\|_k = \|x\|$ $(x \in E)$;

(ii) $\max_{i\in\mathbb{N}_k}\|x_i\| \leq \|(x_1,\dots,x_k)\|_k \leq \sum_{i=1}^{k}\|x_i\| \leq k\ \max_{i\in\mathbb{N}_k}\|x_i\|$ $(x_1,\dots,x_k \in E)$.

It follows from (ii) that if $(E, \|\cdot\|)$ is a Banach space, then $((E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$. In this case, $((E^k, \|\cdot\|_k)k \in \mathbb{N})$ is a multi-Banach space.

By (ii), we get the following lemma.

Lemma 1.2. *Suppose that $k \in \mathbb{N}$ and $(x_1,\dots,x_k) \in E^k$. For each $j \in \mathbb{N}_k$, let $\{x_n^j\}_{n\in\mathbb{N}}$ be a sequence in E such that $\lim_{n\to\infty} x_n^j = x_j$. Then for each $(y_1,\dots,y_k) \in E^k$, one has*

$$\lim_{n\to\infty}\left(x_n^1 - y_1,\dots,x_n^k - y_k\right) = (x_1 - y_1,\dots,x_k - y_k). \tag{1.1}$$

Definition 1.3. Let $((E^k, \|\cdot\|_k)k \in \mathbb{N})$ be a multi-normed space. A sequence $\{x_n\}$ in E is a multinull sequence if, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}}\|(x_n,\dots,x_{n+k-1})\|_k < \epsilon \quad (n \geq n_0). \tag{1.2}$$

Let $x \in E$. We say that $\lim_{n\to\infty} x_n = x$ if $\{x_n - x\}$ is a multi-null sequence.

Definition 1.4. Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra such that $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$ is said to be a multi-normed space. Then $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$ is a multi-normed algebra if

$$\left\|(x_1y_1,\dots,x_ky_k)\right\|_k \leq \|(x_1,\dots,x_k)\|_k\left\|(y_1,\dots,y_k)\right\|_k, \tag{1.3}$$

for $k \in \mathbb{N}$ and $x_1,\dots,x_k,y_1,\dots,y_k \in \mathcal{A}$. Furthermore, if $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$ is a multi-Banach space, then $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$ is a multi-Banach algebra.

Let $\mathcal{A}$ be an algebra and $k_0 \in \{0,1,\dots,\} \cup \{\infty\}$. A family $\{D_j\}_{j=0}^{k_0}$ of linear mappings on $\mathcal{A}$ is said to be a *higher derivation* of rank k_0 if the functional equation $D_j(xy) = \sum_{i=0}^{j} D_i(x)D_{j-i}(y)$ holds for all $x, y \in \mathcal{A}$, $j = 0,1,2,\dots,k_0$. If $D_0 = id_{\mathcal{A}}$, where $id_{\mathcal{A}}$ is the identity map on $\mathcal{A}$, then D_1 is a derivation and $\{D_j\}_{j=0}^{k_0}$ is called a *strong* higher derivation. A standard example of a higher derivation of rank k_0 is $\{D^j/j!\}_{j=0}^{k_0}$, where $D : \mathcal{A} \to \mathcal{A}$ is a derivation. The reader may find more information about higher derivations in [12–18].

A family $\{f_j\}_{j=0}^{k_0}$ of linear mappings on $\mathcal{A}$ is called a *generalized* strong higher derivation if $f_0 = id_{\mathcal{A}}$, and there exists a higher derivation $\{D_j\}_{j=0}^{k_0}$ such that

$$f_j(xy) = xf_j(y) + \sum_{i=1}^{j} D_i(x)f_{j-i}(y), \tag{1.4}$$

for all $x, y \in \mathcal{A}$ and $j = 0,1,2,\dots,k_0$.

The stability of derivations was studied by Park [19, 20]. In this paper, using some ideas from [21, 22], we investigate the superstability of generalized strong higher derivations in multi-Banach algebras.

2. Stability of Generalized Higher Derivations

In this section, we define the notion of an approximate generalized higher derivation. Then we show that an approximate generalized strong higher derivation on a multi-Banach algebra is a strong generalized higher derivation.

Lemma 2.1. *Let $(E, \|\cdot\|)$ be a normed space, and let $((F^k, \|\cdot\|_k : k \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$, $\epsilon > 0$, and $f : E \to F$ a mapping satisfying $f(0) = 0$ and*

$$\sup_{k\in\mathbb{N}} \left\| \left(f\left(\frac{x_1 + y_1}{t}\right) - \frac{f(x_1)}{t} - \frac{f(y_1)}{t}, \dots, f\left(\frac{x_k + y_k}{t}\right) - \frac{f(x_k)}{t} - \frac{f(y_k)}{t} \right) \right\|_k \leq \epsilon, \tag{2.1}$$

for all integer $t > 1$ and all $x_1, \dots, x_k, y_1, \dots, y_k \in E$, then there exists a unique additive mapping $T : E \to F$ such that

$$\left\| (f(x_1) - T(x_1), \dots, f(x_k) - T(x_k)) \right\| \leq \epsilon \quad (x_1, \dots, x_k \in E). \tag{2.2}$$

Proof. Substituting $y_i = 0$ for $i = 1, \dots, k$ and replacing $x_1, \dots, x_k$ by $tx_1, \dots, tx_k$ in (2.1), we get

$$\sup_{k\in\mathbb{N}} \left\| \left(f(x_1) - \frac{f(tx_1)}{t}, \dots, f(x_k) - \frac{f(tx_k)}{t} \right) \right\|_k \leq \epsilon. \tag{2.3}$$

Replacing $x_1, \dots, x_k$ by $t^n x_1, \dots, t^n x_k$ and dividing by t^n in (2.3), it follows that

$$\sup_{k\in\mathbb{N}} \left\| \left(\frac{f(t^n x_1)}{t^n} - \frac{f(t^{n+1} x_1)}{t^{n+1}}, \dots, \frac{f(t^n x_k)}{t^n} - \frac{f(t^{n+1} x_k)}{t^{n+1}} \right) \right\|_k \leq \frac{\epsilon}{t^n}. \tag{2.4}$$

An induction argument implies that

$$\sup_{k\in\mathbb{N}} \left\| \left(\frac{f(t^n x_1)}{t^n} - \frac{f(t^{n+m} x_1)}{t^{n+m}}, \dots, \frac{f(t^n x_k)}{t^n} - \frac{f(t^{n+m} x_k)}{t^{n+m}} \right) \right\|_k \leq \epsilon \left(\frac{1}{t^{n+1}} + \cdots \frac{1}{t^{n+m}} \right), \tag{2.5}$$

for $x \in E$ and $n, m \in \mathbb{N}$. Hence, the sequence $\{f(t^n x)/t^n\}$ is cauchy and hence is convergent in the complete multi-normed space F. Let $T : E \to F$ be the mapping defined by

$$T(x) := \lim_{n\to\infty} \frac{f(t^n x)}{t^n}. \tag{2.6}$$

Hence, for each $r > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}}\left\|\left(\frac{f(t^n x_1)}{t^n} - T(x), \ldots, \frac{f(t^{n+k-1}x_k)}{t^{n+k-1}} - T(x)\right)\right\|_k \le r \ (n \ge N). \tag{2.7}$$

In particular, the property (ii) of multi-norm implies that

$$\lim_{n\to\infty}\left\|\frac{f(t^n x)}{t^n} - T(x)\right\| = 0 \quad (x \in E). \tag{2.8}$$

We show that T is additive. Putting $n = 0$ in (2.5), we get

$$\sup_{k\in\mathbb{N}}\left\|\left(f(x_1) - \frac{f(t^m x_1)}{t^m}, \ldots, f(x_k) - \frac{f(t^m x_k)}{t^m}\right)\right\|_k \le \epsilon. \tag{2.9}$$

Taking the limit as $m \to \infty$, we obtain

$$\sup_{k\in\mathbb{N}}\|(f(x_1) - T(x_1), \ldots, f(x_k) - T(x_k))\|_k \le \epsilon. \tag{2.10}$$

Let $x, y \in E$, put $x_1 = \cdots = x_k = t^n x$, $y_1 = \cdots = y_k = t^n y$ in (2.1), and divide by t^n, Then we have

$$\left\|t^{-n} f\left(\frac{t^n x + t^n y}{t}\right) - t^{-1}\frac{f(t^n x)}{t^n} - t^{-1}\frac{f(t^n y)}{t^n}\right\|_k \le \frac{\epsilon}{t^n}. \tag{2.11}$$

By letting $n \to \infty$, we get

$$T\left(\frac{x+y}{t}\right) = \frac{T(x)}{t} + \frac{T(y)}{t}. \tag{2.12}$$

Letting $y = 0$ in (2.12) yields $T(x/t) = T(x)/t$ for all $x \in E$. Hence, we get $T(x + y) = T(x) + T(y)$, that is, T is additive. Now, if T' is another required additive mapping, we see that

$$\begin{aligned}\|T'(x) - T(x)\| &\le \frac{1}{t^n}\|T'(t^n x\ ) - T(t^n x)\| \\ &\le \frac{1}{t^n}\|T'(t^n x) - f(t^n x)\| + \frac{1}{t^n}\|f(t^n x) - T(t^n x)\| \\ &\le \frac{2}{t^{n-1}(t-1)}\epsilon,\end{aligned} \tag{2.13}$$

for all $x \in E$. By letting $n \to \infty$ in this inequality, we conclude that $T = T'$. This proves the uniqueness assertion. □

Definition 2.2. Let $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$ be a multi-Banach algebra. Suppose that $\epsilon > 0$, $t > 1$ is an integer and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a control function such that

$$\varphi(t^n x, t^m y) \le \alpha^{n+m} \varphi(x, y), \tag{2.14}$$

for some $0 < \alpha < t$, all nonnegative numbers m, n and all $x, y \in \mathcal{A}$. An (ϵ, φ)-approximate generalized strong higher derivation of rank k_0 is a family $\{f_j\}_{j=0}^{k_0}$ of mappings from $\mathcal{A}$ into $\mathcal{A}$ with $f_j(0) = 0$, $f_0 = id_{\mathcal{A}}$, and there exists a family $\{g_j\}_{j=0}^{k_0}$ of mappings from $\mathcal{A}$ into $\mathcal{A}$ such that $g_0 = id_{\mathcal{A}}$ and

$$\begin{aligned}\sup_{k\in\mathbb{N}} \Bigg\| \Bigg(f_j\left(\frac{x_1 + y_1}{t} + z_1 w_1\right) &- \frac{f_j(x_1)}{t} - \frac{f_j(y_1)}{t} - z_1 f_j(w_1) \\ &- g_j(z_1) w_1, \dots, f_j\left(\frac{x_k + k_1}{t} + z_k w_k\right) \\ &- \frac{f_j(x_k)}{t} - \frac{f_j(y_k)}{t} - z_k f_j(w_k) - g_j(z_k) w_k \Bigg) \Bigg\|_k \le \epsilon,\end{aligned} \tag{2.15}$$

for all $0 \le j \le k_0$, $t > 1$ and all $x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k, w_1, \dots, w_k \in \mathcal{A}$, and

$$\left\| f_j(xy) - x f_j(y) - \sum_{i=1}^{j} g_i(x) f_{j-i}(y) \right\| \le \varphi(x, y), \tag{2.16}$$

for all $0 \le j \le k_0$ and $x, y \in \mathcal{A}$.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra with unit e, and let $\{f_j\}_{j=0}^{k_0}$ be a (ϵ, φ)-approximate generalized strong higher derivation on a multi-Banach algebra $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$, then $\{f_j\}_{j=0}^{k_0}$ is a strong higher derivation.*

Proof. Letting $z_i = w_i = 0$ for $i = 1, \dots, k$ in (2.15), Lemma 2.1 implies that for each $0 \le j \le k_0$, there is an additive mapping d_j defined by $d_j(x) = \lim_{n\to\infty}(f_j(t^n x)/t^n)$ such that $\|d_j(x) - f_j(x)\| \le \epsilon$ for all $x \in \mathcal{A}$. If $j = 1$, [21, Theorem 2.2] implies that f_1 and g_1 are a generalized derivation and a derivation, respectively. Also by the proof of [21, Theorem 2.2], we have

$$f_1(xy) = x f_1(y) + g_1(x) y, \qquad \lim_{n\to\infty} \frac{g_1(t^n x)}{t^n} = d_1(x) - x d_1(e) = g_1(x). \tag{2.17}$$

By induction for $1 \le i \le j - 1$, assume that

$$f_i = x f_i(y) + \sum_{l=1}^{i} g_l(x) f_{i-l}(y), \qquad g_i = \sum_{l=0}^{i} g_l(x) g_{i-l}(y), \tag{2.18}$$

for all $x, y \in \mathcal{A}$ such that

$$\lim_{n\to\infty}\frac{g_i(t^n x)}{t^n} = d_i(x) - xd_i(e) - \sum_{l=1}^{i} g_l(x)d_{i-l}(e) = g_i(x). \tag{2.19}$$

It follows from (2.14) and (2.16) that

$$\left\| \frac{f_j(t^{2n}xy)}{t^{2n}} - xf_j(t^n y) - \sum_{i=1}^{j} \frac{g_i(t^n x)}{t^n}\frac{f_{j-i}(t^n y)}{t^n} \right\| \le \frac{\varphi(t^n x, t^n y)}{t^{2n}} \le \left(\frac{\alpha}{t}\right)^{2n}. \tag{2.20}$$

Passing the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} y\frac{g_j(t^n x)}{t^n} = d_j(xy) - xd_j(y) - \sum_{i=1}^{j-1} g_i(x)d_{j-i}(y), \tag{2.21}$$

for all $x, y \in \mathcal{A}$. Put $y = e$ in the above equation, then

$$\lim_{n\to\infty}\frac{g_j(t^n x)}{t^n} = d_j(x) - xd_j(e) - \sum_{i=1}^{j-1} g_i(x)d_{j-i}(e). \tag{2.22}$$

If $D_j(x) = d_j(x) - xd_j(e) - \sum_{i=1}^{j-1} g_i(x)d_{j-i}(e)$, then by additivity of d_i and g_i for $0 \le i \le j-1$, we get

$$\begin{aligned} D_j(a+b) &= d_j(a+b) - (a+b)d_j(e) - \sum_{i=1}^{j} g_i(a+b)d_{j-i}(e) \\ &= d_j(a) + d_j(b) - ad_j(e) - bd_j(e) - \sum_{i=1}^{j} g_i(a)d_{j-i}(e) - \sum_{i=1}^{j} g_i(b)d_{j-i}(e) \\ &= D_j(a) + D_j(b). \end{aligned} \tag{2.23}$$

Therefore, D_j is additive. Now, let $F(x,y) = f_j(xy) - xf_j(y) - \sum_{i=1}^{j} g_i(x)f_{j-i}(y)$, if we take $x_i = y_i = 0$ and $z_i = x$, $w_i = y$ for $i = 1, \ldots, k$ in (2.15), then $\lim_{n\to\infty}(F(t^n x, y)/t^n) = 0$. Hence,

$$\begin{aligned} d_j(xy) &= \lim_{n\to\infty}\frac{f_j(t^n xy)}{t^n} = \lim_{n\to\infty}\frac{f_j(t^n x \cdot y)}{t^n} \\ &= \lim_{n\to\infty}\frac{t^n x f_j(y) + \sum_{i=1}^{j} g_i(t^n x)f_{j-i}(y) + F(t^n x, y)}{t^n} \\ &= xf_j(y) + \sum_{i=1}^{j-1} g_i(x)f_{j-i}(y) + D_j(x)y, \end{aligned} \tag{2.24}$$

for all $x, y \in \mathcal{A}$. Since $g_1, \ldots, g_{j-1}, f_1, \ldots, f_{j-1}$ and D_j are additive, we can write

$$\begin{aligned} t^n x f_j(y) &+ \sum_{i=1}^{j-1} t^n g_i(x) f_{j-i}(y) + t^n D_j(x) y \\ &= d_j(t^n x \cdot y) \\ &= d_j(x \cdot t^n y) \\ &= x f_j(t^n y) + \sum_{i=1}^{j-1} t^n g_i(x) f_{j-i}(y) + t^n D_j(x) y, \end{aligned} \tag{2.25}$$

for all $x, y \in \mathcal{A}$. We conclude that $x f_j(y) = x(f_j(t^n y)/t^n)$, so we can obtain $x f_j(y) = x d_j(y)$, for all $x, y \in \mathcal{A}$ as $n \to \infty$. If $x = e$, we have $f_j = d_j$. Therefore,

$$f_j(xy) = x f_j(y) + \sum_{i=1}^{j-1} g_i(x) f_{j-i}(y) + D_j(x) y, \tag{2.26}$$

for all $x, y \in \mathcal{A}$. Now, we replace y by $t^n y$ in (2.16), then

$$\left\| \frac{f_j(t^n x y)}{t^n} - \frac{x f_j(t^n y)}{t^n} - \sum_{i=1}^{j} g_i(x) f_{j-i}(y) \right\| \le \frac{\varphi(x, t^n y)}{t^n} \le \left(\frac{\alpha}{t}\right)^n, \tag{2.27}$$

for all $x, y \in \mathcal{A}$. We conclude that $x f_j(y) = x(f_j(t^n y)/t^n)$, so we can obtain $x f_j(y) = x d_j(y)$, for all $x, y \in \mathcal{A}$ as $n \to \infty$. If $x = e$, we have $f_j = d_j$. Therefore,

$$f_j(xy) = x f_j(y) + \sum_{i=1}^{j-1} g_i(x) f_{j-i}(y) + D_j(x) y, \tag{2.28}$$

for all $x, y \in \mathcal{A}$. Now, we replace y by $t^n y$ in (2.16), then

$$\left\| \frac{f_j(t^n x y)}{t^n} - \frac{x f_j(t^n y)}{t^n} - \sum_{i=1}^{j} g_i(x) f_{j-i}(y) \right\| \le \frac{\varphi(x, t^n y)}{t^n} \le \left(\frac{\alpha}{t}\right)^n, \tag{2.29}$$

for all $x, y \in \mathcal{A}$. Letting $n \to \infty$, we get

$$d_j(xy) = x d_j(y) + \sum_{i=1}^{j} g_i(x) f_{j-i}(y). \tag{2.30}$$

Thus if $y = e$, we conclude that

$$d_j(x) = x d_j(e) + \sum_{i=1}^{j} g_i(x) f_{j-i}(e), \tag{2.31}$$

for all $x \in \mathcal{A}$. Hence,

$$g_j(x) = d_j(x) - xd_j(e) - \sum_{i=1}^{j-1} g_i(x) f_{j-i}(e) = D_j(x). \tag{2.32}$$

But for all $x, y \in \mathcal{A}$, we have

$$\begin{aligned} D_j(xy) &= f_j(xy) - xyf_j(e) - \sum_{i=1}^{j-1} g_i(xy) f_{j-i}(e) \\ &= xf_j(y) + \sum_{i=1}^{j-1} g_i(x) f_{j-i}(y) + D_j(x)y - xyf_j(e) - \sum_{i=1}^{j-1}\left(\sum_{l=1}^{i} g_l(x) g_{i-l}(y)\right) f_{j-i}(e) \\ &= g_1(x)\left(f_{j-1}(y) - yf_{j-1}(e) - \sum_{l=1}^{j-1} g_l(y) f_{j-l}(e)\right) + \cdots + g_{j-1}(f_1(y) - yf_1(e)) \\ &= xD_j(y) + D_j(x)y + \sum_{i=1}^{j-1} g_i(x) g_{j-i}(y), \end{aligned} \tag{2.33}$$

and by (2.32), it follows that $g_j(xy) = D_j(xy) = \sum_{i=0}^{j} g_i(x) g_{j-i}(y)$; therefore $\{g_j\}$ is a strong higher derivation. By (2.28), we can conclude that $\{f_j\}$ is a generalized strong higher derivation. □

Remark 2.4. Recall that a control function is an operation that controls the recording or processing or transmission of interpretation of data. A typical example of the control function ψ is $\psi(x, y) = \alpha\epsilon(\|x\|^p + \|y\|^q) + \delta\|x\|^p\|y\|^q$, such that $\epsilon, \delta \geq 0$ and $0 \leq p, q < 1$.

Corollary 2.5. *Every (ϵ, ψ)-approximate generalized derivation (regarded as an approximate generalized strong higher derivation of rank 1) on a multi-Banach algebra $((\mathcal{A}^k, \|\cdot\|_k)k \in \mathbb{N})$ is a derivation.*

The following theorem generalizes Theorem 2.3. The arguments are similar to those in the proof of [21, Theorem 2.3].

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra with unit e, and let $\{f_j\}_{j=0}^{k_0}$ be a family $\{f_j\}_{j=0}^{k_0}$ of mappings from $\mathcal{A}$ into $\mathcal{A}$ with $f_j(0) = 0$ and $f_0 = id_{\mathcal{A}}$ for which there exists a family $\{g_j\}_{j=0}^{k_0}$ of mappings in which $g_0 = id_{\mathcal{A}}$ on $\mathcal{A}$ such that*

$$\begin{aligned} \sup_{k\in\mathbb{N}} \Bigg\| \Bigg(& f_j\left(\frac{\beta x_1 + \gamma y_1}{t} + z_1 w_1\right) - \beta\frac{f_j(x_1)}{t} - \gamma\frac{f_j(y_1)}{t} - z_1 f_j(w_1) \\ & - g_j(z_1)w_1, \ldots, f_j\left(\frac{\beta x_k + \gamma y_k}{t} + z_k w_k\right) \\ & -\beta\frac{f_j(x_k)}{t} - \gamma\frac{f_j(y_k)}{t} - z_k f_j(w_k) - g_j(z_k)w_k \Bigg) \Bigg\|_k \leq \epsilon, \end{aligned} \tag{2.34}$$

for all $0 \le j \le k_0$, $t > 1$ *and all* $\beta, \gamma \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ *and*

$$\left\| f_j(xy) - x f_j(y) - \sum_{i=1}^{j} g_i(x) f_{j-i}(y) \right\| \le \psi(x, y), \tag{2.35}$$

for all $0 \le j \le k_0$ *and* $x, y \in \mathcal{A}$, *then* $\{f_j\}_{j=0}^{k_0}$ *is a strong higher derivation.*

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