Research Article

# Multiple Attractors for a Competitive System of Rational Difference Equations in the Plane 

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We investigate global dynamics of the following systems of difference equations $x_{n+1}=\beta_{1} x_{n}$ / $\left(B_{1} x_{n}+y_{n}\right), y_{n+1}=\left(\alpha_{2}+\gamma_{2} y_{n}\right) /\left(A_{2}+x_{n}\right), n=0,1,2, \ldots$, where the parameters $\beta_{1}, B_{1}, \beta_{2}, \alpha_{2}$, $\gamma_{2}, A_{2}$ are positive numbers, and initial conditions $x_{0}$ and $y_{0}$ are arbitrary nonnegative numbers such that $x_{0}+y_{0}>0$. We show that this system has up to three equilibrium points with various dynamics which depends on the part of parametric space. We show that the basins of attractions of different locally asymptotically stable equilibrium points or nonhyperbolic equilibrium points are separated by the global stable manifolds of either saddle points or of nonhyperbolic equilibrium points. We give an example of globally attractive nonhyperbolic equilibrium point and semistable non-hyperbolic equilibrium point.

## 1. Introduction

In this paper we consider the following rational system of difference equations

$$
\begin{align*}
& x_{n+1}=\frac{\beta_{1} x_{n}}{B_{1} x_{n}+y_{n}}, \quad n=0,1,2, \ldots, \\
& y_{n+1}=\frac{\alpha_{2}+\gamma_{2} y_{n}}{A_{2}+x_{n}} \tag{1.1}
\end{align*}
$$

where the parameters $\beta_{1}, B_{1}, \beta_{2}, \alpha_{2}, \gamma_{2}, A_{2}$ are positive numbers, and initial conditions $x_{0}$ and $y_{0}$ are nonnegative numbers such that $x_{0}+y_{0}>0$. System (1.1) was mentioned in [1] as one of three systems of open problem 3 which asked for the description of global dynamics of some rational systems of difference equations. In notation used to labels systems of linear fractional
difference equations used in [1] system (1.1) is known as (3.19) and (4.1). In this paper, we provide the precise description of global dynamics of the system (1.1). We show that the system (1.1) may have between zero and three equilibrium points, which may have different local character. If the system (1.1) has one equilibrium point, then this point is either locally asymptotically stable or saddle point or nonhyperbolic equilibrium point. If the system (1.1) has two equilibrium points, then they are either locally asymptotically stable, and nonhyperbolic, or locally asymptotically stable and saddle point. If the system (1.1) has three equilibrium points then two of the equilibrium points are locally asymptotically stable and the third point, which is between these two points in South-East ordering defined below, is a saddle point. The major problem for global dynamics of the system (1.1) is determining the basins of attraction of different equilibrium points. The difficulty in analyzing the behavior of all solutions of the system (1.1) lies in the fact that there are many regions of parameters where this system possesses different equilibrium points with different local character and that in several cases the equilibrium point is nonhyperbolic. However, all these cases can be handled by using recent results in [2]. The dual of this system is the system where $x_{n}$ and $y_{n}$ replace their role, and it was labeled as system (4.1) and (3.19) in [1]. Dynamics of this system immediately follows from the results proven here, by simply replacing the roles of $x_{n}$ and $y_{n}$.

System (1.1) is a competitive system, and our results are based on recent results about competitive systems in the plane, see $[2,3]$. System (1.1) has a potential to be used as a mathematical model for competition. In fact, the first equation of (1.1) is of Leslie-Gower type, and the second equation can be considered to be of Leslie-Gower type with stocking (or immigration) represented with the term $\alpha_{2}$, see [4-7]. Here $\beta_{1}, \gamma_{2}$ are the inherent birth rates while $B_{1}$ and $A_{2}$ are related to the density-dependent effects on newborn recruitment. Finally, $\alpha_{2}$ affects stocking for species with state variable $y_{n}$.

In Section 2, we present some general results about competitive systems in the plane. In Section 3 contains some basic facts such as the nonexistence of period-two solution of system (1.1). In Section 4 analyzes local stability which is fairly complicated for this system. Finally, in Section 5 gives global dynamics for all values of parameters. This section finishes with an introduction of a new terminology for different type scenarios for competitive systems that can be used to give a simple classification of all possible global behavior for system (1.1). The interesting feature of this paper is that there are five regions of the parameters in which one of the equilibrium points is nonhyperbolic, and yet we are able to describe the global dynamics in all five cases. To achieve this goal, we use new method of proving stability of nonhyperbolic equilibrium points introduced in [2].

## 2. Preliminaries

Consider a first-order system of difference equations of the form

$$
\begin{align*}
& x_{n+1}=f\left(x_{n}, y_{n}\right), \quad n=0,1,2, \ldots, \quad\left(x_{-1}, x_{0}\right) \in \partial \times \partial,  \tag{2.1}\\
& y_{n+1}=g\left(x_{n}, y_{n}\right) \text {, }
\end{align*}
$$

where $f, g: \supset \times \supset \rightarrow \supset$ are continuous functions on an interval $\partial \subset \mathbb{R}, f(x, y)$ is nondecreasing in $x$ and non-increasing in $y$, and $g(x, y)$ is non-increasing in $x$ and nondecreasing in $y$. Such system is called competitive. One may associate a competitive map $T$ to a competitive system (2.1) by setting $T=(f, g)$ and considering $T$ on $B=\Omega \times J$.

We now present some basic notions about competitive maps in plane. Define a partial order $\leq$ on $\mathbb{R}^{2}$ so that the positive cone is the fourth quadrant, that is, $\left(x^{1}, y^{1}\right) \leq\left(x^{2}, y^{2}\right)$ if and only if $x^{1} \leq x^{2}$ and $y^{1} \geq y^{2}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{2}$ the order interval $\llbracket \mathbf{x}, \mathbf{y} \rrbracket$ is the set of all $\mathbf{z}$ such that $\mathbf{x} \leq \mathbf{z} \leq \mathbf{y}$. A set $\mathcal{A}$ is said to be linearly ordered if $\leq$ is a total order on $\mathcal{A}$. If a set $\mathcal{A} \subset \mathbb{R}^{2}$ is linearly ordered by $\preceq$, then the infimum $\mathbf{i}=\inf \mathcal{A}$ and supremum $s=\sup \mathcal{A}$ of $\mathcal{A}$ exist in $\overline{\mathbb{R}}^{2}=[-\infty, \infty] \times[-\infty, \infty]$. If both $\mathbf{i}$ and $\mathbf{s}$ belong to $\mathbb{R}^{2}$, then the linearly ordered set $\mathcal{A}$ is bounded, and conversely. We note that the ordering $\leq$ may be extended to the extended plane $\overline{\mathbb{R}}^{2}$ in a natural way. For example, $(0, \infty) \leq(a, b)$ if $a \geq 0$ or $a=\infty$. If $\mathbf{x} \in \mathbb{R}^{2}$, we denote with $Q_{\ell}(\mathbf{x}), \ell \in\{1,2,3,4\}$, the four quadrants in $\overline{\mathbb{R}}^{2}$ relative to $\mathbf{x}$, that is, $Q_{1}(x, y)=\left\{(u, v) \in \overline{\mathbb{R}}^{2}\right.$ : $u \geq x, v \geq y\}, Q_{2}(x, y)=\left\{(u, v) \in \overline{\mathbb{R}}^{2}: x \geq u, v \geq y\right\}$, and so on.

A map $T$ on a set $\bar{B} \subset \mathbb{R}^{2}$ is a continuous function $T: B \rightarrow B$. The map is smooth on $B$ if the interior of $B$ is nonempty and if $T$ is continuously differentiable on the interior of $\mathcal{B}$. A set $\mathcal{A} \subset \mathbb{B}$ is invariant for the map $T$ if $T(\mathcal{A}) \subset \mathcal{A}$. A point $\mathbf{x} \in \mathbb{B}$ is a fixed point of $T$ if $T(\mathbf{x})=\mathbf{x}$, and a minimal period-two point if $T^{2}(\mathbf{x})=\mathbf{x}$ and $T(\mathbf{x}) \neq \mathbf{x}$. A period-two point is either a fixed point or a minimal period-two point. The orbit of $\mathbf{x} \in \mathcal{B}$ is the sequence $\left\{T^{\ell}(\mathbf{x})\right\}_{\ell=0}^{\infty}$. A minimal period two orbit is an orbit $\left\{\mathbf{x}_{\ell}\right\}_{\ell=0}^{\infty}$ for which $\mathbf{x}_{0} \neq \mathbf{x}_{1}$ and $\mathbf{x}_{0}=\mathbf{x}_{2}$. The basin of attraction of a fixed point $\mathbf{x}$ is the set of all $\mathbf{y}$ such that $T^{n}(\mathbf{y}) \rightarrow \mathbf{x}$. A fixed point $\mathbf{x}$ is a global attractor on a set $\mathcal{A}$ if $\mathcal{A}$ is a subset of the basin of attraction of $\mathbf{x}$. A fixed point $\mathbf{x}$ is a saddle point if $T$ is differentiable at $\mathbf{x}$, and the eigenvalues of the Jacobian matrix of $T$ at $\mathbf{x}$ are such that one of them lies in the interior of the unit circle in $\mathbb{R}^{2}$, while the other eigenvalue lies in the exterior of the unit circle. If $T=\left(T_{1}, T_{2}\right)$ is a map on $\mathcal{R} \subset \mathbb{R}^{2}$, define the sets $\mathcal{R}_{T}(-,+):=\left\{(x, y) \in \mathcal{R}: T_{1}(x, y) \leq\right.$ $\left.x, T_{2}(x, y) \geq y\right\}$ and $\mathcal{R}_{T}(+,-):=\left\{(x, y) \in \mathcal{R}: T_{1}(x, y) \geq x, T_{2}(x, y) \leq y\right\}$. For $\mathcal{A} \subset \mathbb{R}^{2}$ and $x \in \mathbb{R}^{2}$, define the distance from $x$ to $\mathcal{A}$ as $\operatorname{dist}(x, \mathcal{A}):=\inf \{\|x-y\|: y \in \mathcal{A}\}$.

A map $T$ is competitive if $T(\mathbf{x}) \preceq T(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$, and $T$ is strongly competitive if $\mathbf{x} \leq \mathbf{y}$ implies that $T(\mathbf{x})-T(\mathbf{y}) \in\{(u, v): u>0, v<0\}$. If $T$ is differentiable, a sufficient condition for $T$ to be strongly competitive is that the Jacobian matrix of $T$ at any $\mathbf{x} \in \mathcal{B}$ has the sign configuration

$$
\left(\begin{array}{ll}
+ & -  \tag{2.2}\\
- & +
\end{array}\right)
$$

For additional definitions and results (e.g., repeller, hyperbolic fixed points, stability, asymptotic stability, stable and unstable manifolds) see [8,9] for competitive maps, and $[10,11]$ for difference equations.

If $\mathcal{A}$ is any subset of $\mathbb{R}^{k}$, we shall use the notation $\operatorname{clos}(\mathcal{A})$ to denote the closure of $\mathcal{A}$ in $\mathbb{R}^{k}$, and $\mathcal{A}^{\circ}$ to denote the interior of $\mathcal{A}$.

The next results are stated for order-preserving maps on $\mathbb{R}^{n}$ and are known but given here for completeness. See [12] for a more general version valid in ordered Banach spaces.

Theorem 2.1. For a nonempty set $R \subset \mathbb{R}^{n}$ and $\leq$ a partial order on $\mathbb{R}^{n}$, let $T: R \rightarrow R$ be an orderpreserving map, and let $a, b \in R$ be such that $a<b$ and $\llbracket a, b \rrbracket \subset R$. If $a \leq T(a)$ and $T(b) \leq b$, then $\llbracket a, b \rrbracket$ is invariant and
(i) there exists a fixed point of $T$ in $\llbracket a, b \rrbracket$,
(ii) if $T$ is strongly order preserving, then there exists a fixed point in $\llbracket a, b \rrbracket$ which is stable relative to $\llbracket a, b \rrbracket$,
(iii) if there is only one fixed point in $\llbracket a, b \rrbracket$, then it is a global attractor in $\llbracket a, b \rrbracket$ and therefore asymptotically stable relative to $\llbracket a, b \rrbracket$.

Corollary 2.2. If the nonnegative cone of $\leq$ is a generalized quadrant in $\mathbb{R}^{n}$, and if $T$ has no fixed points in $\llbracket u_{1}, u_{2} \rrbracket$ other than $u_{1}$ and $u_{2}$, then the interior of $\llbracket u_{1}, u_{2} \rrbracket$ is either a subset of the basin of attraction of $u_{1}$ or a subset of the basin of attraction of $u_{2}$.

Define a rectangular region $\mathcal{R}$ in $\mathbb{R}^{2}$ to be the cartesian product of two intervals in $\mathbb{R}$.
Remark 2.3. It follows from the Perron-Frobenius theorem and a change of variables [9] that, at each point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly competitive then no eigenvector is aligned with a coordinate axis.

Theorem 2.4. Let $T$ be a competitive map on a rectangular region $R \subset \mathbb{R}^{2}$. Let $\bar{x} \in \mathcal{R}$ be a fixed point of $T$ such that $\Delta:=\mathcal{R} \cap \operatorname{int}\left(Q_{1}(\bar{x}) \cup Q_{3}(\bar{x})\right)$ is nonempty (i.e., $\bar{x}$ is not the NW or SE vertex of $\mathcal{R}$ ), and $T$ is strongly competitive on $\Delta$. Suppose that the following statements are true.
(a) The map $T$ has a $C^{1}$ extension to a neighborhood of $\bar{x}$.
(b) The Jacobian matrix $J_{T}(\bar{x})$ of $T$ at $\bar{x}$ has real eigenvalues $\lambda, \mu$ such that $0<|\lambda|<\mu$, where $|\lambda|<1$, and the eigenspace $E^{\lambda}$ associated with $\lambda$ is not a coordinate axes.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\bar{x}$ that is invariant and a subset of the basin of attraction of $\bar{x}$, such that $\mathcal{C}$ is tangential to the eigenspace $E^{\lambda}$ at $\bar{x}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathbb{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of $T$.

We shall see in Theorem 2.7 and in the examples in [2] that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 2.5. For the curve $\mathcal{C}$ of Theorem 2.4 to have endpoints in $\partial \mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.
(i) The map $T$ has no fixed points nor periodic points of minimal period two in $\Delta$.
(ii) The map $T$ has no fixed points in $\Delta$, $\operatorname{det} J_{T}(\bar{x})>0$, and $T(x)=\bar{x}$ has no solutions $x \in \Delta$.
(iii) The map $T$ has no points of minimal period two in $\Delta$, $\operatorname{det} J_{T}(\bar{x})<0$, and $T(x)=\bar{x}$ has no solutions $x \in \Delta$.

In many cases, one can expect the curve $\mathcal{C}$ to be smooth.
Theorem 2.6. Under the hypotheses of Theorem 2.4, suppose that there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ in $\mathbb{R}^{2}$ such that $T$ is of class $C^{k}$ on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian matrix of $T$ at each $x \in \Delta$ is invertible. Then, the curve $\mathcal{C}$ in the conclusion of Theorem 2.4 is of class $C^{k}$.

In applications, it is common to have rectangular domains $\mathcal{R}$ for competitive maps. If a competitive map has several fixed points, often the domain of the map may be split
into rectangular invariant subsets such that Theorem 2.4 could be applied to the restriction of the map to one or more subsets. For maps that are strongly competitive near the fixed point, hypothesis (b) of Theorem 2.4 reduces just to $|\lambda|<1$. This follows from a change of variables [9] that allows the Perron-Frobenius theorem to be applied to give that at any point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axes.

Smith performed a systematic study of competitive and cooperative maps in $[9,13,14]$ and in particular introduced invariant manifolds techniques in his analysis [13-15] with some results valid for maps on $n$-dimensional space. Smith restricted attention mostly to competitive maps $T$ that satisfy additional constraints. In particular, $T$ is required to be a diffeomorphism of a neighborhood of $\mathbb{R}_{+}^{n}$ that satisfies certain conditions (this is the case if $T$ is orientation preserving or orientation reversing), and that the coordinate semiaxes are invariant under $T$. For such class of maps (as well as for cooperative maps satisfying similar hypotheses), Smith obtained results on invariant manifolds passing through hyperbolic fixed points and a fairly complete description of the phase-portrait when $n=2$, especially for those cases having a unique fixed point on each of the open positive semiaxes. In our results, presented here, we removed all these constraints and added the precise analysis of invariant manifolds of nonhyperbolic equilibrium points. The invariance of coordinate semiaxes seems to be serious restriction in the case of competitive models with constant stocking or harvesting, see [16] for stocking.

The next result is useful for determining basins of attraction of fixed points of competitive maps. Compare to Theorem 4.4 in [13], where hyperbolicity of the fixed point is assumed, in addition to other hypotheses.

Theorem 2.7. (A) Assume the hypotheses of Theorem 2.4, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2.4. If the endpoints of $\mathcal{C}$ belong to $\partial \boldsymbol{R}$, then $\mathcal{C}$ separates $\boldsymbol{R}$ into two connected components, namely,

$$
\begin{align*}
& \mathcal{W}_{-}:=\left\{x \in \mathcal{R} \backslash \mathcal{C}: \exists y \in \mathcal{C} \text { with } x \leq_{\text {se }} y\right\}, \\
& \mathcal{W}_{+}:=\left\{x \in \mathcal{R} \backslash \mathcal{C}: \exists y \in \mathcal{C} \text { with } y \leq_{\text {se }} x\right\}, \tag{2.3}
\end{align*}
$$

such that the following statements are true:
(i) $\mathcal{W}_{-}$is invariant, and $\operatorname{dist}\left(T^{n}(x), Q_{2}(\bar{x})\right) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_{-}$.
(ii) $\mathcal{O}_{+}$is invariant, and $\operatorname{dist}\left(T^{n}(x), Q_{4}(\bar{x})\right) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_{+}$.
(B) If, in addition to the hypotheses of part ( $A$ ), $\bar{x}$ is an interior point of $\mathcal{R}$, and $T$ is $C^{2}$ and strongly competitive in a neighborhood of $\bar{x}$, then $T$ has no periodic points in the boundary of $Q_{1}(\bar{x}) \cup Q_{3}(\bar{x})$ except for $\bar{x}$, and the following statements are true.
(iii) For every $x \in \mathcal{W}_{-}$there exists $n_{0} \in \mathbb{N}$ such that $T^{n}(x) \in \operatorname{int} \mathcal{Q}_{2}(\bar{x})$ for $n \geq n_{0}$.
(iv) For every $x \in \mathcal{W}_{+}$there exists $n_{0} \in \mathbb{N}$ such that $T^{n}(x) \in \operatorname{int} Q_{4}(\bar{x})$ for $n \geq n_{0}$.

Basins of attraction of period-two solutions or period-two orbits of certain systems or maps can be effectively treated with Theorems 2.4 and 2.7. See $[2,6,11]$ for the hyperbolic case; for the nonhyperbolic case, see examples in [2,17].

If $T$ is a map on a set $R$ and if $\bar{x}$ is a fixed point of $T$, the stable set $\mathcal{W}^{s}(\bar{x})$ of $\bar{x}$ is the set $\left\{x \in \mathcal{R}: T^{n}(x) \rightarrow \bar{x}\right\}$, and unstable set $\mathcal{W}^{u}(\bar{x})$ of $\bar{x}$ is the set

$$
\begin{equation*}
\left\{x \in \mathcal{R}: \text { there exists }\left\{x_{n}\right\}_{n=-\infty}^{0} \subset \mathcal{R} \text { s.t. } T\left(x_{n}\right)=x_{n+1}, x_{0}=x, \lim _{n \rightarrow-\infty} x_{n}=\bar{x}\right\} . \tag{2.4}
\end{equation*}
$$

When $T$ is noninvertible, the set $\mathcal{W}^{s}(\bar{x})$ may not be connected and made up of infinitely many curves, or $\mathcal{J}^{u}(\bar{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^{s}(\bar{x})$ and $\mathcal{W}^{u}(\bar{x})$ are the stable and unstable manifolds of $\bar{x}$.

Theorem 2.8. In addition to the hypotheses of part (B) of Theorem 2.7, suppose that $\mu>1$ and that the eigenspace $E^{\mu}$ associated with $\mu$ is not a coordinate axes. If the curve $\mathcal{C}$ of Theorem 2.4 has endpoints in $\partial \mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^{s}(\bar{x})$ of $\bar{x}$, and the unstable set $\mathcal{W}^{u}(\bar{x})$ of $\bar{x}$ is a curve in $R$ that is tangential to $E^{\mu}$ at $\bar{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^{u}(\bar{x})$ in $\mathcal{R}$ are fixed points of $T$.

The following result gives information on local dynamics near a fixed point of a map when there exists a characteristic vector whose coordinates have negative product and such that the associated eigenvalue is hyperbolic. This is a well-known result, valid in much more general setting which we include it here for completeness. A point $(x, y)$ is a subsolution if $T(x, y) \preceq_{\mathrm{se}}(x, y)$, and $(x, y)$ is a supersolution if $(x, y) \preceq_{\mathrm{se}} T(x, y)$. An order interval $\llbracket(a, b),(c, d) \rrbracket$ is the cartesian product of the two compact intervals $[a, c]$ and $[b, d]$.

Theorem 2.9. Let $T$ be a competitive map on a rectangular set $\mathcal{R} \subset \mathbb{R}^{2}$ with an isolated fixed point $\bar{x} \in \mathcal{R}$ such that $\mathcal{R} \cap \operatorname{int}\left(Q_{2}(\bar{x}) \cup Q_{4}(\bar{x})\right) \neq \emptyset$. Suppose that $T$ has a $C^{1}$ extension to a neighborhood of $\bar{x}$. Let $v=\left(v^{(1)}, v^{(2)}\right) \in \mathbb{R}^{2}$ be an eigenvector of the Jacobian matrix of $T$ at $\bar{x}$, with associated eigenvalue $\mu \in \mathbb{R}$. If $v^{(1)} v^{(2)}<0$, then there exists an order interval $\rho$ which is also a relative neighborhood of $\bar{x}$ such that, for every relative neighborhood $\mathcal{U} \subset \supset$ of $\bar{x}$, the following statements are true.
(i) If $\mu>1$, then $\mathcal{U} \cap \operatorname{int} Q_{2}(\bar{x})$ contains a subsolution, and $\mathcal{U} \cap \operatorname{int} Q_{4}(\bar{x})$ contains a supersolution. In this case, for every $x \in \supset \cap \operatorname{int}\left(Q_{2}(\bar{x}) \cup Q_{4}(\bar{x})\right)$, there exists $N$ such that $T^{n}(x) \notin \rho$ for $n \geq N$.
(ii) If $\mu<1$, then $\mathcal{U} \cap \operatorname{int} Q_{2}(\bar{x})$ contains a supersolution and $\mathcal{U} \cap$ int $Q_{4}(\bar{x})$ contains a subsolution. In this case, $T^{n}(x) \rightarrow \bar{x}$ for every $x \in \mathcal{D}$.

In the nonhyperbolic case, we have the following result.
Theorem 2.10. Assume that the hypotheses of Theorem 2.9 hold, that $T$ is real analytic at $\bar{x}$, and that $\mu=1$. Let $c_{j}, d_{j}, j=2,3, \ldots$ be defined by the Taylor series

$$
\begin{equation*}
T(\bar{x}+t v)=\bar{x}+v t+\left(c_{2}, d_{2}\right) t^{2}+\cdots+\left(c_{n}, d_{n}\right) t^{n}+\cdots \tag{2.5}
\end{equation*}
$$

Suppose that there exists an index $\ell \geq 2$ such that $\left(c_{\ell}, d_{\ell}\right) \neq(0,0)$ and $\left(c_{j}, d_{j}\right)=(0,0)$ for $j<l$. If either
(a) $c_{\ell} d_{\ell}<0$, or $(b) c_{\ell} \neq 0, T(\bar{x}+t v)^{(2)}$ is affine in $t$, or $(c) d_{\ell} \neq 0, T(\bar{x}+t v)^{(1)}$ is affine in $t$, then there exists an order interval 3 which is also a relative neighborhood of $\bar{x}$ such that, for every relative neighborhood $\mathcal{U} \subset \supset$ of $\bar{x}$, the following statements are true.
(i) If $\ell$ is odd and $\left(c_{\ell}, d_{\ell}\right) \preceq_{s e}(0,0)$, then $\mathcal{U} \cap \operatorname{int} Q_{4}(\bar{x})$ contains a supersolution, and $\mathcal{U} \cap \operatorname{int} Q_{2}(\bar{x})$ contains a subsolution. In this case, for every $x \in \partial \cap \operatorname{int}\left(Q_{2}(\bar{x}) \cup\right.$ $\left.Q_{4}(\bar{x})\right)$, there exists $N$ such that $T^{n}(x) \notin$ O for $n \geq N$.
(ii) If $\ell$ is odd and $(0,0) \leq_{s e}\left(c_{\ell}, d_{\ell}\right)$, then $\mathcal{U} \cap$ int $Q_{4}(\bar{x})$ contains a subsolution and $\mathcal{U} \cap$ int $Q_{2}(\bar{x})$ contains a supersolution. In this case, $T^{n}(x) \rightarrow \bar{x}$ for every $x \in J$.
(iii) If $\ell$ is even and $\left(c_{\ell}, d_{\ell}\right) \leq_{s e}(0,0)$, then $\mathcal{U} \cap$ int $Q_{4}(\bar{x})$ contains a subsolution and $\mathcal{U}$ Пint $Q_{2}(\bar{x})$ contains a subsolution. In this case, $T^{n}(x) \rightarrow \bar{x}$ for every $x \in \supset \cap Q_{4}(\bar{x})$, and for every $x \in \supset \cap \operatorname{int}\left(Q_{2}(\bar{x})\right)$, there exists $N$ such that $T^{n}(x) \notin \supset$ for $n \geq N$.
(iv) If $\ell$ is even and $(0,0) \preceq_{s e}\left(c_{\ell}, d_{\ell}\right)$, then $\mathcal{U} \cap$ int $Q_{2}(\bar{x})$ contains a supersolution and $\mathcal{U} \cap$ int $Q_{4}(\bar{x})$ contains a supersolution. In this case, $T^{n}(x) \rightarrow \bar{x}$ for every $x \in$ $\partial \cap Q_{2}(\bar{x})$, and, for every $x \in \supset \cap \operatorname{int}\left(Q_{4}(\bar{x})\right)$ there exists, $N$ such that $T^{n}(x) \notin \supset$ for $n \geq N$.

## 3. Some Basic Facts

In this section, we give some basic facts about the nonexistence of period-two solutions, local injectivity of map $T$ at the equilibrium point.

### 3.1. Equilibrium Points

The equilibrium points $(\bar{x}, \bar{y})$ of the system (1.1) satisfy

$$
\begin{align*}
& \bar{x}=\frac{\beta_{1} \bar{x}}{B_{1} \bar{x}+\bar{y}^{\prime}}  \tag{3.1}\\
& \bar{y}=\frac{\alpha_{2}+\gamma_{2} \bar{y}}{A_{2}+\bar{x}}
\end{align*}
$$

Solutions of System (3.1) are
(i) $\bar{x}=0, \bar{y}=\alpha_{2} / A_{2}-\gamma_{2}$ when $A_{2}>\gamma_{2}$, that is,

$$
\begin{equation*}
E_{1}=\left(0, \frac{\alpha_{2}}{A_{2}-\gamma_{2}}\right) \tag{3.2}
\end{equation*}
$$

(ii) If $\bar{x} \neq 0$, then using System (3.1), we obtain

$$
\begin{gather*}
\bar{y}=\beta_{1}-B_{1} \bar{x} \\
0=\bar{x}^{2} B_{1}-\bar{x}\left(B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}\right)-\left(\beta_{1}\left(A_{2}-\gamma_{2}\right)-\alpha_{2}\right) \tag{3.3}
\end{gather*}
$$

Solutions of System (3.3) are

$$
\begin{equation*}
\bar{x}_{3,2}=\frac{B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1} \pm \sqrt{D_{0}}}{2 B_{1}}, \quad \bar{y}_{2,3}=\frac{B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1} \pm \sqrt{D_{0}}}{2} \tag{3.4}
\end{equation*}
$$

Table 1

| $E_{1}$ | $A_{2}>\gamma_{2}, \alpha_{2}>\frac{\left(\beta_{1}+B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}$ or |
| :--- | :--- |
|  | $A_{2}>\gamma_{2}, \alpha_{2} \leq \frac{\left(\beta_{1}+B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}, \beta_{1}<\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, B_{1}>\frac{\beta_{1}}{A_{2}-\gamma_{2}}$ |
| $E_{1} \equiv E_{2} \equiv E_{3}$ | $A_{2}>\gamma_{2}, \beta_{1}=\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, B_{1}=\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}$ |
| $E_{1} \equiv E_{3}$ | $A_{2}>\gamma_{2}, \beta_{1}=\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, B_{1}>\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}$ |
| $E_{1} \equiv E_{2}, E_{3}$ | $A_{2}>\gamma_{2}, \beta_{1}=\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, B_{1}<\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}$ |
| $E_{1}, E_{2}, E_{3}$ | $A_{2}>\gamma_{2}, B_{1}\left(\gamma_{2}-A_{2}\right)+\sqrt{4 B_{1} \alpha_{2}}<\beta_{1}<\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, \alpha_{2}>B_{1}\left(A_{2}-\gamma_{2}\right)^{2}$ |
| $E_{1}, E_{3}$ | $A_{2}>\gamma_{2}, \beta_{1}>\frac{\alpha_{2}}{A_{2}-\gamma_{2}}$ |
| $E_{1}, E_{2}=E_{3}$ | $A_{2}>\gamma_{2},\left(A_{2}-\gamma_{2}\right) B_{1}<\beta_{1},\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}=0$ |
| $E_{2}, E_{3}$ | $A_{2}<\gamma_{2},\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}>0, \beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right)$ |
| $E_{2}=E_{3}$ | $A_{2}<\gamma_{2},\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}=0, \beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right)$ |
| $e_{1}, e_{2}$ | $A_{2}=\gamma_{2}, \beta_{1}^{2}-4 B_{1} \alpha_{2} \geq 0$ |
| No equilibrium | $\gamma_{2} \geq A_{2}$ and $\alpha_{2} \leq \frac{\left(\beta_{1}+B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}, \beta_{1}<B_{1}\left(\gamma_{2}-A_{2}\right)$ or |

where $D_{0}=\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}$, which gives a pair of the equilibrium points $E_{2}=$ $\left(\bar{x}_{2}, \bar{y}_{2}\right)$ and $E_{3}=\left(\bar{x}_{3}, \bar{y}_{3}\right)$.

Geometrically, the equilibrium points are the intersections of two equilibrium curves: $C_{1}: x=0 \cup y=-B_{1} x+\beta_{1}$ and $C_{2}: y=\alpha_{2} /\left(A_{2}-\gamma_{2}+x\right)$. Depending on the values of parameters, $C_{2}$ may have between 0 and 3 intersection points with two lines which constitutes $C_{1}$.

The algebraic criteria for the existence of the equilibrium points are summarized in Table 1.

Where

$$
\begin{align*}
& e_{1}=\left(\frac{\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}}, \frac{\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2}\right), \\
& e_{2}=\left(\frac{\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}}, \frac{\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2}\right) . \tag{3.5}
\end{align*}
$$

Remark 3.1. Observe the following: If the system (1.1) has two or three equilibrium points $E_{1}$, $E_{2}$, and $E_{3}$ then, $E_{1} \preceq E_{2} \preceq E_{3}$. Indeed, consider the critical curve $C_{2}: y=\alpha_{2} /\left(A_{2}-\gamma_{2}+x\right)$. Observe that $y(0)=\alpha_{2} /\left(A_{2}-\gamma_{2}\right), y\left(\bar{x}_{2}\right)=\bar{y}_{2}$, and $y\left(\bar{x}_{3}\right)=\bar{y}_{3}$. It is obvious that the following
holds $0 \leq \bar{x}_{2} \leq \bar{x}_{3}$. Since, the critical curve $C_{2}$ decreases, we have $y(0) \geq y\left(\bar{x}_{2}\right) \geq y\left(\bar{x}_{3}\right)$, that is, $E_{1} \preceq E_{2} \preceq E_{3}$.

Lemma 3.2. Assume that $x_{0}=0, y_{0} \in \mathbb{R}^{+}=(0, \infty)$. Then the following statements are true for solutions of the system (1.1).
(i) If $A_{2}>\gamma_{2}$, then $x_{n}=0$, for all $n \in \mathbb{N}$, and $y_{n} \rightarrow \alpha_{2} /\left(A_{2}-\gamma_{2}\right), n \rightarrow \infty$.
(ii) If $A_{2}<\gamma_{2}$, then $x_{n}=0$, for all $n \in \mathbb{N}$, and $y_{n} \rightarrow \infty, n \rightarrow \infty$.
(iii) If $A_{2}=\gamma_{2}$, then $y_{n}=y_{0}+\left(\alpha_{2} / \gamma_{2}\right) n$, and $x_{n}=0$, for all $n \in \mathbb{N}, y_{n} \rightarrow \infty, n \rightarrow \infty$.

Assume that $x_{0} \neq 0$ and $\left(x_{0}, y_{0}\right) \in \mathbb{R}_{2}^{+}$. Then, the following statements are true for all $n=$ 1,2,...:
(iv) $x_{n} \leq \beta_{1} / B_{1}$.
(v) $y_{n} \leq c\left(\gamma_{2} / A_{2}\right)^{n}+\left(\alpha_{2} /\left(A_{2}-\gamma_{2}\right)\right)$ and
(a) $y_{n} \geq B_{1} \alpha_{2} /\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)+\epsilon_{1}, A_{2}>\gamma_{2}$, where $\epsilon_{1}$ is arbitrarily small positive number.
(b) $y_{n} \leq \alpha_{2} /\left(A_{2}-\gamma_{2}\right)+\varepsilon_{2}, \varepsilon>0, A_{2}>\gamma_{2}$, where $\epsilon_{2}$ is arbitrarily small positive number.

Proof. Since (i)-(iv) are immediate consequences of the system (1.1), we will prove only (v).
Take $x_{0}=0$ and $y_{0} \in \mathbb{R}^{+}$. Then, we have $x_{n}=0$ for all $n \in \mathbb{N}$, and

$$
\begin{equation*}
y_{n+1}=\frac{\alpha_{2}}{A_{2}}+\frac{\gamma_{2}}{A_{2}} y_{n} . \tag{3.6}
\end{equation*}
$$

Solution of (3.6), when $A_{2} \neq \gamma_{2}$ is

$$
\begin{equation*}
y_{n}=c\left(\frac{\gamma_{2}}{A_{2}}\right)^{n}+\frac{\alpha_{2}}{A_{2}-\gamma_{2}} \tag{3.7}
\end{equation*}
$$

which immediately implies (i) and (ii). Statement (iii) follows from (3.6).
Equation

$$
\begin{equation*}
x_{n+1}=\frac{\beta_{1} x_{n}}{B_{1} x_{n}+y_{n}} \tag{3.8}
\end{equation*}
$$

implies that

$$
\begin{equation*}
x_{n} \leq \frac{\beta_{1}}{B_{1}} \tag{3.9}
\end{equation*}
$$

Using the last inequality, we have

$$
\begin{equation*}
y_{n+1}=\frac{\alpha_{2}+\gamma_{2} y_{n}}{A_{2}+x_{n}} \geq \frac{\alpha_{2}+\gamma_{2} y_{n}}{A_{2}+\left(\beta_{1} / B_{1}\right)}=\frac{B_{1} \alpha_{2}}{A_{2} B_{1}+\beta_{1}}+\frac{\gamma_{2} B_{1}}{B_{1} A_{2}+\beta_{1}} y_{n} \tag{3.10}
\end{equation*}
$$

which by difference inequality theorem [18] implies the following

$$
\begin{equation*}
y_{n} \geq c\left(\frac{\gamma_{2} B_{1}}{A_{2} B_{1}+\beta_{1}}\right)^{n}+\frac{B_{1} \alpha_{2}}{B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}} \geq \frac{B_{1} \alpha_{2}}{B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}}+\epsilon_{1}, \quad n \longrightarrow \infty . \tag{3.11}
\end{equation*}
$$

Furthermore, second equation in (1.1) implies that

$$
\begin{equation*}
y_{n+1} \leq \frac{\alpha_{2}}{A_{2}}+\frac{\gamma_{2}}{A_{2}} y_{n} \tag{3.12}
\end{equation*}
$$

which, by the difference inequalities argument, see [18], implies that $y_{n} \leq u_{n}$, where $u_{n}$ satisfies (3.6). In view of (3.7) we obtain our conclusion.

### 3.2. Period-Two Solution

In this section, we prove that System (1.1) has no minimal period-two solution which will be essential for application of Theorems 2.5-2.7. The map $T$ associated to System (1.1) is given by

$$
\begin{equation*}
T(x, y)=\left(\frac{\beta_{1} x}{B_{1} x+y}, \frac{\alpha_{2}+\gamma_{2} y}{A_{2}+x}\right) \tag{3.13}
\end{equation*}
$$

Lemma 3.3. System (1.1) has no minimal period-two solution.
Proof. We have

$$
\begin{align*}
T(T(x, y)) & =T\left(\frac{\beta_{1} x}{B_{1} x+y}, \frac{\alpha_{2}+\gamma_{2} y}{A_{2}+x}\right) \\
& =\left(\frac{\beta_{1}^{2} x\left(A_{2}+x\right)}{B_{1} \beta_{1} x\left(A_{2}+x\right)+\left(\alpha_{2}+\gamma_{2} y\right)\left(B_{1} x+y\right)}, \frac{\left(B_{1} x+y\right)\left(\alpha_{2}\left(A_{2}+x\right)+\gamma_{2}\left(\alpha_{2}+\gamma_{2} y\right)\right)}{\left(A_{2}+x\right)\left(A_{2}\left(B_{1} x+y\right)+\beta_{1} x\right)}\right) \tag{3.14}
\end{align*}
$$

Period-two solution satisfies

$$
\begin{align*}
& \frac{\beta_{1}^{2} x\left(A_{2}+x\right)}{B_{1} \beta_{1} x\left(A_{2}+x\right)+\left(\alpha_{2}+\gamma_{2} y\right)\left(B_{1} x+y\right)}-x=0  \tag{3.15}\\
& \frac{\left(B_{1} x+y\right)\left(\alpha_{2}\left(A_{2}+x\right)+\gamma_{2}\left(\alpha_{2}+\gamma_{2} y\right)\right)}{\left(A_{2}+x\right)\left(A_{2}\left(B_{1} x+y\right)+\beta_{1} x\right)}-y=0 \tag{3.16}
\end{align*}
$$

We show that this system has no other positive solutions except the equilibrium points.

Equation (3.15) is equivalent to

$$
\begin{equation*}
x\left(y \alpha_{2}+x^{2} B_{1} \beta_{1}-A_{2} \beta_{1}^{2}+x\left(-\beta_{1}^{2}+B_{1}\left(\alpha_{2}+A_{2} \beta_{1}\right)\right)+y^{2} \gamma_{2}+x y B_{1} \gamma_{2}\right)=0 \tag{3.17}
\end{equation*}
$$

$$
\begin{align*}
-x y^{2} A_{2} & +x^{2} B_{1} \alpha_{2}+x^{2} y\left(-A_{2} B_{1}-\beta_{1}\right)+y\left(A_{2} \alpha_{2}+\alpha_{2} \gamma_{2}\right)+x\left(A_{2} B_{1} \alpha_{2}+B_{1} \alpha_{2} \gamma_{2}\right) \\
& +y^{2}\left(-A_{2}^{2}+\gamma_{2}^{2}\right)+x y\left(-A_{2}^{2} B_{1}+\alpha_{2}-A_{2} \beta_{1}+B_{1} \gamma_{2}^{2}\right)=0 \tag{3.18}
\end{align*}
$$

If $x=0$ then we obtain the fixed point $E_{1}$. So assume that $x \neq 0$. Then, using (3.17), we have

$$
\begin{equation*}
y \alpha_{2}+x^{2} B_{1} \beta_{1}-A_{2} \beta_{1}^{2}+x\left(-\beta_{1}^{2}+B_{1}\left(\alpha_{2}+A_{2} \beta_{1}\right)\right)+y^{2} \gamma_{2}+x y B_{1} \gamma_{2}=0 \tag{3.19}
\end{equation*}
$$

Equation (3.19) implies that

$$
\begin{equation*}
y^{2}=\frac{-y \alpha_{2}-x^{2} B_{1} \beta_{1}+A_{2} \beta_{1}^{2}-x\left(-\beta_{1}^{2}+B_{1}\left(\alpha_{2}+A_{2} \beta_{1}\right)\right)-x y B_{1} \gamma_{2}}{\gamma_{2}} . \tag{3.20}
\end{equation*}
$$

Substituting (3.20) into (3.18), we have

$$
\begin{equation*}
x+A_{2}=0 \tag{3.21}
\end{equation*}
$$

or

$$
\begin{align*}
& A_{2}\left(\left(y+x B_{1}\right) \alpha_{2}+\left(x+A_{2}\right)\left(x B_{1}-\beta_{1}\right) \beta_{1}\right) \\
& \quad+\left(\left(y+x B_{1}\right) \alpha_{2}-x y \beta_{1}\right) \gamma_{2}+\beta_{1}\left(-x B_{1}+\beta_{1}\right) \gamma_{2}^{2}=0 \tag{3.22}
\end{align*}
$$

Equation (3.21) implies that $x=-A_{2}$, and (3.23) implies that

$$
\begin{equation*}
y=\frac{A_{2}\left(\left(x+A_{2}\right) \beta_{1}^{2}-x B_{1}\left(\alpha_{2}+\left(x+A_{2}\right) \beta_{1}\right)\right)-x B_{1} \alpha_{2} \gamma_{2}+\left(x B_{1}-\beta_{1}\right) \beta_{1} \gamma_{2}^{2}}{A_{2} \alpha_{2}+\left(\alpha_{2}-x \beta_{1}\right) \gamma_{2}} . \tag{3.23}
\end{equation*}
$$

Replacing (3.23) into (3.19), we get

$$
\begin{equation*}
-x^{2} B_{1}-\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)+x\left(-A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)=0 \tag{3.24}
\end{equation*}
$$

$$
\left.-\left(\beta_{1}\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right)\left(A_{2}+\gamma_{2}\right)^{2}\right)+x\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right)\left(A_{2}+\gamma_{2}\right)\left(A_{2} B_{1}-\beta_{1}\right)+B_{1} \gamma_{2}\right)
$$

$$
\begin{equation*}
+x^{2} \beta_{1}\left(A_{2}^{2} B_{1}+A_{2} B_{1} \gamma_{2}+\beta_{1} \gamma_{2}\right)=0 \tag{3.25}
\end{equation*}
$$

Solutions of (3.24) are the equilibrium points.
Consider (3.25). Discriminant of this equation is given by

$$
\begin{align*}
\Delta= & \left(4 \beta_{1}^{2}\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)+\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right)\left(\beta_{1}-B_{1}\left(A_{2} \gamma_{2}\right)\right)^{2}\right)  \tag{3.26}\\
& \times\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right) .
\end{align*}
$$

Now, $\Delta \geq 0$ implies that

$$
\begin{equation*}
x_{1}=-\frac{\left(A_{2}+\gamma_{2}\right)\left(\Delta_{1}+\sqrt{\Delta}\right)}{2 \beta_{1}\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)}, \quad x_{2}=-\frac{\left(A_{2}+\gamma_{2}\right)\left(\Delta_{1}-\sqrt{\Delta}\right)}{2 \beta_{1}\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)}, \tag{3.27}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta_{1}=\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right)\left(-\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right) . \tag{3.28}
\end{equation*}
$$

Using (3.23), we obtain

$$
\begin{equation*}
y_{1}=-\frac{\Delta_{2}-\sqrt{\Delta}}{2\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)}, \quad y_{2}=-\frac{\Delta_{2}+\sqrt{\Delta}}{2\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)}, \tag{3.29}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta_{2}=A_{2}^{2} B_{1} \beta_{1}+\alpha_{2}\left(\beta_{1}-B_{1} \gamma_{2}\right)+\beta_{1} \gamma_{2}\left(\beta_{1}+B_{1} \gamma_{2}\right)+A_{2}\left(\beta_{1}^{2}-B_{1}\left(\alpha_{2}-2 \beta_{1} \gamma_{2}\right)\right) \tag{3.30}
\end{equation*}
$$

We prove the following claims.
Claim 3.4. For all values of parameters $y_{2}<0$.
Proof. If $\Delta_{2}>0$, then $y_{2}<0$. Now, we assume that $\Delta_{2} \leq 0$. Then,

$$
\begin{gather*}
y_{2}<0 \Longleftrightarrow \Delta_{2}+\sqrt{\Delta}>0 \Longleftrightarrow \Delta-\Delta_{2}^{2}>0  \tag{3.31}\\
\Delta-\Delta_{2}^{2}=-4 \beta_{1}\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)\left(A_{2} B_{1}\left(-\alpha_{2}+\beta_{1} \gamma_{2}\right)+\gamma_{2}\left(\beta_{1}^{2}+B_{1}\left(-\alpha_{2}+\beta_{1} \gamma_{2}\right)\right)\right) . \tag{3.32}
\end{gather*}
$$

Equation (3.32) implies that

$$
\begin{equation*}
\Delta-\Delta_{2}^{2}>0 \Longleftrightarrow \alpha_{2}>\frac{A_{2} B_{1} \beta_{1} \gamma_{2}+\beta_{1}^{2} \gamma_{2}+B_{1} \beta_{1} \gamma_{2}^{2}}{B_{1}\left(A_{2}+\gamma_{2}\right)} \tag{3.33}
\end{equation*}
$$

Since $\Delta_{2} \leq 0$ if and only if

$$
\begin{align*}
& B_{1}>\frac{\beta_{1}}{A_{2}+\gamma_{2}}, \quad \alpha_{2} \geq \frac{A_{2}^{2} B_{1} \beta_{1}+A_{2} \beta_{1}^{2}+2 A_{2} B-1 \beta_{1} \gamma_{2}+\beta_{1}^{2} \gamma_{2}+B_{1} \beta_{1} \gamma_{2}^{2}}{A_{2} B_{1}-\beta_{1}+B_{1} \gamma_{2}}, \\
& \frac{A_{2}^{2} B_{1} \beta_{1}+A_{2} \beta_{1}^{2}+2 A_{2} B-1 \beta_{1} \gamma_{2}+\beta_{1}^{2} \gamma_{2}+B_{1} \beta_{1} \gamma_{2}^{2}}{A_{2} B_{1}-\beta_{1}+B_{1} \gamma_{2}}-\frac{A_{2} B_{1} \beta_{1} \gamma_{2}+\beta_{1}^{2} \gamma_{2}+B_{1} \beta_{1} \gamma_{2}^{2}}{B_{1}\left(A_{2}+\gamma_{2}\right)}  \tag{3.34}\\
& \quad=\frac{\beta_{1}\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{B_{1}\left(A_{2}+\gamma_{2}\right)\left(-\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}>0 .
\end{align*}
$$

This implies that (3.33) is true. That is $y_{2}<0$.
Claim 3.5. Assume that $x_{1} \geq 0$. Then that $y_{1}<0$.
Proof. Assume that $x_{1}>0$. This is equivalent to

$$
\begin{gather*}
A_{2}<\gamma_{2}, \quad B_{1}+\frac{\beta_{1}\left(A_{2}+3 \gamma_{2}\right)}{A_{2}^{2}-\gamma_{2}^{2}}>0  \tag{3.35}\\
\alpha_{2} \leq-\left(\frac{\beta_{1}\left(A_{2}\left(A_{2} B_{1}+\beta_{1}\right)+3 \beta_{1} \gamma_{2}-B_{1} \gamma_{2}^{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{\left(\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)\right)}\right) . \tag{3.36}
\end{gather*}
$$

Now

$$
\begin{equation*}
y_{1}<0 \Longleftrightarrow \Delta_{2} \geq 0, \quad \Delta-\Delta_{2}^{2}<0 \tag{3.37}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
\Delta_{2} \geq 0 \Longleftrightarrow B_{1}<\frac{\beta_{1}}{A_{2}+\gamma_{2}}, \quad \alpha_{2} \leq-\left(\frac{\beta_{1}\left(A_{2}+\gamma_{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)}\right) \tag{3.38}
\end{equation*}
$$

or

$$
\begin{equation*}
B_{1} \geq \frac{\beta_{1}}{A_{2}+\gamma_{2}} \tag{3.39}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\Delta-\Delta_{2}^{2}<0 \Longleftrightarrow \alpha_{2}<\frac{\beta_{1} \gamma_{2}\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{B_{1}\left(A_{2}+\gamma_{2}\right)} \tag{3.40}
\end{equation*}
$$

Since,

$$
\begin{align*}
& \frac{\beta_{1}\left(A_{2}\left(A_{2} B_{1}+\beta_{1}\right)+3 \beta_{1} \gamma_{2}-B_{1} \gamma_{2}^{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{\left(\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)\right)}+\frac{\beta_{1} \gamma_{2}\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{B_{1}\left(A_{2}+\gamma_{2}\right)} \\
& \quad=\frac{\beta_{1}\left(A_{2}^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)^{2}}{B_{1}\left(A_{2}+\gamma_{2}\right)\left(\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)\right)^{2}}>0 . \tag{3.41}
\end{align*}
$$

This inequality and (3.36) imply (3.40).
Since

$$
\begin{align*}
& \frac{\beta_{1}\left(A_{2}\left(A_{2} B_{1}+\beta_{1}\right)+3 \beta_{1} \gamma_{2}-B_{1} \gamma_{2}{ }^{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{\left(\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)\right)^{2}}-\left(\frac{\beta_{1}\left(A_{2}+\gamma_{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)}\right) \\
& =\frac{2 \beta_{1}\left(A_{2}{ }^{2} B_{1}+\left(A_{2} B_{1}+\beta_{1}\right) \gamma_{2}\right)\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right)}{\left(\beta_{1}-B_{1}\left(A_{2}+\gamma_{2}\right)\right)^{2}}>0 . \tag{3.42}
\end{align*}
$$

Last inequality, (3.36) and (3.39) imply that $\Delta_{2} \geq 0$. So we prove, if $x_{1}>0$, then $y_{1}<0$. Assume that $x_{1}=0$.
We have

$$
\begin{equation*}
x_{1}=0 \Longleftrightarrow \Delta_{1}+\sqrt{\Delta}=0 . \tag{3.43}
\end{equation*}
$$

Since,

$$
\begin{equation*}
\Delta_{1}^{2}-\Delta=4 \beta_{1}^{2}\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right)\left(A_{2}+\gamma_{2}\right)^{2}\left(B_{1} A_{2}^{2}+B_{1} \gamma_{2} A_{2}+\beta_{1} \gamma_{2}\right) \tag{3.44}
\end{equation*}
$$

we have that $x_{1}=0$ if and only if

$$
\begin{equation*}
A_{2}<\gamma_{2}, \quad \alpha_{2}+A_{2} \beta_{1}=\beta_{1} \gamma_{2}, \quad \Delta_{1} \leq 0, \tag{3.45}
\end{equation*}
$$

which is true, because

$$
\begin{equation*}
\Delta_{1}:=\left(\alpha_{2}+\beta_{1}\left(A_{2}-\gamma_{2}\right)\right)\left(-\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)\right) . \tag{3.46}
\end{equation*}
$$

Replacing $\alpha_{2}$ with $\beta_{1}\left(\gamma_{2}-A_{2}\right)$ in the formula for $y_{1}$, we obtain that $y_{1}=-\beta_{1}$.
Hence, there does not exist period-two solution.

## 4. Linearized Stability Analysis

The Jacobian matrix of the map $T$, given by (3.13), has the form

$$
J_{T}=\left(\begin{array}{cc}
\frac{\beta_{1} y}{\left(B_{1} x+y\right)^{2}}-\frac{\beta_{1} x}{\left(B_{1} x+y\right)^{2}}  \tag{4.1}\\
-\frac{\alpha_{2}+\gamma_{2} y}{\left(A_{2}+x\right)^{2}} & \frac{\gamma_{2}}{A_{2}+x}
\end{array}\right)
$$

The determinant of (4.1) is given by

$$
\begin{equation*}
\operatorname{det} J_{T}(x, y)=\frac{\beta_{1}\left(y A_{2} \gamma_{2}-x \alpha_{2}\right)}{\left(x+A_{2}\right)^{2}\left(y+x B_{1}\right)^{2}} \tag{4.2}
\end{equation*}
$$

The value of the Jacobian matrix of $T$ at the equilibrium point $E=(\bar{x}, \bar{y}), \bar{x} \neq 0$ is

$$
J_{T}(\bar{x}, \bar{y})=\left(\begin{array}{cc}
\frac{\bar{y}}{B_{1} \bar{x}+\bar{y}}-\frac{\bar{x}}{B_{1} \bar{x}+\bar{y}}  \tag{4.3}\\
-\frac{\bar{y}}{A_{2}+\bar{x}} & \frac{\gamma_{2}}{A_{2}+\bar{x}}
\end{array}\right)
$$

The determinant of (4.3) is given by

$$
\begin{equation*}
\operatorname{det} J_{T}(\bar{x}, \bar{y})=\frac{\bar{y}\left(\gamma_{2}-\bar{x}\right)}{\left(A_{2}+\bar{x}\right)\left(B_{1} \bar{x}+\bar{y}\right)} \tag{4.4}
\end{equation*}
$$

and the trace of (4.3) is

$$
\begin{equation*}
\operatorname{Tr} J_{T}(\bar{x}, \bar{y})=\frac{\bar{y}}{B_{1} \bar{x}+\bar{y}}+\frac{\gamma_{2}}{A_{2}+\bar{x}} \tag{4.5}
\end{equation*}
$$

The characteristic equation has the form

$$
\begin{equation*}
\lambda^{2}-\lambda\left(\frac{\bar{y}}{B_{1} \bar{x}+\bar{y}}+\frac{\gamma_{2}}{A_{2}+\bar{x}}\right)+\frac{\bar{y}\left(\gamma_{2}-\bar{x}\right)}{\left(A_{2}+\bar{x}\right)\left(B_{1} \bar{x}+\bar{y}\right)}=0 . \tag{4.6}
\end{equation*}
$$

Theorem 4.1. Assume that $A_{2}>\gamma_{2}$. Then there exists the equilibrium point $E_{1}$ and
(i) $E_{1}\left(0, \alpha_{2} /\left(A_{2}-\gamma_{2}\right)\right)$ is locally asymptotically stable if $\beta_{1}<\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$,
(ii) $E_{1}$ is a saddle point if $\beta_{1}>\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$. The corresponding eigenvalues are

$$
\begin{equation*}
\lambda_{1}=\frac{\gamma_{2}}{A_{2}} \in(0,1), \quad \lambda_{2}=\frac{\beta_{1}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2}} \in(1,+\infty) \tag{4.7}
\end{equation*}
$$

(iii) $E_{1}$ is nonhyperbolic if $\beta_{1}=\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$. The corresponding eigenvalues are

$$
\begin{equation*}
\lambda_{1}=\frac{\gamma_{2}}{A_{2}} \in(0,1), \quad \lambda_{2}=1 \tag{4.8}
\end{equation*}
$$

and the corresponding eigenvectors are $(0,1)$ and $\left(-1 / \alpha_{2}, 1\right)$, respectively.
Proof. The Jacobian matrix (4.1) at the equilibrium point $E_{1}\left(0, \alpha_{2} /\left(A_{2}-\gamma_{2}\right)\right)$,

$$
J_{T}\left(E_{1}\right)=\left(\begin{array}{cc}
\frac{\beta_{1}}{\bar{y}} & 0  \tag{4.9}\\
-\frac{\alpha_{2}+\gamma \bar{y}}{A_{2}^{2}} & \frac{\gamma_{2}}{A_{2}}
\end{array}\right)
$$

Note that the Jacobian matrix (4.9) implies that the map $T$ is not strongly competitive at the equilibrium point $E_{1}$.

The determinant of (4.9) is given by

$$
\begin{equation*}
\operatorname{det} J_{T}(\bar{x}, \bar{y})=\frac{\beta_{1}}{\bar{y}} \frac{\gamma_{2}}{A_{2}}=\frac{\beta_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2} A_{2}} \tag{4.10}
\end{equation*}
$$

Note that, under the hypothesis of Theorem, the determinant is greater than zero.
The trace of (4.9) is

$$
\begin{equation*}
\operatorname{Tr} J_{T}(\bar{x}, \bar{y})=\frac{\beta_{1}}{\bar{y}}+\frac{\gamma_{2}}{A_{2}}=\frac{\beta_{1}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2}}+\frac{\gamma_{2}}{A_{2}} \tag{4.11}
\end{equation*}
$$

An equilibrium point is locally asymptotically stable if the following conditions are satisfied

$$
\begin{equation*}
\left|\operatorname{Tr} J_{T}(\bar{x}, \bar{y})\right|<1+\operatorname{det} J_{T}(\bar{x}, \bar{y})<2 \tag{4.12}
\end{equation*}
$$

Now, these two conditions become

$$
\begin{equation*}
\frac{\beta_{1}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2}}+\frac{\gamma_{2}}{A_{2}}<1+\frac{\beta_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2} A_{2}}<2 \tag{4.13}
\end{equation*}
$$

Condition

$$
\begin{equation*}
\frac{\beta_{1}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2}}+\frac{\gamma_{2}}{A_{2}}<1+\frac{\beta_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2} A_{2}} \tag{4.14}
\end{equation*}
$$

implies that

$$
\begin{equation*}
\left(A_{2}-\gamma_{2}\right) \beta_{1}<\alpha_{2} \tag{4.15}
\end{equation*}
$$

If $\beta_{1}<\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$, then this condition is satisfied.
Condition $1+\left(\beta_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)\right) / \alpha_{2} A_{2}<2$ is equivalent to

$$
\begin{equation*}
\frac{\beta_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2} A_{2}}<1 \tag{4.16}
\end{equation*}
$$

It is easy to see that the condition is satisfied if $\beta_{1}<\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$.
Next, we prove (ii).
An equilibrium point is a saddle if and only if the following conditions are satisfied

$$
\begin{gather*}
\left|\operatorname{Tr} J_{T}(\bar{x}, \bar{y})\right|>\left|1+\operatorname{det} J_{T}(\bar{x}, \bar{y})\right| \\
\operatorname{Tr}^{2} J_{T}(\bar{x}, \bar{y})-4 \operatorname{det} J_{T}(\bar{x}, \bar{y})>0 \tag{4.17}
\end{gather*}
$$

The first condition is equivalent to

$$
\begin{equation*}
\left(A_{2}-\gamma_{2}\right) \beta_{1}>\alpha_{2} \tag{4.18}
\end{equation*}
$$

which is satisfied if $\beta_{1}>\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$. The second condition is equivalent to

$$
\begin{equation*}
\left(\frac{\beta_{1}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2}}-\frac{\gamma_{2}}{A_{2}}\right)^{2}>0 \tag{4.19}
\end{equation*}
$$

Finally, we prove (iii).
An equilibrium point is nonhyperbolic if the following conditions are satisfied

$$
\begin{align*}
& \quad\left|\operatorname{Tr} J_{T}(\bar{x}, \bar{y})\right|=\left|1+\operatorname{det} J_{T}(\bar{x}, \bar{y})\right|  \tag{4.20}\\
&\left(\operatorname{det} J_{T}(\bar{x}, \bar{y})=1\right.\text { or } \left.\quad\left|\operatorname{Tr} J_{T}(\bar{x}, \bar{y})\right| \leq 2\right)
\end{align*}
$$

The first condition is equivalent to

$$
\begin{equation*}
\left(A_{2}-\gamma_{2}\right) \beta_{1}=\alpha_{2} \tag{4.21}
\end{equation*}
$$

which is satisfied if $\beta_{1}=\alpha_{2} / A_{2}-\gamma_{2}$.
The second condition becomes

$$
\begin{equation*}
\frac{\beta_{1}\left(A_{2}-\gamma_{2}\right)}{\alpha_{2}}+\frac{\gamma_{2}}{A_{2}}=1+\frac{\gamma_{2}}{A_{2}}<2 \tag{4.22}
\end{equation*}
$$

establishing part (iii).
We now perform a similar analysis for the other cases in Table 1.

Theorem 4.2. Assume that

$$
\begin{equation*}
A_{2}>\gamma_{2}, \quad B_{1}\left(\gamma_{2}-A_{2}\right)+\sqrt{4 B_{1} \alpha_{2}}<\beta_{1}<\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, \quad \alpha_{2}>B_{1}\left(A_{2}-\gamma_{2}\right)^{2} \tag{4.23}
\end{equation*}
$$

Then $E_{1}, E_{2}$, and $E_{3}$ exist and
(i) the equilibrium point $E_{1}$ is locally asymptotically stable,
(ii) the equilibrium point $E_{2}$ is a saddle point. Furthermore, if $B_{1}\left(\gamma_{2}-A_{2}\right)+\sqrt{4 B_{2} \alpha_{2}}<\beta_{1}<$ $\alpha_{2} /\left(A_{2}+B_{1} \gamma_{2}\right)$ and $\alpha_{2}>B_{1} A_{2}^{2}$, then the smaller eigenvalue belongs to the interval $(-1,0)$, and the larger eigenvalue belongs to $(1,+\infty)$. In all other cases, the smaller eigenvalues is in $(0,1)$.
That is $\left|\lambda_{1}\right|<1$ is given by

$$
\begin{equation*}
\lambda_{1}=\frac{\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}-\sqrt{\left(\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}\right)^{2}+4 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)\left(\bar{x}_{2}-\gamma_{2}\right) \bar{y}_{2}}}{2 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)} \tag{4.24}
\end{equation*}
$$

and the corresponding eigenvector is

$$
\begin{align*}
v_{1}=( & \beta_{1} \gamma_{2}-\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2} \\
& \left.+\sqrt{\left(\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}\right)^{2}+4 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)\left(\bar{x}_{2}-\gamma_{2}\right) \bar{y}_{2}}, 2 \beta_{1} \bar{y}_{2}\right) \tag{4.25}
\end{align*}
$$

Eigenvalue $\lambda_{2}$, where $\left|\lambda_{2}\right|>1$, is given by

$$
\begin{equation*}
\lambda_{2}=\frac{\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}+\sqrt{\left(\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}\right)^{2}+4 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)\left(\bar{x}_{2}-\gamma_{2}\right) \bar{y}_{2}}}{2 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)} \tag{4.26}
\end{equation*}
$$

(iii) The equilibrium point $E_{3}$ is locally asymptotically stable.

Proof. By Theorem 4.1, (i) holds.
Evaluating the Jacobian matrix (4.3) at the equilibrium point $E_{2}$, we obtain

$$
J_{T}=\left(\begin{array}{cc}
\frac{\bar{y}}{\beta_{1}} & -\frac{\bar{x}}{\beta_{1}}  \tag{4.27}\\
-\frac{\gamma_{2}}{A_{2}+\bar{x}} & \frac{A_{2}+\bar{x}}{}
\end{array}\right)
$$

Note that the Jacobian matrix (4.27) implies that the map $T$ is strongly competitive.
The determinant of (4.27) is given by

$$
\begin{equation*}
\operatorname{det}=\frac{\bar{y} \gamma_{2}}{\beta_{1}\left(A_{2}+\bar{x}\right)}-\frac{\bar{x} \bar{y}}{\beta_{1}\left(A_{2}+\bar{x}\right)}, \tag{4.28}
\end{equation*}
$$

and the trace of (4.27) is given by

$$
\begin{equation*}
\operatorname{Tr} J_{T}(\bar{x}, \bar{y})=\frac{\bar{y}}{\beta_{1}}+\frac{\gamma_{2}}{A_{2}+\bar{x}} \tag{4.29}
\end{equation*}
$$

The equilibrium point $E_{2}$ is a saddle if and only if (4.17) is satisfied. The first condition is equivalent to

$$
\begin{equation*}
\frac{\bar{y}}{\beta_{1}}+\frac{\gamma_{2}}{A_{2}+\bar{x}}>\left|1+\frac{\bar{y} \gamma}{\beta_{1}\left(A_{2}+\bar{x}\right)}-\frac{\bar{x} \bar{y}}{\beta_{1}\left(A_{2}+\bar{x}\right)}\right| \tag{4.30}
\end{equation*}
$$

which is equivalent to

$$
\begin{align*}
\bar{y}\left(A_{2}+\bar{x}\right)+\beta_{1} \gamma_{2} & >\beta_{1}\left(A_{2}+\bar{x}\right)+\bar{y}\left(\gamma_{2}-\bar{x}\right) \\
& \Longrightarrow \bar{y}\left(A_{2}+\bar{x}\right)+\beta_{1}\left(\bar{x}-\left(A_{2}-\gamma_{2}\right)\right)-\bar{y}\left(\gamma_{2}-\bar{x}\right)>0 \\
& \Longrightarrow \bar{y}\left(A_{2}+\bar{x}-\gamma_{2}\right)-\beta_{1}\left(A_{2}-\gamma_{2}+\bar{x}\right)>-\bar{x} \bar{y}  \tag{4.31}\\
& \Longrightarrow\left(A_{2}-\gamma_{2}+\bar{x}\right)\left(\bar{y}-\beta_{1}\right)>-\bar{x} \bar{y} \\
& \Longrightarrow\left(\beta_{1}-\bar{y}\right)\left(A_{2}-\gamma_{2}+\bar{x}\right)<\bar{x} \bar{y}
\end{align*}
$$

In light of (3.3) $\beta_{1}-\bar{y}_{2}=B_{1} \bar{x}_{2}$, and by using (3.4), $A_{2}-\gamma_{2}+\bar{x}_{2}=\bar{y}_{3} / B_{1}$. Now, we have

$$
\begin{equation*}
B_{1} \bar{x}_{2} \frac{\bar{y}_{3}}{B_{1}}<\bar{x}_{2} \bar{y}_{2} \tag{4.32}
\end{equation*}
$$

This implies that $\bar{y}_{3}<\bar{y}_{2}$ which is true.
Condition

$$
\begin{equation*}
\operatorname{Tr}^{2} J_{T}(\bar{x}, \bar{y})-4 \operatorname{det} J_{T}(\bar{x}, \bar{y})>0 \tag{4.33}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
\left(\frac{\bar{y}_{2}}{\beta_{1}}-\frac{\gamma_{2}}{\beta_{1}\left(A_{2}+\bar{x}_{2}\right)}\right)^{2}+\frac{\bar{x}_{2} \bar{y}_{2}}{\beta_{2}\left(A_{2}+\bar{x}_{2}\right)}>0 \tag{4.34}
\end{equation*}
$$

which is true.
To prove the second part of the statement (ii), we use the characteristic equation (4.6) of System (1.1) at the equilibrium point. Now, we have

$$
\begin{equation*}
\lambda_{1}+\lambda_{2}=\frac{\bar{y}}{\beta_{1}}+\frac{\bar{y} \gamma_{2}}{\left(A_{2}+\bar{x}\right) \beta_{1}}, \quad \lambda_{1} \lambda_{2}=\frac{\bar{y}\left(\gamma_{2}-\bar{x}\right)}{\left(A_{2}+\bar{x}\right) \beta_{1}} \tag{4.35}
\end{equation*}
$$

Since the map $T$ is strongly competitive, the Jacobian matrix (4.27) has two real and distinct eigenvalues, the larger one in absolute value being positive.

The first equation implies that either both eigenvalues are positive, or the smaller one is negative. First, we show that, under hypothesis (ii) of theorem, the product of these two eigenvalues is less than zero. In order to prove that, it is enough to prove that $\gamma_{2}-\bar{x}_{2}<0$.

We have

$$
\begin{align*}
\gamma_{2}-\bar{x}_{2} & =\gamma_{2}-\frac{B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}-\sqrt{\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}} \\
& =\frac{B_{1}\left(A_{2}+\gamma_{2}\right)-\beta_{1}+\sqrt{\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}} \tag{4.36}
\end{align*}
$$

Now $\gamma_{2}-\bar{x}_{2}<0$ if and only if

$$
\begin{equation*}
B_{1}\left(A_{2}+\gamma_{2}\right)-\beta_{1}+\sqrt{\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}}<0 \tag{4.37}
\end{equation*}
$$

which holds if

$$
\begin{gather*}
\beta_{1}>B_{1}\left(A_{2}+\gamma_{2}\right), \quad\left(B_{1}\left(A_{2}+\gamma_{2}\right)-\beta_{1}\right)^{2}-\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}+4 B_{1} \alpha_{2}>0, \\
\\
B_{1}\left(A_{2}+\gamma_{2}\right)<\beta_{1}<\frac{\alpha_{2}}{A_{2}}+B_{1} \gamma_{2},  \tag{4.38}\\
\frac{\alpha_{2}}{A_{2}}+B_{1} \gamma_{2}-\left(A_{2} B_{1}+\gamma_{2} B_{1}\right)=\frac{\alpha_{2}-A_{2}^{2} B_{1}}{A_{2}}>0 \Longleftrightarrow B_{1}<\frac{\alpha_{2}}{A_{2}^{2}} .
\end{gather*}
$$

Also, we have

$$
\begin{equation*}
\frac{\alpha_{2}}{A_{2}}+B_{1} \gamma_{2}-\left(B_{1}\left(\gamma_{2}-A_{2}\right)+2 \sqrt{B_{1} \alpha_{2}}\right) \geq 0 \tag{4.39}
\end{equation*}
$$

In all other cases $\gamma_{2}-\bar{x}_{2}>0$.
This proves that the smaller eigenvalue is negative. Since the equilibrium point is a saddle point, it has to belong to $(-1,0)$. The larger one belongs to $(1,+\infty)$. The proof of second statement is similar.

Now, we prove that $E_{3}$ is locally asymptotically stable.
Notice that

$$
\begin{equation*}
\left|\operatorname{Tr} J_{T}(\bar{x}, \bar{y})\right|<1+\operatorname{det} J_{T}(\bar{x}, \bar{y})<2 \tag{4.40}
\end{equation*}
$$

implies that $\bar{y}_{2}>\bar{y}_{3}$ which is true.

Theorem 4.3. Assume that

$$
\begin{equation*}
A_{2}>\gamma_{2}, \quad \beta_{1}>\left(A_{2}-\gamma_{2}\right) B_{1}, \quad\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)^{2}-4 \alpha_{2} B_{1}=0 \tag{4.41}
\end{equation*}
$$

Then $E_{1}, E_{2}=E_{3}$ exist and:
(i) The equilibrium point $E_{1}$ is locally asymptotically stable.
(ii) The equilibrium point $E_{2}=E_{3}=\left(B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1} / 2 B_{1}, B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1} / 2\right)$ is nonhyperbolic. The eigenvalues of the Jacobian matrix evaluated at $E_{2}$ are

$$
\begin{equation*}
\lambda_{1}=1, \quad \lambda_{2}=\frac{A_{2}^{2} B_{1}^{2}-\beta_{1}^{2}+2 B_{1} \beta_{1} \gamma_{2}-B_{1}^{2} \gamma_{2}^{2}}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)} \tag{4.42}
\end{equation*}
$$

and the corresponding eigenvectors, respectively, are

$$
\begin{equation*}
\left(-\frac{1}{B_{1}}, 1\right), \quad\left(\frac{\left(-A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)}{2 B_{1} \beta_{1}\left(A_{2} B_{1}+\beta_{1}-B_{1} \gamma_{2}\right)}, 1\right) \tag{4.43}
\end{equation*}
$$

Furthermore, if $\beta_{1}>B_{1}\left(A_{2}+\gamma_{2}\right)$, then $\lambda_{2} \in(-1,0)$. If $B_{1}\left(A_{2}-\gamma_{2}\right)<\beta_{1}<B_{1}\left(A_{2}+\gamma_{2}\right)$, then $\lambda_{2} \in(0,1)$.

Proof. By Theorem 4.1, $E_{1}$ is a locally asymptotically stable.
Now, we prove that $E_{2}$ is nonhyperbolic.
The Jacobian matrix (4.3) at the equilibrium point $E_{2}=E_{3}=\left(B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1} / 2 B_{1}\right.$, $\left.B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1} / 2\right)$ is

$$
J_{T}\left(E_{2}\right)=\left(\begin{array}{cc}
\frac{\bar{y}}{\beta_{1}} & -\frac{\bar{x}}{\beta_{1}}  \tag{4.44}\\
-\frac{\bar{y}}{A_{2}+\bar{x}} & \frac{\gamma_{2}}{A_{2}+\bar{x}}
\end{array}\right) .
$$

The eigenvalues of (4.44) satisfy

$$
\begin{gather*}
\left(\frac{B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}}{2 \beta_{1}}-\lambda\right)\left(\frac{2 B_{1} \gamma_{2}}{2 B_{1} A_{2}+B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}}-\lambda\right)  \tag{4.45}\\
-\frac{B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}}{2 \beta_{1} B_{1}} \frac{B_{1}\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)}{2 B_{1} A_{2}+B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}}=0
\end{gather*}
$$

and are given as

$$
\begin{equation*}
\lambda_{1}=1, \quad \lambda_{2}=\frac{A_{2}^{2} B_{1}^{2}-\beta_{1}^{2}+2 B_{1} \beta_{1} \gamma_{2}-B_{1}^{2} \gamma_{2}^{2}}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)} \tag{4.46}
\end{equation*}
$$

Hence, $E_{2}$ is nonhyperbolic.

Notice that $\left|\lambda_{2}\right|<1$. Now, we show that $\lambda_{2}$ can be in $(-1,0)$ or $(0,1)$.
We have

$$
\begin{equation*}
\lambda_{2}=\frac{A_{2}^{2} B_{1}^{2}-\beta_{1}^{2}+2 B_{1} \beta_{1} \gamma_{2}-B_{1}^{2} \gamma_{2}^{2}}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)}=\frac{\left(B_{1}\left(A_{2}+\gamma_{2}\right)-\beta_{1}\right)\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)}, \tag{4.47}
\end{equation*}
$$

which is negative if $\beta_{1}>B_{1}\left(A_{2}+\gamma_{2}\right)$, and positive if $\beta_{1}<B_{1}\left(A_{2}+\gamma_{2}\right)$.
Theorem 4.4. Assume that

$$
\begin{equation*}
A_{2}>\gamma_{2}, \quad \beta_{1}>\frac{\alpha_{2}}{A_{2}-\gamma_{2}} . \tag{4.48}
\end{equation*}
$$

Then, $E_{1}$ and $E_{3}$ exist and
(i) the equilibrium point $E_{1}$ is a saddle point,
(ii) the equilibrium point $E_{3}$ is locally asymptotically stable.

Proof. By Theorem 4.1, (i) holds. Observe that the assumption of Theorem 4.4 implies that the $x$ coordinate of the equilibrium point $E_{2}$ is less than zero.

The proof that the equilibrium point $E_{3}$ is locally asymptotically stable is similar to the corresponding proof of Theorem 4.2.

Theorem 4.5. The following statements are true.
(a) Assume

$$
\begin{equation*}
A_{2}<\gamma_{2}, \quad\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}>0, \quad \beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right) . \tag{4.49}
\end{equation*}
$$

Then $E_{2}$ and $E_{3}$ exist and
(i) the equilibrium point $E_{2}$ is a saddle. If $\beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right)$, then the larger eigenvalue is in $(1,+\infty)$, the smaller eigenvalue is in $(-1,0)$. If $B_{1}\left(\gamma_{2}-A_{2}\right)<\beta_{1}<B_{1}\left(\gamma_{2}+A_{2}\right)$, then the smaller eigenvalue is in $(0,1)$. That is, $\left|\lambda_{1}\right|<1$ is given by

$$
\begin{equation*}
\lambda_{1}=\frac{\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}-\sqrt{\left(\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}\right)^{2}+4 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)\left(\bar{x}_{2}-\gamma_{2}\right) \bar{y}_{2}}}{2 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)}, \tag{4.50}
\end{equation*}
$$

and the corresponding eigenvector is

$$
\begin{align*}
v_{1}=( & \beta_{1} \gamma_{2}-\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}  \tag{4.51}\\
& \left.+\sqrt{\left(\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}\right)^{2}+4 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)\left(\bar{x}_{2}-\gamma_{2}\right) \bar{y}_{2}}, 2 \beta_{1} \bar{y}_{2}\right) .
\end{align*}
$$

Eigenvalue $\lambda_{2}$, where $\left|\lambda_{2}\right|>1$, is given by

$$
\begin{equation*}
\lambda_{2}=\frac{\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}+\sqrt{\left(\beta_{1} \gamma_{2}+\left(A_{2}+\bar{x}_{2}\right) \bar{y}_{2}\right)^{2}+4 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)\left(\bar{x}_{2}-\gamma_{2}\right) \bar{y}_{2}}}{2 \beta_{1}\left(A_{2}+\bar{x}_{2}\right)} . \tag{4.52}
\end{equation*}
$$

(ii) The equilibrium point $E_{3}$ is locally asymptotically stable.
(b) Assume that $A_{2}<\gamma_{2},\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}=0, \beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right)$. Then there exists a unique positive equilibrium $E_{2}=E_{3}=\left(B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1} / 2 B_{1}, B_{1}\left(A_{2}-\gamma_{2}\right)+\right.$ $\left.\beta_{1} / 2\right)$ which is nonhyperbolic. The eigenvalues are $\lambda_{1}=1$ and $\lambda_{2}=A_{2}^{2} B_{1}^{2}-\beta_{1}^{2}+2 B_{1} \beta_{1} \gamma_{2}-$ $B_{1}^{2} \gamma_{2}^{2} / 2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)$, and the corresponding eigenvectors, respectively, are

$$
\begin{equation*}
\left(-\frac{1}{B_{1}}, 1\right), \quad\left(\frac{\left(-A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)}{2 B_{1} \beta_{1}\left(A_{2} B_{1}+\beta_{1}-B_{1} \gamma_{2}\right)}, 1\right) \tag{4.53}
\end{equation*}
$$

If $\beta_{1}>B_{1}\left(A_{2}+\gamma_{2}\right)$, then $\lambda_{2} \in(-1,0)$. If $B_{1}\left(\gamma_{2}-A_{2}\right)<\beta_{1}<B_{1}\left(\gamma_{2}+A_{2}\right)$ then $\lambda_{2} \in(0,1)$.
Proof. The proof of statements (a) is similar to the proof of the statements (ii) and (iii) of the Theorem 4.2.

Now, we prove statement (b).
The characteristic equation of the system (1.1) at the equilibrium point $E_{2}=E_{3}$ has the form

$$
\begin{gather*}
\left(\frac{B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}}{2 \beta_{1}}-\lambda\right)\left(\frac{2 B_{1} \gamma_{2}}{2 B_{1} A_{2}+B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}}-\lambda\right)  \tag{4.54}\\
-\frac{B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}}{2 \beta_{1} B_{1}} \frac{B_{1}\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)}{2 B_{1} A_{2}+B_{1}\left(\gamma_{2}-A_{2}\right)+\beta_{1}}=0
\end{gather*}
$$

in which solutions are eigenvalues of $J_{T}\left(E_{2}\right)$

$$
\begin{equation*}
\lambda_{1}=1, \quad \lambda_{2}=\frac{A_{2}^{2} B_{1}^{2}-\beta_{1}^{2}+2 B_{1} \beta_{1} \gamma_{2}-B_{1}^{2} \gamma_{2}^{2}}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)} \tag{4.55}
\end{equation*}
$$

Hence, $E_{2}=E_{3}$ is nonhyperbolic. Notice that $\left|\lambda_{2}\right|<1$.
Now, we determine that the sign of $\lambda_{2}$

$$
\begin{equation*}
\lambda_{2}=\frac{A_{2}^{2} B_{1}^{2}-\beta_{1}^{2}+2 B_{1} \beta_{1} \gamma_{2}-B_{1}^{2} \gamma_{2}^{2}}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)}=\frac{\left(B_{1}\left(A_{2}+\gamma_{2}\right)-\beta_{1}\right)\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)}{2 \beta_{1}\left(A_{2} B_{1}+\beta_{1}+B_{1} \gamma_{2}\right)} \tag{4.56}
\end{equation*}
$$

is negative if $\beta_{1}>B_{1}\left(A_{2}+\gamma_{2}\right)$, and positive if $B_{1}\left(\gamma_{2}-A_{2}\right)<\beta_{1}<B_{1}\left(\gamma_{2}+A_{2}\right)$.

Now, we consider the special case of the system (1.1) when $A_{2}=\gamma_{2}$.
In this case, the system (1.1) becomes

$$
\begin{align*}
& x_{n+1}=\frac{\beta_{1} x_{n}}{B_{1} x_{n}+y_{n}}, \quad n=0,1,2, \ldots \\
& y_{n+1}=\frac{\alpha_{2}+A_{2} y_{n}}{A_{2}+x_{n}}, \tag{4.57}
\end{align*}
$$

If the following condition holds

$$
\begin{equation*}
\beta_{1}^{2}-4 B_{1} \alpha_{2} \geq 0 \tag{4.58}
\end{equation*}
$$

then the system (4.57) has two positive equilibrium points

$$
\begin{align*}
& e_{1}=\left(\frac{\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}}, \frac{\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2}\right)  \tag{4.59}\\
& e_{2}=\left(\frac{\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}}, \frac{\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2}\right)
\end{align*}
$$

We prove the following.
Theorem 4.6. Assume that

$$
\begin{equation*}
A_{2}=\gamma_{2} \tag{4.60}
\end{equation*}
$$

Then the following statements hold.
(i) If $\beta_{1}^{2}-4 B_{1} \alpha_{2}=0$, then the system (4.57) has the unique equilibrium point $e=$ $\left(\beta_{1} / 2 B_{1}, \beta_{1} / 2\right)$ which is nonhyperbolic. The following holds:
(a) If $\alpha_{2}<B_{1} A_{2}^{2}$, then $\lambda_{1}=1$, and $\lambda_{2}=2 A_{2} B_{1}-\beta_{1} / 2\left(2 A_{2} B_{1}+\beta_{1}\right) \in(0,1)$, and the corresponding eigenvectors, respectively, are

$$
\begin{equation*}
\left(-\frac{1}{B_{1}}, 1\right), \quad\left(\frac{\left(-2 A_{2} B_{1}-\beta_{1}\right)}{2 B_{1} \beta_{1}}, 1\right) \tag{4.61}
\end{equation*}
$$

(b) If $\alpha_{2}>B_{1} A_{2}^{2}$, then $\lambda_{1}=1$ and $\lambda_{2}=2 A_{2} B_{1}-\beta_{1} / 2\left(2 A_{2} B_{1}+\beta_{1}\right) \in(-1,0)$, and the corresponding eigenvectors, respectively, are

$$
\begin{equation*}
\left(-\frac{1}{B_{1}}, 1\right), \quad\left(\frac{\left(-2 A_{2} B_{1}-\beta_{1}\right)}{2 B_{1} \beta_{1}}, 1\right) . \tag{4.62}
\end{equation*}
$$

(c) If $\alpha_{2}=B_{1} A_{2}^{2}$, then $\lambda_{1}=1$ and $\lambda_{2}=0$, and the corresponding eigenvectors, respectively, are

$$
\begin{equation*}
\left(-\frac{1}{B_{1}}, 1\right), \quad\left(\frac{1}{B_{1}}, 1\right) \tag{4.63}
\end{equation*}
$$

(ii) If $\beta_{1}^{2}-4 B_{1} \alpha_{2}>0$, then the system (4.57) has two positive equilibrium points: $e_{1}$ is locally asymptotically stable and $e_{2}$ is a saddle point. The following holds.
(d) If $2 A_{2} B_{1}>\beta_{1}$ or $2 A_{2} B_{1} \leq \beta_{1}$ and $B_{1} A_{2}^{2}+\alpha_{2}<A_{2} \beta_{1}$, then $\lambda_{2} \in(1,+\infty)$ and $\lambda_{1} \in(0,1)$.
(e) If $2 A_{2} B_{1}<\beta_{1}$ and $B_{1} A_{2}^{2}+\alpha_{2}>A_{2} \beta_{1}$, then $\lambda_{2} \in(1,+\infty)$ and $\lambda_{1} \in(-1,0)$, where

$$
\begin{equation*}
\lambda_{1}=\frac{2 \alpha_{2} B_{1}+3 A_{2} \beta_{1} B_{1}+A_{2} B_{1} \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}-\sqrt{\mathscr{f}}}{2 \beta_{1}\left(2 A_{2} B_{1}+\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}\right)} \tag{4.64}
\end{equation*}
$$

where

$$
\begin{align*}
\mathscr{F}=( & \left.2 \alpha_{2} B_{1}+3 A_{2} \beta_{1} B_{1}+A_{2} \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}} B_{1}\right)^{2}  \tag{4.65}\\
& -8 B_{1} \beta_{1}\left(B_{1} \beta_{1} A_{2}^{2}-\alpha_{2} \beta_{1}+\left(B_{1} A_{2}^{2}+\alpha_{2}\right) \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}\right)
\end{align*}
$$

and $\left|\lambda_{1}\right|<1$.
The corresponding eigenvector for both cases (c) and (d) is $v_{1}=\left(v_{1}^{(1)}, v_{2}^{(1)}\right)$, where

$$
\begin{align*}
& v_{1}^{(1)}=\left(2 A_{2} B_{1}+\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}\right)\left(-2 \alpha_{2} B_{1}+A_{2} \beta_{1} B_{1}-A_{2} B_{1} \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}+\sqrt{\Phi}\right) \\
& v_{2}^{(1)}=4 A_{2} \beta_{1}^{2} B_{1}^{2}+8 \alpha_{2} \beta_{1} B_{1}^{2}+4 A_{2} \beta_{1} B_{1}^{2} \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}} \tag{4.66}
\end{align*}
$$

where

$$
\begin{gather*}
\mathscr{\Phi}=2 B_{1}\left(-2 A_{2}^{2} \alpha_{2} B_{1}^{2}+2 \alpha_{2}^{2} B_{1}+A_{2}^{2} \beta_{1}^{2} B_{1}+6 A_{2} \alpha_{2} \beta_{1} B_{1}+4 \alpha_{2} \beta_{1}^{2}\right.  \tag{4.67}\\
\\
\left.+\left(-B_{1} \beta_{1} A_{2}^{2}+2 B_{1} \alpha_{2} A_{2}-4 \alpha_{2} \beta_{1}\right) \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}\right)
\end{gather*}
$$

Proof. Assume that $\beta_{1}^{2}-4 B_{1} \alpha_{2}=0$. Then we have that

$$
\begin{equation*}
e_{1}=e_{2}=e=\left(\frac{\beta_{1}}{2 B_{1}}, \frac{\beta_{1}}{2}\right) \tag{4.68}
\end{equation*}
$$

The characteristic equation associated to the system (4.57) at the equilibrium point $e$ is given by

$$
\begin{equation*}
\lambda^{2}-\lambda\left(\frac{1}{2}-\frac{2 B_{1} A_{2}}{2 B_{1} A_{2}+\beta_{1}}\right)+\frac{B_{1} A_{2}}{2 B_{1} A_{2}+\beta_{1}}-\frac{\beta_{1}}{2\left(2 B_{1} A_{2}+\beta_{1}\right)}=0 \tag{4.69}
\end{equation*}
$$

Solutions of (4.69) are

$$
\begin{equation*}
\lambda_{1}=1, \quad \lambda_{2}=\frac{2 A_{2} B_{1}-\beta_{1}}{2\left(2 A_{2} B_{1}+\beta_{1}\right)} \tag{4.70}
\end{equation*}
$$

It is easy to see that $\left|\lambda_{2}\right|<1$.
Now, assume that $\alpha_{2}<B_{1} A_{2}^{2}$. Since $\beta_{1}>0$, from $\beta_{1}^{2}-4 B_{1} \alpha_{2}=0$, we have $\beta_{1}=2 \sqrt{B_{1} \alpha_{2}}$. The numerator of $\lambda_{2}$ is $2 A_{2} B_{1}-2 \sqrt{B_{1} \alpha_{2}}>0$. Assume the opposite, that is, $2 A_{2} B_{1}-2 \sqrt{B_{1} \alpha_{2}}<0$. Then, we have

$$
\begin{equation*}
A_{2}^{2} B_{1}^{2}<B_{1} \alpha_{2} \Longrightarrow \alpha_{2}>B_{1} A_{2}^{2} \tag{4.71}
\end{equation*}
$$

which is a contradiction. So, we confirmed (a).
Assume that $\alpha_{2}>B_{1} A_{2}^{2}$. Then the numerator of $\lambda_{2}<0$. If $2 A_{2} B_{1}-2 \sqrt{B_{1} \alpha_{2}}>0$, then we have

$$
\begin{equation*}
A_{2}^{2} B_{1}^{2}>B_{1} \alpha_{2} \Longrightarrow \alpha_{2}<B_{1} A_{2}^{2} \tag{4.72}
\end{equation*}
$$

which is a contradiction. So, (b) holds.
Assume that $\beta_{1}^{2}-4 B_{1} \alpha_{2}>0$. Then there are two positive equilibrium points

$$
\begin{align*}
& e_{1}=\left(\frac{\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}}, \frac{\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2}\right) \\
& e_{2}=\left(\frac{\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}}, \frac{\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2}\right) \tag{4.73}
\end{align*}
$$

Now, we prove that $e_{1}$ is a locally asymptotically stable equilibrium point.
We check the conditions for locally asymptotically stable equilibrium point. We have

$$
\begin{equation*}
\frac{\bar{y}}{\beta_{1}}+\frac{A_{2}}{A_{2}+x}<1+\frac{\bar{y} A_{2}}{\beta_{1}\left(A_{2}+x\right)}-\frac{\bar{x} \bar{y}}{\beta_{1}\left(A_{2}+\bar{x}\right)} \tag{4.74}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
y\left(A_{2}+\bar{x}\right)+\beta_{1} A_{2}<\beta_{1}\left(A_{2}+\bar{x}\right)+\bar{y} A_{2}-\bar{x} \bar{y} \tag{4.75}
\end{equation*}
$$

which is equivalent to $\bar{x}\left(2 \bar{y}-\beta_{1}\right)<0$ and

$$
\begin{equation*}
2 \bar{y}-\beta_{1}=\beta_{1}-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}-\beta_{1}=-\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}<0 \tag{4.76}
\end{equation*}
$$

which is true.
Now, we check condition $1+\operatorname{det} J_{T}(\bar{x}, \bar{y})<2$. We have that

$$
\begin{equation*}
\frac{\bar{y} A_{2}}{\beta_{1}\left(A_{2}+\bar{x}\right)}-\frac{\bar{x} \bar{y}}{\beta_{1}\left(A_{2}+\bar{x}\right)}<1 \tag{4.77}
\end{equation*}
$$

This implies that $\bar{y} A_{2}-\bar{x} \bar{y}<\beta_{1} A_{2}+\beta_{1} \bar{x}$ which is true, since

$$
\begin{equation*}
-\frac{A_{2} \sqrt{B_{1}^{2}-4 B_{1} \alpha_{2}}}{2}-\alpha_{2}<\frac{\beta_{1} A_{2}}{2}+\frac{\beta_{1}^{2}+\beta_{1} \sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}} \tag{4.78}
\end{equation*}
$$

Hence, $e_{1}$ is a locally asymptotically stable equilibrium point.
Now, we prove that $e_{2}$ is a saddle. We check the condition (4.17).
Condition $\left|\operatorname{Tr} J_{T}(\bar{x}, \bar{y})\right|>\left|1+\operatorname{det} J_{T}(\bar{x}, \bar{y})\right|$ is equivalent to $\bar{x}\left(2 \bar{y}-\beta_{1}\right)>0$. This is true, since

$$
\begin{equation*}
2 \bar{y}-\beta_{1}=\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}-\beta_{1}=\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}>0 \tag{4.79}
\end{equation*}
$$

Condition

$$
\begin{equation*}
\operatorname{Tr}^{2} J_{T}(\bar{x}, \bar{y})-4 \operatorname{det} J_{T}(\bar{x}, \bar{y})>0 \tag{4.80}
\end{equation*}
$$

is equivalent to

$$
\begin{equation*}
\left(\frac{\bar{y}}{\beta_{1}}-\frac{A_{2}}{A_{2}+\bar{x}}\right)^{2}+\frac{\bar{x} \bar{y}}{\beta_{1}\left(A_{2}+\bar{x}\right)}>0 \tag{4.81}
\end{equation*}
$$

Hence, $e_{2}$ is a saddle.
Now, we prove the statements (c) and (d).
The characteristic equation associated to the system (4.57) at the equilibrium point has the following form

$$
\begin{equation*}
\lambda^{2}-\lambda\left(\frac{\bar{y}}{B_{1} \bar{x}+\bar{y}}+\frac{A_{2}}{A_{2}+\bar{x}}\right)+\frac{\bar{y}\left(A_{2}-\bar{x}\right)}{\left(A_{2}+\bar{x}\right)\left(B_{1} \bar{x}+\bar{y}\right)}=0 \tag{4.82}
\end{equation*}
$$

Now, we have that

$$
\begin{equation*}
\lambda_{1}+\lambda_{2}=\frac{\bar{y}}{B_{1} \bar{x}+\bar{y}}+\frac{A_{2}}{A_{2}+\bar{x}}>0, \quad \lambda_{1} \lambda_{2}=\frac{\bar{y}\left(A_{2}-\bar{x}\right)}{\left(A_{2}+\bar{x}\right)\left(B_{1} \bar{x}+\bar{y}\right)} \tag{4.83}
\end{equation*}
$$

Consider $A_{2}-\bar{x}$. We have that

$$
\begin{gather*}
A_{2}-\bar{x}=\frac{2 B_{1} A_{2}-\beta_{1}+\sqrt{\beta_{1}^{2}-4 B_{1} \alpha_{2}}}{2 B_{1}},  \tag{4.84}\\
\beta_{1}^{2}-4 B_{1} \alpha_{2}-\left(\beta_{1}-2 B_{1} A_{2}\right)^{2}=-4 B_{1}\left(B_{1} A_{2}^{2}-\beta_{1} A_{2}+\alpha_{2}\right)
\end{gather*}
$$

which implies statements (c) and (d).

## 5. Global Behavior

In this section, we present the results on global behavior of the system (1.1).
Theorem 5.1. Table 2 describes the global behavior of the system (1.1)
Proof. Throughout the proof of theorem $\leq$ will denote $\leq_{\text {se }}$.
$\left(\boldsymbol{R}_{1}, \boldsymbol{R}_{2}\right)$ In view of Lemma 3.2, the map $T$ which corresponds to the system (1.1) has an attractive and invariant box $\mathcal{B}=\left[0, \beta_{1} / B_{1}\right] \times\left[L_{y}, U_{y}\right]$, where $L_{y}=B_{1} \alpha_{2} / B_{1}\left(A_{2}-\right.$ $\left.\gamma_{2}\right)+\beta_{1}, U_{y}=\alpha_{2} /\left(A_{2}-\gamma_{2}\right)$, which contains a unique fixed point $E_{1}$. By Theorem 2.1, every solution of the system (1.1) converges to $E_{1}$. Clearly, the basin of attraction of the equilibrium point $E_{1}$ is given by $[0,+\infty)^{2} \backslash\{(0,0)\}$.
$\left(\mathcal{R}_{3}\right)$ By Lemma $3.2 x_{0}=0$ implies that $x_{n}=0$, for all $n \in \mathbb{N}$, and $y_{n} \rightarrow \alpha_{2} /\left(A_{2}-\right.$ $\left.\gamma_{2}\right), n \rightarrow \infty$, which shows that $y$-axes is a subset of the basin of attraction $B\left(E_{1}\right)$. Furthermore, every solution of (1.1) enters and stays in the box $\mathcal{B}$, and so we can restrict our attention to solutions that starts in $\mathcal{B}$. Clearly the set $Q_{4}\left(E_{3}\right) \cap \mathcal{B}$ is an invariant set with a single equilibrium point $E_{3}$, and so every solution that starts there is attracted to $E_{3}$. In view of Corollary 2.2, the interior of rectangle $\llbracket E_{1}, E_{3} \rrbracket$ is attracted to either $E_{1}$ or $E_{3}$, and because $E_{3}$ is the local attractor, it is attracted to $E_{3}$. If $(x, y) \in \mathcal{B} \backslash\left(\llbracket E_{1}, E_{3} \rrbracket \cup\left(Q_{4}\left(E_{3}\right) \cap \mathcal{B}\right) \cup\{(0, y): y \geq 0\}\right)$, then there exist the points $\left(x_{l}, y_{l}\right) \in \llbracket E_{1}, E_{3} \rrbracket$ and $\left(x_{u}, y_{u}\right) \in Q_{4}\left(E_{3}\right) \cap \mathbb{B}$ such that $\left(x_{l}, y_{l}\right) \leq_{\text {se }}(x, y) \leq_{\text {se }}\left(x_{u}, y_{u}\right)$. Consequently, $T^{n}\left(\left(x_{l}, y_{l}\right)\right) \leq_{\mathrm{se}} T^{n}((x, y)) \leq_{\mathrm{se}} T^{n}\left(\left(x_{u}, y_{u}\right)\right)$ for all $n=1,2, \ldots$ and so $T^{n}((x, y)) \rightarrow E_{3}$ as $n \rightarrow \infty$, which completes the proof.
$\left(\boldsymbol{R}_{4}\right)$ The first part of this Theorem is proven in Theorem 4.4.
Now, we prove a global result

$$
J_{T}\left(E_{1}\right)=\left(\begin{array}{cc}
\frac{\bar{y}}{\beta_{1}} & 0  \tag{5.1}\\
-\frac{\gamma_{y}}{A_{2}} & \frac{\gamma_{2}}{A_{2}}
\end{array}\right)
$$

The eigenvalues of $J_{T}\left(E_{1}\right)$ are given by $\lambda_{1}=\beta_{1} / y$ and $\lambda_{2}=\gamma_{2} / A_{2}$ and so

$$
\begin{equation*}
\beta_{1}>\frac{\alpha_{2}}{A_{2}-\gamma_{2}} \Longrightarrow \lambda_{1}>1, \quad A_{2}>\gamma_{2} \Longrightarrow \lambda_{2}<1 . \tag{5.2}
\end{equation*}
$$

The eigenvector of $T$ at $E_{1}$ that corresponds to the eigenvalue $\lambda_{2}<1$ is $(0, y)$.

Table 2: Global behavior of system (1.1). GAS stands for globally asymptotically stable.

\begin{tabular}{|c|c|c|}
\hline Region \& \& Global behavior <br>
\hline $\mathcal{R}_{1}$

$\mathcal{R}_{2}$ \& $$
\begin{gathered}
A_{2}>\gamma_{2}, \alpha_{2}>\frac{\left(\beta_{1}-B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}, \\
A_{2}>\gamma_{2}, \alpha_{2} \leq \frac{\left(\beta_{1}-B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}, \\
\quad \beta_{1}<\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, B_{1}>\frac{\beta_{1}}{A_{2}-\gamma_{2}}
\end{gathered}
$$ \& There exists a unique equilibrium $E_{1}\left(0, \alpha_{2} /\left(A_{2}-\gamma_{2}\right)\right)$, and it is G.A.S.. The basin of attraction of $E_{1}$ is $[0,+\infty)^{2} \backslash\{(0,0)\}$ <br>

\hline $\mathcal{R}_{3}$ \& $$
\begin{gathered}
A_{2}>\gamma_{2}, \beta_{1}=\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)}, \\
B_{1}<\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}
\end{gathered}
$$ \& There exist two equilibrium points $E=E_{1}=E_{2}=$ $\left(0, \alpha_{2} /\left(A_{2}-\gamma_{2}\right)\right)=\left(0, \beta_{1}\right)$ which is nonhyperbolic, and $E_{3}$, which is locally asymptotically stable. Furthermore, the positive part of the $y$-axes is the basin of attraction $\mathcal{B}\left(E_{1}\right)$ of $E_{1}$. The equilibrium point $E_{3}$ is globally asymptotically stable with the basin of attraction $\mathcal{B}\left(E_{3}\right)=(0, \infty) \times[0, \infty)$ <br>

\hline $\mathcal{R}_{4}$ \& $$
\begin{gathered}
A_{2}>\gamma_{2}, \beta_{1}=\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, \\
B_{1}<\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}
\end{gathered}
$$ \& There exist two equilibrium points $E_{1}$ which are a saddle, and $E_{3}$, which is a locally asymptotically stable equilibrium point. Furthermore, the positive part of $y$-axes is the global stable manifold of $\chi^{s}\left(E_{1}\right)$. The equilibrium point $E_{3}$ is G.A.S. with the basin of attraction $\mathcal{B}\left(E_{3}\right)=(0, \infty) \times[0, \infty)$ <br>

\hline $\mathcal{R}_{5}$ \& $$
\begin{gathered}
\gamma_{2}<A_{2} \\
B_{1}\left(\gamma_{2}-A_{2}\right)+\sqrt{4 B_{1} \alpha_{2}}<\beta_{1}< \\
\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, \\
B_{1}<\frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}
\end{gathered}
$$ \& There exist three equilibrium points $E_{1}$ and $E_{3}$ which are locally asymptotically stable and $E_{2}$ which is a saddle. The global stable manifold $\mathcal{W}^{s}\left(E_{2}\right)$ separates the positive quadrant so that all orbits which start below this manifold are attracted to the point $E_{3}$, and all orbits which start above this manifold are attracted to the equilibrium point $E_{1}$. All orbits that start on $\mathcal{W}^{s}\left(E_{2}\right)$ are attracted to $E_{2}$. The global unstable manifold $\mathcal{W}^{u}\left(E_{2}\right)$ is the graph of a continuous strictly decreasing function, and $\mathcal{W}^{u}\left(E_{2}\right)$ has endpoints $E_{a}$ and $E_{3}$ <br>

\hline $\mathcal{R}_{6}$ \& $$
\begin{gathered}
\gamma_{2}<A_{2},\left(A_{2}-\gamma_{2}\right) B_{1}<\beta_{1}, \\
\left(B_{1}\left(A_{2}-\gamma_{2}\right)+\beta_{1}\right)^{2}-4 \alpha_{2} B_{1}=0
\end{gathered}
$$ \& There exist two equilibrium points $E_{1}$, which is locally asymptotically stable, and $E_{2} \equiv E_{3}$, which is nonhyperbolic. Furthermore, there exists an unbounded increasing invariant curve $\mathcal{W}_{E_{2}}$ which is a part of the basin of attraction of $E_{2}$. Every solution that starts above this curve is attracted to the equilibrium point $E_{1}$; every solution that starts below this curve converges to $E_{2}$ <br>

\hline $\mathrm{R}_{7}$ \& $$
\begin{gathered}
\gamma_{2}<A_{2}, \beta_{1}=\frac{\alpha_{2}}{A_{2}-\gamma_{2}}, \\
B_{1} \geq \frac{\alpha_{2}}{\left(A_{2}-\gamma_{2}\right)^{2}}
\end{gathered}
$$ \& There exists a unique equilibrium point $E_{1}=E_{2}=E_{3}$ which is nonhyperbolic. All orbits are attracted to the equilibrium point $E_{1}$ <br>

\hline $\mathrm{R}_{8}$ \& $$
\begin{gathered}
A_{2}<\gamma_{2} \\
\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}>0, \\
\beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right)
\end{gathered}
$$ \& There exist two equilibrium points $E_{2}$, which is a saddle, and $E_{3}$, which is a locally asymptotically stable equilibrium point. Furthermore, there exists the global stable manifold $\mathcal{W}^{2}\left(E_{2}\right)$ that separates the positive quadrant so that all orbits below this manifold are attracted to the equilibrium point $E_{3}$, and all orbits above this manifold are asymptotic to $(0, \infty)$. All orbits that start on $\mathcal{W}^{s}\left(E_{2}\right)$ are attracted to $E_{2}$. The global unstable manifold $\mathfrak{W}^{u}\left(E_{2}\right)$ is the graph of a continuous strictly decreasing function, and $\mathcal{W}^{u}\left(E_{2}\right)$ has endpoints $E_{2}$ and $E_{3}$ <br>

\hline
\end{tabular}

Table 2: Continued.

\begin{tabular}{|c|c|c|}
\hline Region \& \& Global behavior <br>
\hline R9 \& $$
\begin{gathered}
A_{2}<\gamma_{2} \\
\left(B_{1}\left(\gamma_{2}-A_{2}\right)-\beta_{1}\right)^{2}-4 B_{1} \alpha_{2}>0 \\
\beta_{1}>B_{1}\left(\gamma_{2}-A_{2}\right)
\end{gathered}
$$ \& There exists a unique equilibrium point $E=E_{2}=E_{3}$ which is nonhyperbolic. Furthermore, there exists an unbounded increasing invariant curve $\mathcal{W}_{E}$ which is a part of the basin of attraction of $E$. Every solution that stars below this curve is attracted to $E$; every solution that starts above this curve is asymptotic to $(0, \infty)$ <br>
\hline $\boldsymbol{R}_{10}$ \& $A_{2}=\gamma_{2}, \beta_{1}^{2}-4 B_{1} \alpha_{2}=0$, \& There exists a unique equilibrium $e=\left(\beta_{1} / 2 B_{1}, \beta_{1} / 2\right)$ which is nonhyperbolic. Furthermore, there exists an unbounded increasing invariant curve $\mathcal{W}_{E}$, which is a part of the basin of attraction of $E$. Every solution that starts below this curve is attracted to the equilibrium point: every solution that starts above this curve is asymptotic to $(0, \infty)$ <br>
\hline $\boldsymbol{R}_{11}$ \& $A_{2}=\gamma_{2}, \beta_{1}^{2}-4 B_{1} \alpha_{2}>0$ \& There exist two equilibrium points $e_{1}$, which is locally asymptotically stable, and $e_{2}$, which is a saddle equilibrium point. Furthermore, there exists the global stable manifold $W^{s}\left(e_{2}\right)$ that separates the positive quadrant so that all orbits below this manifold are attracted to the equilibrium point $e_{1}$, and allorbits above this manifold are asymptotic to $(0, \infty)$. All orbits that starts on $W^{s}\left(e_{2}\right)$ are attracted to $e_{2}$. The global unstable manifold $W^{u}\left(e_{2}\right)$ is the graph of a continuous strictly decreasing function, and $W^{u}\left(e_{2}\right)$ has an endpoint at $e_{1}$ <br>
\hline $\boldsymbol{R}_{12}$

$\boldsymbol{R}_{13}$ \& $$
\begin{gathered}
\gamma_{2} \geq A_{2} \text { and either } \\
\alpha_{2} \leq \frac{\left(\beta_{1}+B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}, \\
\beta_{1}<B_{1}\left(\gamma_{2}-A_{2}\right) \\
\text { or } \\
\alpha_{2}>\frac{\left(\beta_{1}+B_{1}\left(A_{2}-\gamma_{2}\right)\right)^{2}}{4 B_{1}}
\end{gathered}
$$ \& The system (1.1) does not possess an equilibrium point. Its global behavior is described as follows

$$
x_{n} \rightarrow 0 \quad y_{n} \rightarrow \infty, \quad n \rightarrow \infty
$$ <br>

\hline
\end{tabular}

The rest of the proof is similar to the proof of part $\left(\mathcal{R}_{3}\right)$ and uses some continuity arguments.
$\left(\boldsymbol{R}_{5}\right)$ The first part of this Theorem is proven in Theorem 4.2. Lemma 3.3 states that the system (1.1) has no minimal period-two solution. Take that $\mathcal{R}=\mathbb{R}_{+}^{2} \backslash\{(0,0)\} . T$ is strongly monotone in $\mathcal{R}$ and differentiable in int $R=\mathcal{R}^{\circ}$ (interior of $\mathcal{R}$ ). $E_{2}$ is a saddle point and $E_{2} \in \mathcal{R}^{\circ}$. Then, all hypothesis (a)-(d) of Theorem 2.6 are satisfied. In light of Theorems 2.6 and 2.7, there exist the global stable manifold $W^{s}\left(E_{2}\right)$ and the global unstable manifold $W^{u}\left(E_{2}\right)$ which are the graphs of a continuous strictly monotonic functions. The global stable manifold $W^{s}\left(E_{2}\right)$ separates the first quadrant into two invariant regions $W_{-}$(below the stable) manifold and $W_{+}$(above the stable manifold) which are connected. Each orbit starting above $W^{s}\left(E_{2}\right)$ remains above and is asymptotic to $E_{1}$. Each orbit starting below $W^{s}\left(E_{2}\right)$ remains below and is asymptotic to $E_{3}$. This implies that $E_{1}$ and $E_{3}$ are global attractors. Theorem 4.2 implies that they are globally asymptotically stable.
$\left(\boldsymbol{R}_{6}\right)$ Notice that in this case the eigenvector which corresponds to nonhyperbolic eigenvalue $\lambda_{1}=1$ at $E_{2}$ is $\mathbf{v}_{2}=\left(-1 / B_{1}, 1\right)$, see Theorem 4.3. Thus, the hypotheses of Theorems 2.4 and 2.7 are satisfied at the equilibrium point $E_{2}$, and the conclusions
of Theorems 2.4, 2.5, and 2.7 follow. Let $\mathcal{C}, \mathcal{O}_{-}$and $\mathcal{O}_{+}$be the sets given in the conclusion of Theorems 2.4 and 2.7. Let $S:=\left\{(x, y): 0 \leq x \leq \beta_{1} / B_{1}, 0 \leq y\right\}$. Since $\beta_{1} x / B_{1} x+y \leq \beta_{1} / B_{1}$ for $x \geq 0, y \geq 0, x+y>0$, the map $T$ satisfies $T\left([0, \infty)^{2} \backslash(0,0)\right) \subset S$. Thus, $T\left(\mathcal{C} \cup \mathcal{W}_{+}\right) \subset\left(\mathcal{C} \cup \mathcal{O}_{+}\right) \cap S$, which implies that $T\left(\mathcal{C} \cup \mathcal{W}_{+}\right)$ is bounded. In view of Theorem 2.7, every solution which starts in $\mathcal{W}_{+}$eventually enters $Q_{4}\left(E_{2}\right)$, and so is in rectangle $S \cap Q_{4}\left(E_{2}\right)$, which by Theorem 2.1, implies that all such solutions converge to the equilibrium point $E_{2}$.
If $(x, y)$ is in $\mathcal{W}_{-}$, by Theorem 2.7, the orbit of $(x, y)$ eventually enters $\mathcal{Q}_{2}\left(E_{2}\right)$. Assume (without loss of generality) that $(x, y) \in$ int $Q_{2}\left(E_{2}\right)$.
In view of Corollary 2.2 and the fact that $E_{1}$ is a local attractor $\llbracket E_{1}, E_{2} \rrbracket$ is a subset of the basin of attraction of $E_{1}$. Let $\left(x_{0}, y_{0}\right)$ be any point in $\mathcal{W}_{+}$. Then there exists $\tilde{y}_{0} \geq$ $\max \left\{y_{0}, \alpha_{2} / A_{2}-\gamma_{2}\right\}$ such that $\left(0, \tilde{y}_{0}\right) \leq_{\text {se }}\left(x_{0}, y_{0}\right)$. Now $\left(0, \tilde{y}_{0}\right) \leq_{\text {se }}\left(0, \tilde{y}_{0}\right)=\left(0, \alpha_{2}+\right.$ $\left.r_{2} \tilde{y}_{0} / A_{2}\right)$, which implies that $\left\{T^{n}\left(0, \tilde{y}_{0}\right)\right\}=\left\{\left(0, \tilde{y}_{n}\right)\right\}$ is an increasing sequence, and so $\left\{\tilde{y}_{n}\right\}$ is a decreasing sequence and thus is convergent to $\alpha_{2} / A_{2}-\gamma_{2}$. In view of $\left(0, \tilde{y}_{0}\right) \preceq_{\text {se }}\left(x_{0}, y_{0}\right)$, we conclude that $T^{n}\left(\left(0, \tilde{y}_{0}\right)\right) \leq_{\text {se }} T^{n}\left(\left(x_{0}, y_{0}\right)\right)$ and so $T^{n}\left(\left(x_{0}, y_{0}\right)\right)$ eventually enters $\llbracket E_{1}, E_{2} \rrbracket$, and so it converges to $E_{1}$.
$\left(\mathcal{R}_{7}\right)$ Let $\left(x_{0}, y_{0}\right)$ be any point in $[0, \infty)^{2} \backslash\{(0,0)\}$. Then there exist points $\left(\tilde{x}_{0}, 0\right)$ and $\left(0, \tilde{y}_{0}\right), \tilde{x}_{0} \geq \max \left\{x_{0}, \beta_{1} / B_{1}\right\}, \tilde{y}_{0} \geq \max \left\{y_{0}, \alpha_{2} /\left(A_{2}-\gamma_{2}\right)\right\}$ such that $\left(0, \tilde{y}_{0}\right) \leq_{\text {se }}\left(x_{0}, y_{0}\right) \leq\left(\tilde{x}_{0}, 0\right)$. This gives $T^{n}\left(\left(0, \tilde{y}_{0}\right)\right) \leq_{\mathrm{se}} T^{n}\left(\left(x_{0}, y_{0}\right)\right) \leq T^{n}\left(\left(\tilde{x}_{0}, 0\right)\right)$ for all $n \geq 1$. Clearly, $T\left(\widetilde{x}_{0}, 0\right)=\left(\beta_{1} / B_{1}, \alpha_{2} /\left(A_{2}+\tilde{x}_{0}\right)\right) \leq\left(\widetilde{x}_{0}, 0\right)$, which implies that $\left\{T^{n}\left(\tilde{x}_{0}, 0\right)\right\}=\left\{\left(\widetilde{x}_{n}, \tilde{y}_{n}\right)\right\}$ is a decreasing sequence bounded below by $E_{1}$ and so is convergent to $E_{1}$. Proof that $\left\{T^{n}\left(0, \tilde{y}_{0}\right)\right\}$ is convergent to $E_{1}$ is carried in a same way as in the proof of $\left(\mathcal{R}_{6}\right)$. In view of $T^{n}\left(\left(0, \tilde{y}_{0}\right)\right) \leq_{\mathrm{se}} T^{n}\left(\left(x_{0}, y_{0}\right)\right) \leq T^{n}\left(\left(\tilde{x}_{0}, 0\right)\right)$ we conclude that $\left\{T^{n}\left(\left(x_{0}, y_{0}\right)\right)\right\}$ converges to $E_{1}$.
$\left(\mathcal{R}_{8}\right) \operatorname{Put} T_{1}(x, y)=\beta_{1} x /\left(B_{1} x+y\right), T_{2}(x, y)=\left(\alpha_{2}+\gamma_{2} y\right) /\left(A_{2}+x\right)$. Take $x=\left(x_{0}, y_{0}\right) \in$ $W_{+}\left(E_{2}\right) \cap \mathcal{R}(-,+)$, where $\mathcal{R}(-,+)=\left\{(x, y) \in \mathcal{R}: T_{1}(x, y)<x, T_{2}(x, y)>y\right\}$. It is known that $\mathcal{R}(-,+)$ is an invariant set, see [11].
Then we have

$$
\begin{equation*}
T_{1}\left(x_{0}, y_{0}\right)=\frac{\beta_{1} x_{0}}{B_{1} x_{0}+y_{0}}<x_{0}, \quad T_{2}\left(x_{0}, y_{0}\right)=\frac{\alpha_{2}+\gamma_{2} y_{0}}{A_{2}+x_{0}}>y_{0} . \tag{5.3}
\end{equation*}
$$

This implies

$$
\begin{equation*}
\left(T_{1}\left(x_{0}, y_{0}\right), T_{2}\left(x_{0}, y_{0}\right)\right) \preceq_{\text {se }}\left(x_{0}, y_{0}\right) \Longleftrightarrow T\left(x_{0}, y_{0}\right) \preceq_{\text {se }}\left(x_{0}, y_{0}\right) . \tag{5.4}
\end{equation*}
$$

By using monotonicity $T^{2}\left(x_{0}, y_{0}\right) \leq_{\mathrm{se}} T\left(x_{0}, y_{0}\right)$. By using induction $T^{n+1}\left(x_{0}, y_{0}\right) \leq_{\mathrm{se}}$ $T^{n}\left(x_{0}, y_{0}\right)$. This implies that sequence $\left\{x_{n}\right\}$ is non-increasing and $\left\{y_{n}\right\}$ is nondecreasing. By Lemma 3.2, $\left\{x_{n}\right\}$ is bounded, hence it must converges. By using equation for $x_{n+1}$ we see that the limit is zero. Since, $\left\{y_{n}\right\}$ is unbounded and nondecreasing then $y_{n} \rightarrow \infty, n \rightarrow \infty$.
By Theorems 2.4 and 2.7, all orbits below this manifold are attracted to the equilibrium point $E_{3}$.
$\left(\boldsymbol{R}_{9}\right)$ Since the hypotheses of Theorems 2.4, and 2.7 are satisfied at the equilibrium point $E_{2}$, the conclusions of Theorems 2.4, 2.5, and 2.7 follow. Let $\mathcal{C}, \mathcal{O}_{-}$and $\mathcal{O}_{+}$be the
sets given in the conclusion of Theorems 2.4 and 2.7. Let $S:=\{(x, y): 0 \leq x \leq$ $\left.\beta_{1} / B_{1}, \quad 0 \leq y\right\}$. Since $\beta_{1} x / B_{1} x+y \leq \beta_{1} / B_{1}$ for $x \geq 0, y \geq 0, x+y>0$, the map $T$ satisfies $T\left([0, \infty)^{2} \backslash(0,0)\right) \subset S$. Thus $T\left(\mathcal{C} \cup \mathcal{W}_{+}\right) \subset\left(\mathcal{C} \cup \mathcal{W}_{+}\right) \cap S$, which implies that $T\left(\mathcal{C} \cup \mathcal{W}_{+}\right)$is bounded. In view of Theorem 2.7 every solution which starts in $\mathcal{W}_{+}$ eventually enters $Q_{4}\left(E_{2}\right)$ and so is in rectangle $S \cap Q_{4}\left(E_{2}\right)$, which by Theorem 2.1, implies that all such solutions converge to the equilibrium point $E_{2}$.
If $(x, y)$ is in $\mathcal{W}_{-}$, by Theorem 2.7 the orbit of $(x, y)$ eventually enters $\mathcal{Q}_{2}\left(E_{2}\right)$. Assume (without loss of generality) that $(x, y) \in$ int $Q_{2}\left(E_{2}\right)$.
A calculation gives

$$
\begin{equation*}
T\left(E_{2}+t \mathbf{v}_{2}\right)=\left(\frac{\beta_{1}+B_{1}\left(\gamma_{2}-A_{2}\right)-2 t}{2 B_{1}}, \frac{B_{1}\left(2 \alpha_{2}+\gamma_{2}\left(\beta_{1}+B_{1}\left(A_{2}-\gamma_{2}\right)+2 t\right)\right.}{\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)-2 t}\right) \tag{5.5}
\end{equation*}
$$

for all $t$ and

$$
\begin{gather*}
\frac{d}{d t} T\left(E_{2}+t \mathbf{v}_{2}\right)=\left(-\frac{1}{B_{1}}, \frac{2 B_{1}\left(\beta_{1}+2 \alpha_{2}+B_{1}\left(A_{2}+\gamma_{2}\right)+\beta_{1} \gamma_{2}+B_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)\right)}{\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)-2 t\right)^{2}}\right) \\
\left.\frac{1}{2} \frac{d^{2}}{d t^{2}} T\left(E_{2}+t \mathbf{v}_{2}\right)\right|_{t=0}=\left(0, \frac{4 B_{1}\left(\beta_{1}+2 \alpha_{2}+B_{1}\left(A_{2}+\gamma_{2}\right)+\beta_{1} \gamma_{2}+B_{1} \gamma_{2}\left(A_{2}-\gamma_{2}\right)\right)}{\left(\beta_{1}+B_{1}\left(A_{2}+\gamma_{2}\right)-2 t\right)^{3}}\right) \tag{5.6}
\end{gather*}
$$

Since in expansion (2.5) we have $\left(c_{2}, d_{2}\right)$ with $d_{2}>0$ and $T\left(E_{2}+t \mathbf{v}_{2}\right)^{(1)}$ is affine in $t$, by Theorem 2.10 in any relative neighborhood of $E_{2}$ there exists a subsolution $(w, z) \in Q_{2}\left(E_{2}\right)$, that is, $T(w, z) \preceq_{\text {se }}(w, z)$. Choose one such $(w, z)$ so that $(x, y) \preceq_{\text {se }}(w, z)$. Since $T$ is competitive, $T^{n+1}(w, z) \leq T^{n}(w, z)$ for $n=0,1,2, \ldots$. The monotonically decreasing sequence $\left\{T^{n}(w, z)\right\}$ in $Q_{2}(E)$ is unbounded below, since if it were not it would converge to the unique fixed point in $Q_{2}(E)$, namely $E$, which is not possible. Let $\left(w_{n}, z_{n}\right):=T^{n}(w, z), n=0,1, \ldots$ Then $\left(w_{n}, z_{n}\right) \in S$ for $n=1,2, \ldots$, hence $\left\{w_{n}\right\}$ is bounded. It follows that $\left\{z_{n}\right\}$ is monotone and unbounded. From (1.1) it follows that $w_{n} \rightarrow 0$. Since $T^{n}(x, y) \preceq_{\text {se }}\left(w_{n}, z_{n}\right)$, it follows that $T^{n}(x, y) \rightarrow(0, \infty)$.
$\left(\boldsymbol{R}_{10}\right)$ The eigenvalues of the map $T$ are $\lambda_{1}=1$ and $\lambda_{2}=2 A_{2} B_{1}-\beta_{1} / 2\left(2 A_{2} B_{1}+\beta_{1}\right)$. The corresponding eigenvectors are

$$
\begin{equation*}
\left(-\frac{1}{B_{1}}, 1\right), \quad\left(\frac{2 A_{2} B_{1}+\beta_{1}}{2 B_{1} \beta_{1}}, 1\right) \tag{5.7}
\end{equation*}
$$

The existence of the continuous curve $\mathcal{W}_{E}$ follows as in the proof of $\left(\mathcal{R}_{9}\right)$ as well as convergence of all points which start in $\mathcal{W}_{+}$.
If $(x, y)$ is in $\mathcal{W}_{-}$, by Theorem 2.7 the orbit of $(x, y)$ eventually enters $Q_{2}\left(E_{2}\right)$. Assume (without loss of generality) that $(x, y) \in$ int $Q_{2}\left(E_{2}\right)$.

A straightforward calculation gives

$$
\begin{equation*}
T\left(E_{2}+t \mathbf{v}_{2}\right)=\left(\frac{\beta_{1}-2 t}{2 B_{1}}, \frac{B_{1}\left(2 \alpha_{2}+A_{2}\left(\beta_{1}+2 t\right)\right)}{\beta_{1}+2 B_{1} A_{2}-2 t}\right) \tag{5.8}
\end{equation*}
$$

for all $t$ and

$$
\begin{gather*}
\frac{d}{d t} T\left(E_{2}+t \mathbf{v}_{2}\right)=\left(-\frac{1}{B_{1}}, \frac{\left(\beta_{1}+2 B_{1} A_{2}\right)^{2}}{\left(\beta_{1}+2 B_{1} A_{2}-2 t\right)^{2}}\right)  \tag{5.9}\\
\left.\frac{1}{2} \frac{d^{2}}{d t^{2}} T\left(E_{2}+t \mathbf{v}_{2}\right)\right|_{t=0}=\left(0, \frac{2}{\beta_{1}+2 B_{1} A_{2}}\right)
\end{gather*}
$$

Since in expansion (2.5) we have $\left(c_{2}, d_{2}\right)$ with $d_{2}>0$ and $T\left(E_{2}+t \mathbf{v}_{2}\right)^{(1)}$ is affine in $t$, by Theorem 2.10 in any relative neighborhood of $E_{2}$ there exists a subsolution $(w, z) \in Q_{2}\left(E_{2}\right)$, that is, $T(w, z) \preceq_{\text {se }}(w, z)$. The rest of the proof is same as the proof of the case $\left(\mathcal{R}_{10}\right)$.
$\left(\mathcal{R}_{11}\right)$ The proof, which is similar to the proof of $\left(\mathcal{R}_{8}\right)$, follows as an immediate application of Lemmas 3.2 and 3.3, and Theorems 2.4, 2.7, and 4.6.
$\left(\mathcal{R}_{12}\right)$ For every point $\left(x_{0}, y_{0}\right) \in[0, \infty)^{2} \backslash\{(0,0)\}$, there exists $\tilde{x}_{0} \geq \beta_{1} / B_{1}$ such that $\left(x_{0}, y_{0}\right) \leq\left(\tilde{x}_{0}, 0\right)$. Clearly, $T\left(\tilde{x}_{0}, 0\right)=\left(\beta_{1} / B_{1}, \alpha_{2} /\left(A_{2}+\tilde{x}_{0}\right)\right) \leq\left(\tilde{x}_{0}, 0\right)$, which implies that $\left\{T^{n}\left(\tilde{x}_{0}, 0\right)\right\}=\left\{\left(\tilde{x}_{n}, \tilde{y}_{n}\right)\right\}$ is a decreasing sequence, and so $\tilde{y}_{n}$ is an increasing sequence. If $\tilde{y}_{n}$ would be convergent, then $\left\{\left(\tilde{x}_{n}, \tilde{y}_{n}\right)\right\}$ would converge to an equilibrium of system (1.1) which is impossible. Thus, $\lim _{n \rightarrow \infty} \tilde{y}_{n}=\infty$, which implies that $\lim _{n \rightarrow \infty} \tilde{y}_{n}=0$. In view of $\left(x_{0}, y_{0}\right) \leq\left(\tilde{x}_{0}, 0\right)$, we obtain that $T^{n}\left(\left(x_{0}, y_{0}\right)\right) \leq T^{n}\left(\left(\tilde{x}_{0}, 0\right)\right)$ and so

$$
\begin{equation*}
x_{n} \longrightarrow 0, \quad y_{n} \longrightarrow \infty, \quad n \longrightarrow \infty \tag{5.10}
\end{equation*}
$$

follows.

Remark 5.2. We can see from Theorem 5.1 that system (1.1) exhibits variety of behaviors in different ranges of parameters. These behaviors can be classified in a few categories that verbally describe situation. Here, we use some terminology introduced in [19].

A coexistence attractor is one in which both species are present. An exclusion attractor is one in which one species is absent and the other species is present. By multiple mixed-type attractors, we mean a scenario that includes at least one coexistence attractor and at least one exclusion attractor. Park in [20,21] observed the coexistence case in an experimental treatment that also included cases of competitive exclusion, that is, he observed a case termed to be multiple mixed-type attractors. Competition theory is primarily an equilibrium theory that is exemplified, by its limited number of asymptotic outcomes: a globally attracting coexistence equilibrium; a globally attracting exclusion equilibrium; at least two attracting exclusion equilibria; at least two attracting coexistence equilibria; a continuum of nonhyperbolic equilibria. (In this context, globally attracting means within the positive cone
of state space.) Four of these five asymptotic alternatives are illustrated by the LeslieGower model (the discrete analog of the famous LotkaVolterra differential equation model), see [2, 5, 17]

$$
\begin{equation*}
x_{n+1}=\frac{b_{1} x_{n}}{1+x_{n}+c_{1} y_{n}}, \quad y_{n+1}=\frac{b_{2} y_{n}}{1+c_{2} x_{n}+y_{n}}, \quad n=0,1, \ldots \tag{5.11}
\end{equation*}
$$

and fifth alternative, precisely two attracting coexistence equilibria are illustrated by the LeslieGower model with stockings, see [16]

$$
\begin{equation*}
x_{n+1}=\frac{b_{1} x_{n}}{1+x_{n}+c_{1} y_{n}}+h_{1}, \quad y_{n+1}=\frac{b_{2} y_{n}}{1+c_{2} x_{n}+y_{n}}+h_{2}, \quad n=0,1, \ldots \tag{5.12}
\end{equation*}
$$

where the parameters $b_{1}, b_{2}, c_{1}, c_{2}, h_{1}$, and $h_{2}$ are positive numbers, and the initial conditions $x_{0}, y_{0}$ are arbitrary nonnegative numbers. Here, $b_{1}$ and $b_{2}$ are the inherent birth rates, $c_{1}$ and $c_{2}$ the density-dependent effects on newborn recruitment, and $h_{1}$ and $h_{2}$ are constant stockings. With this in mind, we introduce the following terminology. Let $E_{1}, E_{2}$ and $E_{3}$ be three equilibrium points of general competitive system (2.1) in south-east ordering $E_{1} \preceq_{\text {se }} E_{2} \preceq_{\text {se }} E_{3}$ and assume that $E_{1}$ and $E_{3}$ are attractors, and $E_{2}$ is a saddle point or nonhyperbolic equilibrium having one characteristic values in $(-1,1)$. If $E_{1}$ and $E_{3}$ belong to $y$-axes and $x$-axes and $E_{2}$ is a saddle point (resp., nonhyperbolic equilibrium having one characteristic values in $(-1,1)$ ) then we say that system (2.1) exhibits saddle competitive exclusion (resp., nonhyperbolic competitive exclusion). If $E_{1}$ and $E_{3}$ belong to the interior of first quadrant and $E_{2}$ is a saddle point (resp., nonhyperbolic equilibrium having one characteristic values in $(-1,1)$ ) then we say that system (2.1) exhibits saddle competitive coexistence (resp., nonhyperbolic competitive coexistence). If one of the equilibrium points $E_{1}$ and $E_{3}$ is on the axes and the other is in the interior of first quadrant and $E_{2}$ is a saddle point (resp., nonhyperbolic equilibrium having one characteristic values in $(-1,1)$ ), then we say that system (2.1) exhibits saddle competitive exclusion to coexistence (resp., nonhyperbolic competitive exclusion-to-coexistence). If there exists a single attractor in the interior of first quadrant which attracts all points where it is defined except eventually the points on the axes, we say that system (2.1) exhibits global competitive coexistence or global competitive exclusion depending on whether the attractor is on the axes or in the interior of first quadrant. If any of the attractors one the axes is the point $(0, \infty)$ or $(\infty, 0)$, such a situation will be named singular.

Using this terminology, we can describe the global behavior of system (1.1) in a concise way as follows. System (1.1) exhibits global competitive exclusion if the parameters belong to the regions $\left(\boldsymbol{R}_{1}\right),\left(\boldsymbol{R}_{2}\right)$, or $\left(\boldsymbol{R}_{7}\right)$ and global competitive coexistence if the parameters belong to the region $\left(\mathcal{R}_{4}\right)$.

System (1.1) exhibits saddle competitive exclusion to coexistence if the parameters belong to the region $\left(\mathcal{R}_{5}\right)$. System (1.1) exhibits nonhyperbolic competitive exclusion to coexistence if the parameters belong to the region $\left(\mathcal{R}_{6}\right)$. System (1.1) exhibits singular saddle competitive exclusion to coexistence if the parameters belong to the regions $\left(\mathcal{R}_{8}\right)$ or ( $\mathcal{R}_{9}$ ). System (1.1) exhibits singular nonhyperbolic competitive exclusion to coexistence if the parameters belong to the region $\left(\mathcal{R}_{10}\right)$ or $\left(\mathcal{R}_{11}\right)$. Finally, System (1.1) exhibits singular competitive exclusion when the parameters belong to the region $\left(\mathcal{R}_{12}\right)$. With this terminology the possible scenarios for this system are limited, and transition from one scenario to another could be possible explained by using the bifurcation theory. In particular, transition from
global competitive exclusion to global competitive coexistence was explained in [3], and some related results can be found in [19], where an attempt has been made to explain the transitions from one scenario to another by using evolutionary game theory.

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