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Research Article

Boundary Value Problems for *q*-Difference Inclusions

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We investigate the existence of solutions for a class of second-order q-difference inclusions with nonseparated boundary conditions. By using suitable fixed-point theorems, we study the cases when the right-hand side of the inclusions has convex as well as nonconvex values.

1. Introduction

The discretization of the ordinary differential equations is an important and necessary step towards finding their numerical solutions. Instead of the standard discretization based on the arithmetic progression, one can use an equally efficient q-discretization related to geometric progression. This alternative method leads to q-difference equations, which in the limit $q \to 1$ correspond to the classical differential equations. q-difference equations are found to be quite useful in the theory of quantum groups [1]. For historical notes and development of the subject, we refer the reader to [2, 3] while some recent results on q-difference equations can be found in [4–6]. However, the theory of boundary value problems for nonlinear q-difference equations is still in the initial stages, and many aspects of this theory need to be explored.

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, stochastic analysis, and so forth and are widely studied by many authors; see [7–13] and the references therein. For some works concerning difference inclusions and dynamic inclusions on time scales, we refer the reader to the papers [14–17].

In this paper, we study the existence of solutions for second-order q-difference inclusions with nonseparated boundary conditions given by

$$D_q^2 u(t) \in F(t, u(t)), \quad 0 \le t \le T, \tag{1.1}$$

$$u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T), \tag{1.2}$$

where $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact valued multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, T is a fixed constant, and $\eta \neq 1$ is a fixed real number.

The aim of our paper is to establish some existence results for the Problems (1.1)-(1.2), when the right-hand side is convex as well as nonconvex valued. First of all, an integral operator is found by applying the tools of q-difference calculus, which plays a pivotal role to convert the given boundary value problem to a fixed-point problem. Our approach is simpler as it does not involve the typical series solution form of q-difference equations. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we will combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we will use the fixed-point theorem for generalized contraction multivalued maps due to Wegrzyk. The methods used are standard; however, their exposition in the framework of Problems (1.1)-(1.2) is new.

The paper is organized as follows: in Section 2, we recall some preliminary facts that we need in the sequel, and we prove our main results in Section 3.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which we need for the forthcoming analysis.

2.1. q-Calculus

Let us recall some basic concepts of q-calculus [1–3].

For $0 < q < 1$, we define the q-derivative of a real-valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t). \tag{2.1}$$

The higher-order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{2.2}$$

The q-integral of a function f defined in the interval $[a,b]$ is given by

$$\int_a^x f(t) d_q t := \sum_{n=0}^{\infty} x(1-q) q^n f(xq^n) - af(q^n a), \quad x \in [a,b], \tag{2.3}$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t) d_q t = \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n), \tag{2.4}$$

provided the series converges. If $a \in [0,b]$ and f is defined in the interval $[0,b]$, then

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \tag{2.5}$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{2.6}$$

Observe that

$$D_q I_q f(x) = f(x), \tag{2.7}$$

and if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0). \tag{2.8}$$

In q-calculus, the integration by parts formula is

$$\int_0^x f(t) D_q g(t) d_q t = \left[f(t)g(t)\right]_0^x - \int_0^x D_q f(t) g(qt) d_q t. \tag{2.9}$$

2.2. Multivalued Analysis

Let us recall some basic definitions on multivalued maps [18, 19].

Let X denote a normed space with the norm $|\cdot|$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all bounded sets B in X (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every bounded set B in X. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed-point if there is $x \in X$ such that $x \in G(x)$. The fixed-point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

For more details on multivalued maps, see the books of Aubin and Cellina [20], Aubin and Frankowska [21], Deimling [18], and Hu and Papageorgiou [19].

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm

$$\|u\|_{\infty} = \sup\{|u(t)| : t \in [0,T]\}. \tag{2.10}$$

Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $u : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|u\|_{L^1} = \int_0^T |u(t)|dt, \quad \forall u \in L^1([0,T],\mathbb{R}). \tag{2.11}$$

Definition 2.1. A multivalued map $G : [0,T] \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for any $x \in \mathbb{R}$, the function

$$\mathrm{t} \longmapsto d(x, F(t)) = \inf\{|x - z| : z \in F(t)\} \tag{2.12}$$

is measurable.

Definition 2.2. A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t) \tag{2.13}$$

for all $\|x\|_\infty \le \alpha$ and for a.e. $t \in [0,T]$.

Let E be a Banach space, let X be a nonempty closed subset of E, and let $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,T]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.3. If $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values and $u(\cdot) \in C([0,\mathrm{T}],\mathbb{R})$, then $F(\cdot,\cdot)$ is of lower semicontinuous type if

$$S_F(u) = \left\{ w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,u(t)) \quad \text{for a.e. } t \in [0,T] \right\} \tag{2.14}$$

is lower semicontinuous with closed and decomposable values.

Let (X,d) be a metric space associated with the norm $|\cdot|$. The Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},d^*(A,B)=\sup\{d(a,B):a\in A\},\tag{2.15}$$

where $d(x,B)=\inf_{y\in B}d(x,y)$.

Definition 2.4. A function $l:\mathbb{R}_+\to\mathbb{R}_+$ is said to be a strict comparison function (see [25]) if it is continuous strictly increasing and $\sum_{n=1}^{\infty}l^n(t)<\infty$, for each $t>0$.

Definition 2.5. A multivalued operator N on X with nonempty values in X is called

(a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$d_H(N(x),N(y))\le\gamma d(x,y),\quad\text{for each } x,y\in X,\tag{2.16}$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma<1$,

(c) a generalized contraction if and only if there is a strict comparison function $l:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$d_H(N(x),N(y))\le l(d(x,y)),\quad\text{for each } x,y\in X.\tag{2.17}$$

The following lemmas will be used in the sequel.

Lemma 2.6 (see [22]). *Let X be a Banach space. Let $F:[0,T]\times X\to\mathcal{P}(X)$ be an L^1-Carathéodory multivalued map with $S_F\neq\emptyset$, and let Γ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$, then the operator*

$$\Gamma\circ S_F:C([0,T],X)\longrightarrow\mathcal{P}(C([0,T],X))\tag{2.18}$$

defined by $(\Gamma\circ S_F)(x)=\Gamma(S_F(x))$ has compact convex values and has a closed graph operator in $C([0,T],X)\times C([0,T],X)$.

In passing, we remark that if $\dim X<\infty$, then $S_F(x)\neq\emptyset$ for any $x(\cdot)\in C([0,T],X)$ with $F(\cdot,\cdot)$ as in Lemma 2.6.

Lemma 2.7 (nonlinear alternative for Kakutani maps [23]). *Let E be a Banach space, C, a closed convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to\mathcal{P}_{c,cv}(C)$ is an upper semicontinuous compact map; here, $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C, then either*

(i) *F has a fixed-point in $\overline{U}$,*

(ii) *or there is a $u\in\partial U$ and $\lambda\in(0,1)$ with $u\in\lambda F(u)$.*

Lemma 2.8 (see [24]). *Let Y be a separable metric space, and let $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a lower semicontinuous multivalued map with closed decomposable values, then $N(\cdot)$ has a continuous* *selection; that is, there exists a continuous mapping (single-valued)* $g : Y \to L^1([0,T],\mathbb{R})$ *such that* $g(y) \in N(y)$ *for every* $y \in Y$.

Lemma 2.9 (Wegrzyk's fixed-point theorem [25, 26]). *Let* (X, d) *be a complete metric space. If* $N : X \to \mathcal{P}(X)$ *is a generalized contraction with nonempty closed values, then* $\operatorname{Fix} N \neq \emptyset$.

Lemma 2.10 (Covitz and Nadler's fixed-point theorem [27]). *Let* (X, d) *be a complete metric space. If* $N : X \to \mathcal{P}(X)$ *is a multivalued contraction with nonempty closed values, then* N *has a fixed-point* $z \in X$ *such that* $z \in N(z)$, *that is,* $\operatorname{Fix} N \neq \emptyset$.

3. Main Results

In this section, we are concerned with the existence of solutions for the Problems (1.1)-(1.2) when the right-hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

To define the solution for the Problems (1.1)-(1.2), we need the following result.

Lemma 3.1. *Suppose that* $\sigma : [0,T] \to \mathbb{R}$ *is continuous, then the following problem*

$$\begin{gathered} D_q^2 u(t) = \sigma(t), \quad \text{a.e. } t \in [0,T], \\ u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T) \end{gathered} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^T G(t,qs)\sigma(s)d_q s, \tag{3.2}$$

where $G(t,qs)$ *is the Green's function given by*

$$G(t,qs) = \frac{1}{(\eta-1)^2}\begin{cases} \eta(\eta-1)(qs-t)+\eta T, & \text{if } 0 \le t < s \le T, \\ (\eta-1)(qs-t)+\eta T, & \text{if } 0 \le s \le t \le T. \end{cases} \tag{3.3}$$

Proof. In view of (2.7) and (2.9), the solution of $D_q^2 u = \sigma(t)$ can be written as

$$u(t) = \int_0^t (t-qs)\sigma(s)d_q s + a_1 t + a_2, \tag{3.4}$$

where a_1, a_2 are arbitrary constants. Using the boundary conditions (1.2) and (3.4), we find that

$$a_1 = \frac{-\eta}{(\eta-1)} \int_0^T \sigma(s) d_q s,$$
$$a_2 = \frac{\eta^2 T}{(\eta-1)^2} \int_0^T \sigma(s) d_q s - \frac{\eta}{(\eta-1)} \int_0^T (T-qs)\sigma(s) d_q s. \tag{3.5}$$

Substituting the values of a_1 and a_2 in (3.4), we obtain (3.2). □

Let us denote

$$G_1 = \max_{t,s\in[0,T]} \left|G(t,qs)\right|. \tag{3.6}$$

Definition 3.2. A function $u \in C([0,T],\mathbb{R})$ is said to be a solution of (1.1)-(1.2) if there exists a function $v \in L^1([0,T],\mathbb{R})$ with $v(t) \in F(t,x(t))$ a.e. $t \in [0,T]$ and

$$u(t) = \int_0^T G(t,qs) v(s) d_q s, \tag{3.7}$$

where $G(t,qs)$ is given by (3.3).

Theorem 3.3. *Suppose that*

(H1) *the map* $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact convex values and is Carathéodory,*

(H2) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|_\infty) \tag{3.8}$$

for each $(t,u) \in [0,T]\times\mathbb{R}$,

(H3) *there exists a number* $M > 0$ *such that*

$$\frac{M}{G_1 \psi(M) \|p\|_{L^1}} > 1, \tag{3.9}$$

then the BVP (1.1)-(1.2) has at least one solution.

Proof. In view of Definition 3.2, the existence of solutions to (1.1)-(1.2) is equivalent to the existence of solutions to the integral inclusion

$$u(t) \in \int_0^T G(t,qs) F(s,u(s)) d_q s, \quad t \in [0,T]. \tag{3.10}$$

Let us introduce the operator

$$N(u) := \left\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^T G(t,qs)v(s)d_qs,\ v \in S_{F,u} \right\}. \tag{3.11}$$

We will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1 ($N(u)$ is convex for each $u \in C([0,T],\mathbb{R})$). Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $t \in [0,T]$, we have

$$h_i(t) = \int_0^T G(t,qs)v_i(s)d_qs, \quad (i = 1,2). \tag{3.12}$$

Let $0 \le d \le 1$, then, for each $t \in [0,T]$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^T G(t,qs)[dv_1(s) + (1-d)v_2(s)]d_qs. \tag{3.13}$$

Since $S_{F,u}$ is convex (because F has convex values); therefore,

$$dh_1 + (1-d)h_2 \in N(u). \tag{3.14}$$

Step 2 (N maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$). Let $B_m = \{u \in C([0,T],\mathbb{R}) : \|u\|_\infty \le m, m > 0\}$ be a bounded set in $C([0,T],\mathbb{R})$ and $u \in B_m$, then for each $h \in N(u)$, there exists $v \in S_{F,u}$ such that

$$h(t) = \int_0^T G(t,qs)v(s)d_qs. \tag{3.15}$$

Then, in view of (H2), we have

$$\begin{aligned} |h(t)| &\le \int_0^T \left|G(t,qs)\right| |v(s)| d_qs \\ &\le G_1 \int_0^T p(s)\psi(\|u\|_\infty)d_qs \\ &\le G_1\psi(m)\int_0^T p(s)d_qs. \end{aligned} \tag{3.16}$$

Thus,

$$\|h\|_\infty \le G_1\psi(m)\left\|p\right\|_{L^1}. \tag{3.17}$$

Step 3 (N maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$). Let $r_1, r_2 \in [0, T]$, $r_1 < r_2$ and B_m be a bounded set of $C([0,T],\mathbb{R})$ as in Step 2 and $x \in B_m$. For each $h \in N(u)$

$$
\begin{aligned}
|h(r_2)-h(r_1)| &\leq \int_0^T |G(r_2,s)-G(r_1,s)||v(s)|d_qs \\
&\leq \psi(\|u\|_\infty)\int_0^T |G(r_2,s)-G(r_1,s)|p(s)d_qs \\
&\leq \psi(m)\int_0^T |G(r_2,s)-G(r_1,s)|p(s)d_qs.
\end{aligned}
\tag{3.18}
$$

The right-hand side tends to zero as $r_2 - r_1 \to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli Theorem, we can conclude that $N : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

Step 4 (N has a closed graph). Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$. $h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that, for each $t \in [0,T]$,

$$h_n(t) = \int_0^T G(t,qs)v_n(s)d_qs. \tag{3.19}$$

We must show that there exists $h_* \in S_{F,u_*}$ such that, for each $t \in [0,T]$,

$$h_*(t) = \int_0^T G(t,qs)v_*(s)d_qs. \tag{3.20}$$

Clearly, we have

$$\|h_n - h_*\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

Consider the continuous linear operator

$$\Gamma : L^1([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R}), \tag{3.22}$$

defined by

$$v \longmapsto (\Gamma v)(t) = \int_0^T G(t,qs)v(s)d_qs. \tag{3.23}$$

From Lemma 2.6, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) \in \Gamma(S_{F,u_n}). \tag{3.24}$$

Since $u_n \to u_*$, it follows from Lemma 2.6 that

$$h_*(t) = \int_0^T G(t, qs) v_*(s) d_q s \tag{3.25}$$

for some $v_* \in S_{F,u_*}$.

Step 5 (a priori bounds on solutions). Let u be a possible solution of the Problems (1.1)-(1.2), then there exists $v \in L^1([0,T], \mathbb{R})$ with $v \in S_{F,u}$ such that, for each $t \in [0,T]$,

$$u(t) = \int_0^T G(t, qs) v(s) d_q s. \tag{3.26}$$

For each $t \in [0,T]$, it follows by (H2) and (H3) that

$$\begin{aligned} |u(t)| &\leq G_1 \int_0^T p(s)\psi(\|u\|_\infty) d_q s \\ &\leq G_1 \psi(\|u\|_\infty) \int_0^T p(s) d_q s. \end{aligned} \tag{3.27}$$

Consequently,

$$\frac{\|u\|_\infty}{G_1 \psi(\|u\|_\infty) \|p\|_{L^1}} \leq 1. \tag{3.28}$$

Then by (H3), there exists M such that $\|u\|_\infty \neq M$.

Let

$$U = \{u \in C([0,T], \mathbb{R}) : \|u\|_\infty < M + 1\}. \tag{3.29}$$

The operator $N : \overline{U} \to \mathcal{P}(C([0,T], \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. Consequently, by Lemma 2.7, it follows that N has a fixed-point u in $\overline{U}$ which is a solution of the Problems (1.1)-(1.2). This completes the proof. □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the nonlinear alternative of Leray-Schauder type combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values.

Theorem 3.4. *Suppose that the conditions (H2) and (H3) hold. Furthermore, it is assumed that*

(H4) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact values and*

(a) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $u \mapsto F(t,u)$ *is lower semicontinuous for a.e.* $t \in [0,T]$, (H5) *for each* $\rho > 0$, *there exists* $\varphi_\rho \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le \varphi_\rho(t) \quad \forall \|u\|_\infty \le \rho \text{ and for a.e. } t \in [0,T]. \tag{3.30}$$

then, the BVP (1.1)-(1.2) *has at least one solution.*

Proof. Note that (H4) and (H5) imply that F is of lower semicontinuous type. Thus, by Lemma 2.8, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(u) \in \mathcal{F}(u)$ for all $u \in C([0,T],\mathbb{R})$. So we consider the problem

$$\begin{gathered} D_q^2 u(t) = f(u(t)), \quad 0 \le t \le T, \\ u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T). \end{gathered} \tag{3.31}$$

Clearly, if $u \in C([0,T],\mathbb{R})$ is a solution of (3.31), then u is a solution to the Problems (1.1)-(1.2). Transform the Problem (3.31) into a fixed-point theorem

$$u(t) = \left(\overline{N}u\right)(t), \quad t \in [0,T], \tag{3.32}$$

where

$$\left(\overline{N}u\right)(t) = \int_0^T G(t,qs) f(u(s)) d_q s, \quad t \in [0,T]. \tag{3.33}$$

We can easily show that $\overline{N}$ is continuous and completely continuous. The remainder of the proof is similar to that of Theorem 3.3. □

Now, we prove the existence of solutions for the Problems (1.1)-(1.2) with a nonconvex valued right-hand side by applying Lemma 2.9 due to Wegrzyk.

Theorem 3.5. *Suppose that*

(H6) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact values and* $F(\cdot,u)$ *is measurable for each* $u \in \mathbb{R}$,

(H7) $d_H(F(t,u),F(t,\overline{u})) \le k(t) l(|u-\overline{u}|)$ *for almost all* $t \in [0,1]$ *and* $u,\overline{u} \in \mathbb{R}$ *with* $k \in L^1([0,1],\mathbb{R}_+)$ *and* $d(0,F(t,0)) \le k(t)$ *for almost all* $t \in [0,1]$, *where* $l : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing,*

then the BVP (1.1)-(1.2) *has at least one solution on* $[0,T]$ *if* $\gamma l : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strict comparison function, where* $\gamma = G_1 \|k\|_{L^1}$.

Proof. Suppose that $\gamma l : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function. Observe that by the assumptions (H6) and (H7), $F(\cdot, u(\cdot))$ is measurable and has a measurable selection $v(\cdot)$ (see Theorem 3.6 [28]). Also $k \in L^1([0,1],\mathbb{R})$ and

$$\begin{aligned} |v(t)| &\leq d(0, F(t,0)) + H_d(F(t,0), F(t,u(t))) \\ &\leq k(t) + k(t)l(|u(t)|) \\ &\leq (1 + l(\|u\|_\infty))k(t). \end{aligned} \tag{3.34}$$

Thus, the set $S_{F,u}$ is nonempty for each $u \in C([0,T],\mathbb{R})$.

As before, we transform the Problems (1.1)-(1.2) into a fixed-point problem by using the multivalued map N given by (3.11) and show that the map N satisfies the assumptions of Lemma 2.9. To show that the map $N(u)$ is closed for each $u \in C([0,T],\mathbb{R})$, let $(u_n)_{n\geq 0} \in N(u)$ such that $u_n \to \tilde{u}$ in $C([0,T],\mathbb{R})$, then $\tilde{u} \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,u}$ such that, for each $t \in [0,T]$,

$$u_n(t) = \int_0^T G(\mathrm{t}, qs) v_n(s) d_q s. \tag{3.35}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,u}$ and for each $t \in [0,T]$,

$$u_n(t) \longrightarrow \tilde{u}(t) = \int_0^T G(t, qs) v(s) d_q s. \tag{3.36}$$

So, $\tilde{u} \in N(u)$ and hence $N(u)$ is closed.

Next, we show that

$$d_H(N(u), N(\overline{u})) \leq \gamma l(\|u - \overline{u}\|_\infty) \quad \text{for each } u, \overline{u} \in C([0,T],\mathbb{R}). \tag{3.37}$$

Let $u, \overline{u} \in C([0,T],\mathbb{R})$ and $h_1 \in N(u)$. Then, there exists $v_1(t) \in S_{F,u}$ such that for each $t \in [0,T]$,

$$h_1(t) = \int_0^T G(t, qs) v_1(s) d_q s. \tag{3.38}$$

From (H7), it follows that

$$d_H(F(t, u(t)), F(t, \overline{u}(t))) \leq k(t) l(|u(t) - \overline{u}(t)|). \tag{3.39}$$

So, there exists $w \in F(t, \overline{u}(t))$ such that

$$|v_1(t) - w| \leq k(t) l(|u(t) - \overline{u}(t)|), \quad t \in [0,T]. \tag{3.40}$$

Define $U : [0,T] \to \mathcal{P}(\mathbb{R})$ as

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le k(t)l(|u(t) - \overline{u}(t)|)\}. \tag{3.41}$$

Since the multivalued operator $U(t) \cap F(t,\overline{u}(t))$ is measurable (see Proposition 3.4 in [28]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t,\overline{u}(t))$. So, $v_2(t) \in F(t,\overline{u}(t))$, and for each $t \in [0,T]$,

$$|v_1(t) - v_2(t)| \le k(t)l(|u(t) - \overline{u}(t)|). \tag{3.42}$$

For each $t \in [0,T]$, let us define

$$h_2(t) = \int_0^T G(t,qs)v_2(s)d_qs, \tag{3.43}$$

then

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \int_0^T |G(t,qs)||v_1(s) - v_2(s)|d_qs \\ &\le G_1 \int_0^T k(s)l(\|u - \overline{u}\|)d_qs. \end{aligned} \tag{3.44}$$

Thus,

$$\|h_1 - h_2\|_\infty \le G_1\|k\|_{L^1} l(\|u - \overline{u}\|_\infty) = \gamma l(\|u - \overline{u}\|_\infty). \tag{3.45}$$

By an analogous argument, interchanging the roles of u and $\overline{u}$, we obtain

$$d_H(N(u), N(\overline{u})) \le G_1\|k\|_{L^1} l(\|u - \overline{u}\|_\infty) = \gamma l(\|u - \overline{u}\|_\infty) \tag{3.46}$$

for each $u, \overline{u} \in C([0,T],\mathbb{R})$. So, N is a generalized contraction, and thus, by Lemma 2.9, N has a fixed-point u which is a solution to (1.1)-(1.2). This completes the proof. □

Remark 3.6. We notice that Theorem 3.5 holds for several values of the function l, for example, $l(t) = \ln(1+t)/\chi$, where $\chi \in (0,1)$, $l(t) = t$, and so forth. Here, we emphasize that the condition (H7) reduces to $d_H(F(t,u), F(t,\overline{u})) \le k(t)|u - \overline{u}|$ for $l(t) = t$, where a contraction principle for multivalued map due to Covitz and Nadler [27] (Lemma 2.10) is applicable under the condition $G_1\|k\|_{L^1} < 1$. Thus, our result dealing with a nonconvex valued right-hand side of (1.1) is more general, and the previous results for nonconvex valued right-hand side of the inclusions based on Covitz and Nadler's fixed-point result (e.g., see [8]) can be extended to generalized contraction case.

Remark 3.7. Our results correspond to the ones for second-order q-difference inclusions with antiperiodic boundary conditions $(u(0) = -u(T), D_q u(0) = -D_q u(T))$ for $\eta = -1$. The results for an initial value problem of second-order q-difference inclusions follow for $\eta = 0$. These results are new in the present configuration.

Remark 3.8. In the limit $q \to 1$, the obtained results take the form of their "continuous" (i.e., differential) counterparts presented in Sections 4 (ii) for $\lambda_1 = \lambda_2 = \eta, \mu_1 = 0 = \mu_2$ of [29].

Example 3.9. Consider a boundary value problem of second-order q-difference inclusions given by

$$\begin{gathered} D_q^2 u(t) \in F(t, u(t)), \quad 0 \le t \le 1 \\ u(0) = -u(1), \qquad D_q u(0) = -D_q u(1), \end{gathered} \tag{3.47}$$

where $\eta = -1$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$(t, u) \longrightarrow F(t, u) = \left[\frac{u^3}{u^3 + 3} + t^3 + 3, \frac{u}{u+1} + t + 1\right]. \tag{3.48}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{u^3}{u^3 + 3} + t^3 + 3, \frac{u}{u+1} + t + 1\right) \le 5, \quad u \in \mathbb{R}. \tag{3.49}$$

Thus,

$$\|F(t, u)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, u)\} \le 5 = p(t)\psi(\|u\|_\infty), \quad u \in \mathbb{R}, \tag{3.50}$$

with $p(t) = 1, \psi(\|u\|_\infty) = 5$. Further, using the condition

$$\frac{M}{G_1 \psi(M) \|p\|_{L^1}} > 1, \tag{3.51}$$

we find that $M > 5G_2$, where $G_2 = G_1|_{\eta=-1, T=1}$. Clearly, all the conditions of Theorem 3.3 are satisfied. So, the conclusion of Theorem 3.3 applies to the Problem (3.47).

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