Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 341520, 22 pages
doi:10.1155/2011/341520

Research Article

Uniformly Almost Periodic Functions and Almost Periodic Solutions to Dynamic Equations on Time Scales

Yongkun Li and Chao Wang

Department of Mathematics, Yunnan University, Yunnan, Kunming 650091, China

Correspondence should be addressed to Yongkun Li, yklie@ynu.edu.cn

Received 29 April 2011; Revised 27 July 2011; Accepted 25 August 2011

Academic Editor: Detlev Buchholz

Firstly, we propose a concept of uniformly almost periodic functions on almost periodic time scales and investigate some basic properties of them. When time scale $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, our definition of the uniformly almost periodic functions is equivalent to the classical definitions of uniformly almost periodic functions and the uniformly almost periodic sequences, respectively. Then, based on these, we study the existence and uniqueness of almost periodic solutions and derive some fundamental conditions of admitting an exponential dichotomy to linear dynamic equations. Finally, as an application of our results, we study the existence of almost periodic solutions for an almost periodic nonlinear dynamic equations on time scales.

1. Introduction

In recent years, researches in many fields on time scales have received much attention. The theory of calculus on time scales (see [1, 2] and references cited therein) was initiated by Hilger in his Ph.D. thesis in 1988 [3] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since his fundamental work. It has been created in order to unify the study of differential and difference equations. Many papers have been published on the theory of dynamic equations on time scales [4–10]. Also, the existence of almost periodic, asymptotically almost periodic, and pseudo-almost periodic solutions is among the most attractive topics in qualitative theory of differential equations and difference equations due to their applications, especially in biology, economics and physics [11–29]. However, there are no concepts of almost periodic functions on time scales so that it is impossible for us to study almost periodic solutions for dynamic equations on time scales.

Motivated by the above, our main purpose of this paper is firstly to propose a concept of uniformly almost periodic functions on time scales and investigate some basic properties of them. Then we study the existence and uniqueness of almost periodic solutions to linear dynamic equations on almost time scales. Finally, as an application of our results, we study the existence of almost periodic solutions for almost periodic nonlinear dynamic equations on time scales.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in the later sections. In Section 3, we propose the concept of uniformly almost periodic functions on almost periodic time scales and investigate the basic properties of uniformly almost periodic functions on almost periodic time scales. In Section 4, we study the existence and uniqueness of almost periodic solutions and derive some fundamental conditions of admitting an exponential dichotomy to linear dynamic equations on time scales. In Section 5, as an application of our results, we study the existence of almost periodic solutions for almost periodic nonlinear dynamic equations on time scales.

2. Preliminaries

In this section, we will first recall some basic definitions lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess μ: $\mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t. \tag{2.1}$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that, for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left| \left[y(\sigma(t)) - y(s) \right] - y^\Delta(t) [\sigma(t) - s] \right| < \varepsilon |\sigma(t) - s| \tag{2.2}$$

for all $s \in U$.

Let y be right-dense continuous; if $Y^\Delta(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s) \Delta s = Y(t) - Y(a). \tag{2.3}$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{for all } t \in \mathbb{T}\}$.

An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive provided

$$I + \mu(t)A(t) \text{ is invertible} \quad \forall t \in \mathbb{T}, \tag{2.4}$$

and the class of all such regressive and rd-continuous functions is denoted, similar to the above scalar case, by $\mathcal{R} = \mathbb{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\} \tag{2.5}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases} \tag{2.6}$$

Definition 2.1 (see [1, 2]). Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions; define

$$p \oplus q = p + q + \mu pq, \qquad \ominus p = -\frac{p}{1+\mu p}, \qquad p \ominus q = p \oplus (\ominus q). \tag{2.7}$$

Lemma 2.2 (see [1, 2]). *Assume that $p, q : \mathbb{T} \to R$ are two regressive functions, then*

(i) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(v) $(e_{\ominus p}(t,s))^{\Delta} = (\ominus p)(t)e_{\ominus p}(t,s)$;

(vi) if $a, b, c \in \mathbb{T}$, then $\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c,a) - e_p(c,b)$.

Lemma 2.3 (see [1, 2]). *Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a function sequence on J such that*

(i) *$\{f_n\}_{n\in\mathbb{N}}$ is a uniformly bounded on J;*

(ii) *$\{f_n^{\Delta}\}_{n\in\mathbb{N}}$ is a uniformly bounded on J.*

Then, there is a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ which converges uniformly on J where J is an arbitrary compact subset of $\mathbb{T}$.

3. Uniformly Almost Periodic Functions

Let $\mathbb{T}$ be a given time scale, and $\mathbb{T}$ is a complete metric space with the metric (distance) d defined by

$$d(t, t_0) = |t - t_0| \quad \text{for } t, t_0 \in \mathbb{T}. \tag{3.1}$$

For a given $\delta > 0$, the δ-neighborhood $U(t_0, \delta)$ of a given point $t_0 \in \mathbb{T}$ is the set of all points $t \in \mathbb{T}$ such that $d(t, t_0) < \delta$.

Throughout this paper, $\mathbb{E}^n$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$, D denotes an open set in $\mathbb{E}^n$ or $D = \mathbb{E}^n$, and S denotes an arbitrary compact subset of D.

Definition 3.1. $f : X \to \mathbb{E}^n$ is called continuous at $t_0 \in X \subseteq \mathbb{T}$ if, and only if for any $\varepsilon > 0$, there exists $U(t_0, \delta)$ such that, for any $s \in U(t_0, \delta)$,

$$\left| f(s) - f(t_0) \right| < \varepsilon. \tag{3.2}$$

f is called continuous on X provided that it is continuous for every $t \in X$.

Definition 3.2. $f : X \to \mathbb{E}^n$ is called uniformly continuous on $X \subseteq \mathbb{T}$ if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that, for any $t_1, t_2 \in X$ with $|t_1 - t_2| < \delta(\varepsilon)$ it is implied that

$$\left| f(t_1) - f(t_2) \right| < \varepsilon. \tag{3.3}$$

Similar to the finite covering theorem in functional analysis (see [30]), one can easily show that the following.

Lemma 3.3. *Let $([a,b] \cap \mathbb{T}) \subset \mathbb{T}$ be a closed interval. If $([a,b] \cap \mathbb{T}) \subseteq \bigcup_{\alpha \in I} (G_\alpha \cap \mathbb{T})$, where I is an index set, and for every $\alpha \in I$, G_α is an open set in $\mathbb{R}$, then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$, such that $([a,b] \cap \mathbb{T}) \subseteq \bigcup_{k=1}^{n} (G_{\alpha_k} \cap \mathbb{T})$.*

Also, one can easily prove the following two lemmas.

Lemma 3.4. *If $\{S_n(x)\}$ converges uniformly to $S(x)$ on $[a,b] \cap \mathbb{T}$ and each $S_n(x)$ is continuous on $[a,b] \cap \mathbb{T}$, then $S(x)$ is continuous on $[a,b] \cap \mathbb{T}$, $\lim_{n\to\infty} \int_a^b S_n(x)\Delta x = \int_a^b S(x)\Delta x = \int_a^b \lim_{n\to\infty} S_n(x)\Delta x$, and $\{\int_a^x S_n(t)\Delta t\}$ converges uniformly to $\int_a^x S(t)\Delta t$.*

Lemma 3.5. *If a sequence $\{S_n(x)\}$ converges to $S(x)$ on $[a,b] \cap \mathbb{T}$ and, for each $n \in \mathbb{N}$, $S_n(x)$ has continuous derivative $S_n^\Delta(x)$ and $\{S_n^\Delta(x)\}$ converges uniformly to $Q(x)$, then $S^\Delta(x) = Q(x)$, that is, $\lim_{n\to\infty} S_n^\Delta(x) = (\lim_{n\to\infty} S_n(x))^\Delta$, and $\{S_n(x)\}$ converges to $S(x)$ uniformly on $[a,b] \cap \mathbb{T}$.*

By using Lemma 3.3, one can easily show that the following.

Lemma 3.6. *Let $f \in C([a,b] \cap \mathbb{T}, \mathbb{E}^n)$, then f is uniformly continuous on $[a,b] \cap \mathbb{T}$.*

Definition 3.7. A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}. \tag{3.4}$$

Obviously, if $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$ and if T is an almost periodic time scale, then $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$.

Example 3.8. If $\mathbb{T} = \bigcup_{k \in \mathbb{Z}} [k(a+b), k(a+b)+b]$, where $a \neq -b$, then

$$\sigma(t) = \begin{cases} t & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+b), \\ t+a & \text{if } t \in \bigcup_{k=0}^{\infty} \{k(a+b)+b\}, \end{cases}$$
$$\mu(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+b), \\ a & \text{if } t \in \bigcup_{k=0}^{\infty} \{k(a+b)+b\}. \end{cases} \tag{3.5}$$

$a + b \in \Pi \neq \{0\}$. Hence, it is an almost periodic time scale. Obviously, if $b = 0, a = 1$, then $\mathbb{T} = \mathbb{Z}$. If $b = 1, a = 0$, then $\mathbb{T} = \mathbb{R}$.

Definition 3.9. Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T}, \mathbb{E}^n)$ is called an almost periodic function if the ε-translation set of f

$$E\{\varepsilon, f\} = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\} \tag{3.6}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \quad \forall t \in \mathbb{T}. \tag{3.7}$$

τ is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

Definition 3.10. Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of f

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t+\tau, x) - f(t, x)| < \varepsilon, \ \forall (t, x) \in \mathbb{T} \times S\} \tag{3.8}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \quad \forall t \in \mathbb{T} \times \mathrm{S}. \tag{3.9}$$

τ is called the ε-translation number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Obviously, an almost periodic function can be regarded as a special case of a uniformly almost periodic function. So, in the following, we mainly discuss the basic properties of uniformly almost periodic functions. The basic properties of almost periodic functions can be derived directly from the corresponding ones of uniformly almost periodic functions.

Remark 3.11. If $\mathbb{T} = \mathbb{R}$, then $\Pi = \mathbb{R}$ in this case, Definitions 3.4 and 3.5 are equivalent to the definitions of the almost periodic functions and the uniformly almost periodic functions in [16], respectively. If $\mathbb{T} = \mathbb{Z}$, then $\Pi = \mathbb{Z}$, in this case, Definitions 3.4 and 3.5 are equivalent to the definitions of the almost periodic functions and the uniformly almost periodic sequences in [17, 18].

For convenience, we denote $AP(\mathbb{T}) = \{f : f \in C(\mathbb{T}, \mathbb{E}^n),\ f$ is almost periodic$\}$ and introduce some notations: let $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$ be two sequences. Then, $\beta \subset \alpha$ means that β is a subsequence of α, $\alpha+\beta = \{\alpha_n+\beta_n\}$, $-\alpha = \{-\alpha_n\}$, and α and β are common subsequences of α' and β', respectively, which means that $\alpha_n = \alpha'_{n(k)}$ and $\beta_n = \beta'_{n(k)}$ for some given function $n(k)$.

We will introduce the translation operator $T, T_\alpha f(t,x) = g(t,x)$ which means that $g(t,x) = \lim_{n\to+\infty} f(t+\alpha_n, x)$ and is written only when the limit exists. The mode of convergence, for example, pointwise, uniform, and so forth, will be specified at each use of the symbol.

Similar to the proof of Theorem 1.13 in [16], one can show that.

Theorem 3.12. *Let $f \in C(\mathbb{T}\times D, \mathbb{E}^n)$ be almost periodic in t uniformly for $x \in D$, then it is uniformly continuous and bounded on $\mathbb{T} \times S$.*

Theorem 3.13. *Let $f \in C(\mathbb{T}\times D, \mathbb{E}^n)$ be almost periodic in t uniformly for $x \in D$, then, for any given sequence $\alpha' \subset \Pi$, there exist a subsequence $\beta \subset \alpha'$ and $g \in C(\mathbb{T}\times D, \mathbb{E}^n)$ such that $T_\beta f(t,x) = g(t,x)$ holds uniformly on $\mathbb{T} \times S$ and $g(t,x)$ is almost periodic in t uniformly for $x \in D$.*

Proof. For any $\varepsilon > 0$ and $S \subset D$, let $l = l(\varepsilon/4, S)$ be an inclusion length of $E\{\varepsilon/4, f, S\}$. For any given subsequence $\alpha' = \{\alpha'_n\} \subset \Pi$, we denote $\alpha'_n = \tau'_n + \gamma'_n$, where $\tau'_n \in E\{\varepsilon/4, f, S\}, \gamma'_n \in \Pi$, and $0 \le \gamma'_n \le l, n = 1, 2, \ldots$. (In fact, for any interval with length of l, there exists $\tau'_n \in E\{\varepsilon/4, f, S\}$, thus, we can choose a proper interval with length of l such that $0 \le \alpha'_n - \tau'_n \le l$, from the definition of Π, it is easy to see that $\gamma'_n = \alpha'_n - \tau'_n \in \Pi$.) Therefore, there exists a subsequence $\gamma = \{\gamma_n\} \subset \gamma' = \{\gamma'_n\}$ such that $\gamma_n \to s$ as $n \to \infty$, $0 \le s \le l$.

Also, it follows from Theorem 3.12 that $f(t,x)$ is uniformly continuous on $\mathbb{T}\times S$. Hence, there exists $\delta(\varepsilon, S) > 0$ so that $|t_1 - t_2| < \delta$, for $x \in S$, implies

$$\left|f(t_1,x) - f(t_2,x)\right| < \frac{\varepsilon}{2}. \tag{3.10}$$

Since γ is a convergent sequence, there exists $N = N(\delta)$ so that $p, m \ge N$ implies $|\gamma_p - \gamma_m| < \delta$. Now, one can take $\alpha \subset \alpha', \tau \subset \tau' = \{\tau'_n\}$ such that α, τ common with γ then; for any integers $p, m \ge N$, we have

$$\begin{aligned}\left|f(t+\tau_p-\tau_m,x) - f(t,x)\right| &\le \left|f(t+\tau_p-\tau_m,x) - f(t+\tau_p,x)\right| + \left|f(t+\tau_p,x) - f(t,x)\right| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},\end{aligned} \tag{3.11}$$

that is,

$$(\alpha_p - \alpha_m) - (\gamma_p - \gamma_m) = \tau_p - \tau_m \in E\left\{\frac{\varepsilon}{2}, f, S\right\}. \tag{3.12}$$

Hence, we can obtain

$$\begin{aligned}
\left|f(t+\alpha_p,x) - f(t+\alpha_m,x)\right| &\le \sup_{(t,x)\in\mathbb{T}\times S}\left|f(t+\alpha_p,x) - f(t+\alpha_m,x)\right| \\
&\le \sup_{(t,x)\in\mathbb{T}\times S}\left|f(t+\alpha_p-\alpha_m,x) - f(t,x)\right| \\
&\le \sup_{(t,x)\in\mathbb{T}\times S}\left|f(t+\alpha_p-\alpha_m,x) - f(t+\gamma_p-\gamma_m,x)\right| \\
&\quad + \sup_{(t,x)\in\mathbb{T}\times S}\left|f(t+\gamma_p-\gamma_m,x) - f(t,x)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{3.13}$$

Thus, we can take sequences $\alpha^{(k)} = \{\alpha_n^{(k)}\}$, $k = 1,2,\ldots$, and $\alpha^{(k+1)} \subset \alpha^{(k)} \subset \alpha$ such that, for any integers m, p, and all $(t,x) \in \mathbb{T}\times S$, the following holds:

$$\left|f\left(t+\alpha_p^{(k)},x\right) - f\left(t+\alpha_m^{(k)},x\right)\right| < \frac{1}{k}, \qquad k = 1,2,\ldots. \tag{3.14}$$

For all sequences $\alpha^{(k)}, k = 1,2,\ldots$, we can take a sequence $\beta = \{\beta_n\},\ \beta_n = \alpha_n^{(n)}$ then, it is easy to see that $\{f(t+\beta_n,x)\} \subset \{f(t+\alpha_n,x)\}$ for any integers p, m with $p < m$ and all $(t,x) \in \mathbb{T}\times S$ the following holds:

$$\left|f(t+\beta_p,x) - f(t+\beta_m,x)\right| < \frac{1}{p}. \tag{3.15}$$

Therefore, $\{f(t+\beta_n,x)\}$ converges uniformly on $\mathbb{T}\times S$ that is, $T_\beta f(t,x) = g(t,x)$ holds uniformly on $\mathbb{T}\times S$, where $\beta = \{\beta_n\} \subset \alpha$.

Next, we will prove that $g(t,x)$ is continuous on $\mathbb{T}\times D$. If this is not true,then there must exist $(t_0,x_0) \in \mathbb{T}\times D$ such that $g(t,x)$ is not continuous at this point. Then there exist $\varepsilon_0 > 0$ and sequences $\{\delta_m\}, \{t_m\}$, and $\{x_m\}$, where $\delta_m > 0, \delta_m \to 0$ as $m \to +\infty$, $|t_0 - t_m| + |x_0 - x_m| < \delta_m$ and

$$\left|g(t_0,x_0) - g(t_m,x_m)\right| \ge \varepsilon_0. \tag{3.16}$$

Let $X = \{x_m\} \bigcup \{x_0\}$; obviously, X is a compact subset of D. Hence, there exists positive integer $N = N(\varepsilon_0, X)$ so that $n > N$ implies

$$\left|f\left(t_m + \beta_n, x_m\right) - g(t_m, x_m)\right| < \frac{\varepsilon_0}{3} \quad \forall m \in \mathbb{Z}^+,$$
$$\left|f\left(t_0 + \beta_n, x_0\right) - g(t_0, x_0)\right| < \frac{\varepsilon_0}{3}. \tag{3.17}$$

According to the uniform continuity of $f(t,x)$ on $\mathbb{T} \times X$, for sufficiently large m, we have

$$\left|f\left(t_0 + \beta_n, x_0\right) - f\left(t_m + \beta_n, x_m\right)\right| < \frac{\varepsilon_0}{3}. \tag{3.18}$$

From (3.17)–(3.18), we get

$$\left|g(t_0, x_0) - g(t_m, x_m)\right| < \varepsilon_0, \tag{3.19}$$

this contradicts (3.16). Therefore, $g(t,x)$ is continuous on $\mathbb{T} \times D$.

Finally, for any compact set $S \subset D$ and given $\varepsilon > 0$, one can take $\tau \in E\{\varepsilon, f, S$; then, for all $(t,x) \in \mathbb{T} \times S$, the following holds:

$$\left|f\left(t + \beta_n + \tau, x\right) - f\left(t + \beta_n, x\right)\right| < \varepsilon. \tag{3.20}$$

Let $n \to +\infty$, for all $(t,x) \in \mathbb{T} \times S$; we have

$$\left|g(t + \tau, x) - g(t,x)\right| \leq \varepsilon, \tag{3.21}$$

which implies that $E\{\varepsilon, g, S\}$ is relatively dense. Therefore, $g(t,x)$ is almost periodic in t uniformly for $x \in D$. This completes the proof. □

Theorem 3.14. *Let $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$; if, for any sequence $\alpha' \subset \Pi$, there exists $\alpha \subset \alpha'$ such that $T_\alpha f(t,x)$ exists uniformly on $\mathbb{T} \times S$, then $f(t,x)$ is almost periodic in t uniformly for $x \in D$.*

Proof. For contradiction, if this is not true, then there exist $\varepsilon_0 > 0$ and $S \subset D$ such that, for any sufficiently large $l > 0$, we can find an interval with length of l and there is no ε_0-translation numbers of $f(t,x)$ in this interval; that is, every point in this interval is not in $E\{\varepsilon_0, f, S\}$.

One can take a number $\alpha'_1 \in \Pi$ and find an interval (a_1, b_1) with $b_1 - a_1 > 2|\alpha'_1|$, where $a_1, b_1 \in \Pi$ such that there is no ε_0-translation numbers of $f(t,x)$ in this interval. Next, taking $\alpha'_2 = (1/2)(a_1 + b_1)$, obviously, $\alpha'_2 - \alpha'_1 \in (a_1, b_1)$, so $\alpha'_2 - \alpha'_1 \notin E\{\varepsilon_0, f, S\}$; then, one can find an interval (a_2, b_2) with $b_2 - a_2 > 2(|\alpha'_1| + |\alpha'_2|)$, where $a_2, b_2 \in \Pi$ such that there is no ε_0-translation numbers of $f(t,x)$ in this interval. Next, taking $\alpha'_3 = (1/2)(a_2 + b_2)$, obviously, $\alpha'_3 - \alpha'_2,\ \alpha'_3 - \alpha'_1 \notin E\{\varepsilon_0, f, S\}$. One can repeat these processes, again and again one can find $\alpha'_4,\ \alpha'_5,\ \ldots$, such that $\alpha'_i - \alpha'_j \notin E\{\varepsilon_0, f, S\},\ i > j$. Hence, for any $i \neq j,\ i, j = 1, 2, \ldots$, without loss of generality, let $i > j$, for $x \in S$; we have

$$\sup_{(t,x)\in\mathbb{T}\times S}\left|f\left(t + \alpha'_i, x\right) - f\left(t + \alpha'_j, x\right)\right| = \sup_{(t,x)\in\mathbb{T}\times S}\left|f\left(t + \alpha'_i - \alpha'_j, x\right) - f(t,x)\right| \geq \varepsilon_0. \tag{3.22}$$

Therefore, there is no uniformly convergent subsequence of $\{f(t+\alpha'_n, x)\}$ for $(t, x) \in \mathbb{T}\times S$; this is a contradiction. Thus, $f(t, x)$ is almost periodic in t uniformly for $x \in D$. This completes the proof. □

From Theorems 3.13 and 3.14, we can obtain the following equivalent definition of uniformly almost periodic functions.

Definition 3.15. Let $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$; if, for any given sequence $\alpha' \subset \Pi$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_\alpha f(t, x)$ exists uniformly on $\mathbb{T} \times S$, then $f(t, x)$ is called an almost periodic function in t uniformly for $x \in D$.

Similar to the proof of Theorem 2.11 in [16], one can prove that the following.

Theorem 3.16. *If $f \in C(\mathbb{T}\times D, \mathbb{E}^n)$ is almost periodic in t uniformly for $x \in D$ and $\varphi(t)$ is almost periodic with $\{\varphi(t) : t \in \mathbb{T}\} \subset S$, then $f(t, \varphi(t))$ is almost periodic.*

Definition 3.17. Let $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$; $H(f) = \{g : \mathbb{T} \times D \to \mathbb{E}^n \mid$ there exits $\alpha \in \Pi$ such that $T_\alpha f(t, x) = g(t, x)$ exists uniformly on $\mathbb{T} \times S\}$ is called the hull of f.

Similar to the proofs of Theorems 1.6 and 1.8 in [19], one can prove the following two theorems, respectively.

Theorem 3.18. *$H(f)$ is compact if and only if f is almost periodic in t uniformly for $x \in D$.*

Theorem 3.19. *If $f(t, x)$ is almost periodic in t uniformly for $x \in D$, then for any $g(t, x) \in H(f)$, $H(f) = H(g)$.*

From Definition 3.17 and Theorem 3.19, one can easily show that.

Theorem 3.20. *If $f(t, x)$ is almost periodic in t uniformly for $x \in D$, then, for any $g(t, x) \in H(f)$, $g(t, x)$ is almost periodic in t uniformly for $x \in D$.*

Theorem 3.21. *If $f(t, x)$ is almost periodic in t uniformly for $x \in D$, then, for any $\varepsilon > 0$, there exists a positive constant $L = L(\varepsilon, S)$ and for any $a \in \mathbb{R}$, there exist a constant $\eta > 0$ and $\alpha \in \mathbb{R}$ such that $([\alpha, \alpha + \eta] \cap \Pi) \subset [a, a + L]$ and $([\alpha, \alpha + \eta] \cap \Pi) \subset E(\varepsilon, f, S)$.*

Proof. Since $f(t, x)$ is uniformly continuous on $\mathbb{T} \times S$, for any $\varepsilon > 0$, there exists $\delta(\varepsilon_1, S) > 0$ so that $|t_1 - t_2| < \delta(\varepsilon_1, S)$ implies

$$\left|f(t_1, x) - f(t_2, x)\right| < \varepsilon_1, \quad \forall x \in S, \tag{3.23}$$

where $\varepsilon_1 = \varepsilon/2$.

We take $\eta = \delta(\varepsilon/2, S) = \delta(\varepsilon_1, S)$, and $L = l(\varepsilon_1, S) + \eta$, where $l(\varepsilon_1, S)$ is the inclusion length of $E(\varepsilon_1, f, S)$.

For any $a \in \mathbb{R}$, consider an interval $[a, a + L]$, take

$$\tau \in E(f, \varepsilon_1, S) \cap \left[a - \frac{\eta}{2}, a + \frac{\eta}{2} + l(\varepsilon_1, S)\right], \tag{3.24}$$

and we have

$$\left(\left[\tau-\frac{\eta}{2},\tau+\frac{\eta}{2}\right]\cap\Pi\right)\subset[a,a+L]. \tag{3.25}$$

Hence, for all $\xi\in[\tau-(\eta/2),\tau+(\eta/2)]\cap\Pi$, we have $|\xi-\tau|\leq\eta$. Therefore, for any $(t,x)\in\mathbb{T}\times S$,

$$\begin{aligned}\left|f(t+\xi,x)-f(t,x)\right|&\leq\left|f(t+\xi,x)-f(t+\tau,x)\right|+\left|f(t+\tau,x)-f(t,x)\right|\\&\leq\varepsilon.\end{aligned} \tag{3.26}$$

So, we let $\alpha=\tau-(\eta/2)$, then $([\alpha,\alpha+\eta]\cap\Pi)\subset E(\varepsilon,f,S)$. This completes the proof. □

Theorem 3.22. *If f,g are almost periodic in t uniformly for $x\in D$, then, for any $\varepsilon>0$, $E(f,\varepsilon,S)\cap E(g,\varepsilon,S)$ is a nonempty relatively dense set in $\mathbb{T}$.*

Proof. Since f,g are almost periodic in t uniformly for $x\in D$, they are uniformly continuous on $\mathbb{T}\times S$. For any given $\varepsilon>0$, one can take $\delta_i=\delta_i(\varepsilon/2,S)$ $(i=1,2)$, and $l_1=l_1(\varepsilon/2,S)$, $l_2=l_2(\varepsilon/2,S)$ are inclusion lengths of $E(f,\varepsilon/2,S),E(g,\varepsilon/2,S)$, respectively.

According to Theorem 3.18, we can take

$$\eta=\eta(\varepsilon,S)=\min(\delta_1,\delta_2)\in\Pi,\qquad L_i=l_i+\eta(i=1,2),\qquad L=\max(L_1,L_2). \tag{3.27}$$

Hence, we can find $\varepsilon/2$-translation numbers of $f(t,x)$ and $g(t,x)$: $\tau_1=m\eta$ and $\tau_2=n\eta$, respectively, where $\tau_1,\tau_2\in[a,a+L]\cap\Pi$, m,n are integers, and $|\tau_1-\tau_2|\leq L$.

Let $m-n=s$, then s can only be taken from a finite number set $\{s_1,s_2,\ldots,s_p\}$. When $m-n=s_j, j=1,2,\ldots,p$, denote the $\varepsilon/2$-translation numbers of $f(t)$ and $g(t)$ by τ_1^j, τ_2^j, respectively, that is, $\tau_1^j-\tau_2^j=s_j\eta$, $j=1,2,\ldots,p$, and we take $T=\max_j\{|\tau_1^j|,|\tau_2^j|\}$.

For any $a\in\mathbb{R}$, on the interval $[a+T,a+T+L]$, we can take $\varepsilon/2$-translation numbers of $f(t,x)$ and $g(t,x)$: τ_1 and τ_2, respectively; there must exist some integer s_j such that

$$\tau_1-\tau_2=s_j\eta=\tau_1^j-\tau_2^j. \tag{3.28}$$

Set

$$\tau(\varepsilon,S)=\tau_1-\tau_1^j=\tau_2-\tau_2^j, \tag{3.29}$$

then $\tau(\varepsilon,S)\in[a,a+L+2T]\cap\Pi$, and, for any $(t,x)\in\mathbb{T}\times S$, we have

$$\begin{aligned}\left|f(t+\tau,x)-f(t,x)\right|&\leq\left|f\left(t+\tau_1-\tau_1^j,x\right)-f\left(t-\tau_1^j,x\right)\right|+\left|f\left(t-\tau_1^j,x\right)-f(t,x)\right|<\varepsilon,\\\left|g(t+\tau,x)-g(t,x)\right|&\leq\left|g\left(t+\tau_2-\tau_2^j,x\right)-g\left(t-\tau_2^j,x\right)\right|+\left|g\left(t-\tau_2^j,x\right)-g(t,x)\right|<\varepsilon.\end{aligned} \tag{3.30}$$

Therefore, there exists at least a $\tau=\tau(\varepsilon,S)$ on any interval $[a,a+L+2T]$ with the length $(L+2T)$ such that $\tau\in E(f,\varepsilon,S)\cap E(g,\varepsilon,S)$. The proof is complete. □

According to Definition 3.10, one can easily prove the following.

Theorem 3.23. *If $f(t,x)$ is almost periodic in t uniformly for $x \in D$, then, for any $\alpha \in \mathbb{R}, b \in \Pi$, functions $\alpha f(t,x), f(t+b,x)$ are almost periodic in t uniformly for $x \in D$.*

Theorem 3.24. *If f, g are almost periodic in t uniformly for $x \in D$, then $f+g, fg$ are almost periodic in t uniformly for $x \in D$ and if $\inf_{t\in\mathbb{T}}|g(t,x)| > 0$, then $f(t,x)/g(t,x)$ are almost periodic in t uniformly for $x \in D$.*

Proof. From Theorem 3.22, for any $\varepsilon > 0$, $E(f,\varepsilon/2,S) \cap E(g,\varepsilon/2,S)$ is a nonempty relatively dense set. It is easy to see that if $\tau \in E(f,\varepsilon/2,S) \cap E(g,\varepsilon/2,S)$, then $\tau \in E(f+g,\varepsilon,S)$. Hence,

$$\left(E\left(f,\frac{\varepsilon}{2},S\right) \cap E\left(g,\frac{\varepsilon}{2},S\right)\right) \subset E(f+g,\varepsilon,S). \tag{3.31}$$

Therefore, $E(f+g,\varepsilon,S)$ is a relatively dense set, so $f+g$ is almost periodic in t uniformly for $x \in D$.

On the other hand, denote $\sup_{(t,x)\in\mathbb{T}\times S}|f(t,x)| = M_1$, $\sup_{(t,x)\in\mathbb{T}\times S}|g(t,x)| = M_2$, take $\tau \in E(f,\varepsilon/2,S) \bigcap E(g,\varepsilon/2,S)$, then, for all $(t,x) \in \mathbb{T}\times S$, we have

$$\begin{aligned}
&\left|f(t+\tau,x)g(t+\tau,x) - f(t,x)g(t,x)\right| \\
&\quad\le \left|g(t+\tau,x)\right|\left|f(t+\tau,x) - f(t,x)\right| + \left|f(t,x)\right|\left|g(t+\tau,x) - g(t,x)\right| \\
&\quad\le (M_1+M_2)\varepsilon \equiv \varepsilon_1.
\end{aligned} \tag{3.32}$$

Therefore, $\tau \in E(fg,\varepsilon_1,S), E(fg,\varepsilon_1,S)$ is a relatively dense set, so fg is almost periodic in t uniformly for $x \in D$.

Faunally, denote $\inf_{(t,x)\in\mathbb{T}\times S}|g(t,x)| = N$ and take $\tau \in E(g,\varepsilon,S)$, then, for all $(t,x) \in \mathbb{T} \times S$ we have

$$\left|\frac{1}{g(t+\tau,x)} - \frac{1}{g(t,x)}\right| = \left|\frac{g(t+\tau,x) - g(t,x)}{g(t+\tau,x)g(t,x)}\right| < \frac{\varepsilon}{N^2} \equiv \varepsilon_2, \tag{3.33}$$

that is, $\tau \in E(1/g,\varepsilon_2,S)$. Hence, $1/g$ is almost periodic in t uniformly for $x \in D$, so f/g is almost periodic in t uniformly for $x \in D$. The proof is complete. □

Theorem 3.25. *If $F \in C(\mathbb{R}\times D, \mathbb{E}^n)$ is almost periodic in t uniformly for $x \in D$, then $F(t,x)$ is also continuous on $\mathbb{T}\times D$ and almost periodic in t uniformly for $x \in D$.*

Proof. Let $F(r,x) \in C(\mathbb{R}\times D, \mathbb{E}^n)$ be uniformly almost periodic, then, for any sequence $\alpha' \subset \mathbb{T}_p$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_\alpha F(t+\alpha_n,x)$ exists uniformly on $\mathbb{R}\times S$, where S is any compact set in D. Consequently, $T_\alpha f(t+\alpha_n,x) = T_\alpha F(t+\alpha_n,x)$ exists uniformly on $\mathbb{T}\times S$. In view of Theorem 3.14, this shows that $f(t,x)$ is uniformly almost periodic. □

Corollary 3.26. *If $F \in C(\mathbb{R}, \mathbb{E}^n)$ is an almost periodic function, then $F(t)$ is an almost periodic function on $\mathbb{T}$.*

Theorem 3.27. *If $f_n \in C(\mathbb{T}\times D, \mathbb{E}^n), n = 1,2,\ldots$ are almost periodic in t for $x \in D$ and the sequence $\{f_n(t,x)\}$ uniformly converges to $f(t,x)$ on $\mathbb{T}\times S$, then $f(t,x)$ is almost periodic in t uniformly for $x \in D$.*

Proof. For any $\varepsilon > 0$, there exists sufficiently large n_0 such that, for all $(t,x) \in \mathbb{T}\times S$,

$$\left|f(t,x) - f_{n_0}(t,x)\right| < \frac{\varepsilon}{3}. \tag{3.34}$$

Take $\tau \in E\{f_{n_0}, \varepsilon/3, S\}$, then, for all $(t,x) \in \mathbb{T}\times S$, we have

$$\begin{aligned} \left|f(t+\tau,x) - f(t,x)\right| &\le \left|f(t+\tau,x) - f_{n_0}(t+\tau,x)\right| + \left|f_{n_0}(t+\tau,x) - f_{n_0}(t,x)\right| \\ &\quad + \left|f_{n_0}(t,x) - f(t,x)\right| < \varepsilon, \end{aligned} \tag{3.35}$$

that is, $\tau \in E(f,\varepsilon,S)$. Therefore, $E(f,\varepsilon,S)$ is also a relatively dense set; $f(t,x)$ is almost periodic in t uniformly for $x \in D$. This completes the proof. □

Theorem 3.28. *If $f(t,x)$ is almost periodic in t uniformly for $x \in D$, denote $F(t,x) = \int_0^t f(s,x)\Delta s$, then $F(t,x)$ is almost periodic in t uniformly for $x \in D$ if and only if $F(t,x)$ is bounded on $\mathbb{T}\times S$.*

Proof. If $F(t,x)$ is almost periodic in t uniformly for $x \in D$, then it is easy to see that $F(t,x)$ is bounded on $\mathbb{T}\times S$.

If $F(t,x)$ is bounded, without loss of generality, then we can assume that $F(t,x)$ is a real-valued function. Denote

$$G := \sup_{(t,x)\in\mathbb{T}\times S} F(t,x) > g := \inf_{(t,x)\in\mathbb{T}\times S} F(t,x), \tag{3.36}$$

for any $\varepsilon > 0$, there exist t_1 and t_2 such that

$$F(t_1,x) < g + \frac{\varepsilon}{6}, \quad F(t_2,x) > G - \frac{\varepsilon}{6}, \quad \forall x \in S. \tag{3.37}$$

Let $l = l(\varepsilon_1, S)$ be an inclusion length of $E(f,\varepsilon_1,S)$, where $\varepsilon_1 = \varepsilon/6d, d = |t_1 - t_2|$. For any $\alpha \in \mathbb{T}$, take $\tau \in E(f,\varepsilon_1,S) \cap [\alpha - t_1, \alpha - t_1 + l]$. Denote $s_i = t_i + \tau, (i = 1,2), L = l + d$, so $s_1, s_2 \in [\alpha, \alpha + L] \cap \mathbb{T}$, for all $x \in s$,

$$\begin{aligned} F(s_2,x) - F(s_1,x) &= F(t_2,x) - F(t_1,x) - \int_{t_1}^{t_2} f(t,x)\Delta t + \int_{t_1+\tau}^{t_2+\tau} f(t,x)\Delta t \\ &= F(t_2,x) - F(t_1,x) + \int_{t_1}^{t_2} \left[f(t+\tau,x) - f(t,x)\right]\Delta t \\ &> G - g - \frac{\varepsilon}{3} - \varepsilon_1 d = G - g - \frac{\varepsilon}{2}, \end{aligned} \tag{3.38}$$

that is

$$\left(F(s_1, x) - g\right) + (G - F(s_2, x)) < \frac{\varepsilon}{2}. \tag{3.39}$$

Since

$$F(s_1, x) - g \geq 0, \qquad G - F(s_2, x) \geq 0, \tag{3.40}$$

in any interval with length L, there exist s_1, s_2 such that

$$F(s_1, x) < g + \frac{\varepsilon}{2}, \qquad F(s_2, x) > G - \frac{\varepsilon}{2}. \tag{3.41}$$

Now, we denote $\varepsilon_2 = \varepsilon/2L$; in the following, we will prove that if $\tau \in E(f, \varepsilon_2, S)$, then $\tau \in E(F, \varepsilon, S)$. In fact, for any $(t, x) \in \mathbb{T} \times S$, one can take $s_1, s_2 \in [t, t + L] \cap \mathbb{T}$ such that

$$F(s_1, x) < g + \frac{\varepsilon}{2}, \qquad F(s_2, x) > G - \frac{\varepsilon}{2}, \tag{3.42}$$

so for $\tau \in E(f, \varepsilon_2, S)$, we have

$$\begin{aligned} F(t+\tau, x) - F(t, x) &= F(s_1 + \tau, x) - F(s_1, x) + \int_t^{s_1} f(t, x)\Delta t - \int_{t+\tau}^{s_1+\tau} f(t, x)\Delta t \\ &> g - \left(g + \frac{\varepsilon}{2}\right) - \int_t^{t_1} \left[f(t+\tau, x) - f(t, x)\right]\Delta t \\ &> -\frac{\varepsilon}{2} - \varepsilon_2 L = -\varepsilon, \\ F(t+\tau, x) - F(t, x) &= F(s_2 + \tau, x) - F(s_2, x) + \int_t^{s_2} f(t, x)\Delta t - \int_{t+\tau}^{s_2+\tau} f(t, x)\Delta t \\ &< \frac{\varepsilon}{2} + \varepsilon_2 L = \varepsilon. \end{aligned} \tag{3.43}$$

That is, for $\tau \in E(f, \varepsilon_2, S)$, we have $\tau \in E(F, \varepsilon, S)$, so $F(t, x)$ is almost periodic in t uniformly for $x \in D$. The proof is complete. □

Theorem 3.29. *If $f(t, x)$ is almost periodic in t uniformly for $x \in D$ and $F(\cdot)$ is uniformly continuous on the value field of $f(t, x)$, then $F \circ f$ is almost periodic in t uniformly for $x \in D$.*

Proof. In fact, since F is uniformly continuous on the value field of $f(t, x)$ and $f(t, x)$ is almost periodic in t uniformly for $x \in D$, there exists a real sequence $\alpha = \{\alpha_n\} \subseteq \Pi$ such that

$$T_\alpha(F \circ f) = \lim_{n \to +\infty} F(f(t + \alpha_n, x)) = F\left(\lim_{n \to +\infty} f(t + \alpha_n, x)\right) = F(T_\alpha f) \tag{3.44}$$

holds uniformly on $\mathbb{T} \times S$. Hence, $F \circ f$ is almost periodic in t uniformly for $x \in D$. The proof is complete. □

Similar to the proof of Theorem 1.17 in [19], one can easily get the following.

Theorem 3.30. *A function $f(t,x)$ is almost periodic in t uniformly for $x \in D$ if and only if, for every pair of sequences α', $\beta' \subseteq \Pi$, there exist common subsequences $\alpha \subset \alpha'$, $\beta \subset \beta'$ such that*

$$T_{\alpha+\beta} f(t,x) = T_\alpha T_\beta f(t,x). \tag{3.45}$$

Definition 3.31. If every element of matrix function $M(t,x) = (m_{ij}(t,x))_{n\times m}$, where $m_{ij}(t,x) \in C(\mathbb{T},\mathbb{E})(1,2,\ldots,n; j = 1,2,\ldots,m)$ is almost periodic in t uniformly for $x \in D$, then $M(t,x)$ is called almost periodic in t uniformly for $x \in D$.

If we use matrix norm: $|M(t,x)| = \sqrt{\sum_{ij} m_{ij}^2(t,x)}$, then the definition above can be rewritten as.

Definition 3.32. A matrix function $M(t,x)$ is almost periodic in t uniformly for $x \in D$ if and only if, for any $\varepsilon > 0$, the translation set

$$E(M,\varepsilon,S) = \{\tau \in \Pi : |M(t+\tau,x) - M(t,x)| < \varepsilon,\ \forall (t,x) \in \mathbb{T} \times S\} \tag{3.46}$$

is a relatively dense set.

Theorem 3.33. *Definition 3.31 is equivalent to Definition 3.32.*

Proof. In fact, if $M(t,x)$ is almost periodic in t uniformly for $x \in D$, by Definition 3.31, then every element $m_{ij}(t,x)$ is almost periodic in t uniformly for $x \in D$. Thus, for any $\varepsilon > 0$, there exists nonempty relatively dense set $\theta = \bigcap_{i,j} E(m_{ij}(t,x), \varepsilon/\sqrt{mn}, S)$. For any $\tau \in \theta$, we have

$$|M(t+\tau,x) - M(t,x)| = \left[\sum_{i,j} \left|m_{ij}(t+\tau,x) - m_{ij}(t,x)\right|^2\right]^{1/2} < \varepsilon. \tag{3.47}$$

On the other hand, if, for any $\varepsilon > 0$, $E(M,\varepsilon,S)$ is a relatively dense set, then, for any $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$ and $\tau \in E(M,\varepsilon,S)$, we have

$$\left|m_{ij}(t+\tau,x) - m_{ij}(t,x)\right| < |M(t+\tau,x) - M(t,x)| < \varepsilon, \quad \forall (t,x) \in \mathbb{T} \times S. \tag{3.48}$$

Hence, every element $m_{ij}(t,x)$ is almost periodic in t uniformly for $x \in D$ that is, $M(t,x)$ is almost periodic in t uniformly for $x \in D$. The proof is complete. □

Definition 3.34. A continuous matrix function $M(t,x)$ is called normal if, for any sequence $\alpha' \subseteq \Pi$, thenthere exists subsequence $\alpha \subset \alpha'$ such that $T_\alpha M(t,x)$ exists uniformly on $\mathbb{T} \times S$.

Theorem 3.35. *A continuous matrix function $M(t,x)$ is normal if and only if $M(t,x)$ is almost periodic in t uniformly for $x \in D$.*

Proof. If $M(t,x)$ is normal, then every element $m_{ij}(t,x)$ satisfies Definition 3.15, so $M(t)$ is almost periodic.

On the other hand, if $M(t,x)$ is almost periodic in t uniformly for $x \in D$, by Definition 3.31, for any sequence $\alpha' \subseteq \Pi$, thenthere exists subsequence $\alpha_1 \subset \alpha'$ such that $T_{\alpha_1} m_{11}(t,x)$ exists uniformly on $\mathbb{T} \times S$. Hence, there exists $\alpha_2 \subset \alpha_1$, such that $T_{\alpha_2} m_{12}(t,x)$ exists uniformly on $\mathbb{T} \times S$; we can repeat this step mn times, then we can get a series of subsequences satisfying

$$\alpha = \{\alpha_k\} = \alpha_{mn} \subset \alpha_{mn-1} \subset \ldots \subset \alpha_2 \subset \alpha_1 \subset \alpha' \tag{3.49}$$

such that

$$T_\alpha m_{ij}(t,x), \quad i = 1,2,\ldots,n, \ j = 1,2,\ldots,m \tag{3.50}$$

exist uniformly on $\mathbb{T} \times S$. Therefore, there exists subsequence $\alpha \subset \alpha'$ such that $T_\alpha M(t,x)$ exists uniformly on $\mathbb{T} \times S$; that is, $M(t,x)$ is normal. The proof is complete. □

4. Almost Periodic Dynamic Equations on Almost Periodic Time Scales

Consider the following nonlinear dynamic equation

$$x^\Delta = f(t,x), \tag{4.1}$$

where $f \in C(\mathbb{T} \times \mathbb{E}^n, \mathbb{E}^n)$; let $\Omega = \{x(t) : x(t) \text{ is a bounded solution to (4.1)}\}$.

Definition 4.1. If $\Omega \neq \phi$, then $\lambda = \inf_{x\in\Omega}\|x\|$ exists λ is called the least-value of solutions to (4.1). If there exists $\varphi(t) \in \Omega$ such that $\|\varphi\| = \lambda$, then $\varphi(t)$ is called a minimum norm solution to (4.1), where $\|\cdot\| = \sup_{t\in\mathbb{T}}|\cdot|$.

Similar to the proof of Theorem 5.1 in [19], one can easily get the following.

Lemma 4.2. *If $f(t,x) \in C(\mathbb{T}\times S, \mathbb{E}^n)$ is bounded on $\mathbb{T}\times S$ and* (4.1) *has a bounded solution $\varphi(t)$ such that $\{\varphi(t),\, t \in \mathbb{T}\} \subset S$ and $\mathbf{0} \in S$, then* (4.1) *must have a minimum norm solution.*

Lemma 4.3. *If $f(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{E}^n$, $S = \overline{\{\varphi(t) : t \geq t_0\}}$ and* (4.1) *has a bounded solution $\varphi(t)$ on $[t_0,\infty) \cap \mathbb{T}$, then* (4.1) *has a bounded solution $\psi(t)$ on $\mathbb{T}$ and $\{\psi(t),\, t \in \mathbb{T}\} \subset S$.*

Proof. In fact, we may take $\alpha' = \{\alpha'_k\} \subset \Pi$ such that $\lim_{n\to+\infty} \alpha'_k = +\infty$ and $T_{\alpha'} f(t,x) = f(t,x)$ holds uniformly on $\mathbb{T}\times S$. For any fixed a, consider the interval $(a,\infty)\cap\mathbb{T}$ and $\varphi_k(t) = \varphi(t+\alpha'_k)$. It is easy to see that, for k sufficiently large, $\{\varphi_k\}$ is defined on $(a,\infty) \cap \mathbb{T}$ and is a solution to $x^\Delta = f(t + \alpha'_k, x)$ and $\{\varphi_k(t)\}$ is uniformly bounded and equicontinuous on $(a,\infty) \cap \mathbb{T}$. Then let α be a sequence which goes to $-\infty$, according to Lemma 2.3, there must exist $\alpha \subset \alpha'$ such that $T_\alpha \varphi(t) = \psi(t)$ holds uniformly on any compact subset of $\mathbb{T}$ and, for all $t \in \mathbb{T}$, we have $\psi(t) \in S$. Since $T_\alpha f(t,x) = f(t,x)$, by Lemma 3.5, $\psi(t)$ is a solution to (4.1). This completes the proof. □

Lemma 4.4. *Let* $f(t,x) \in C(\mathbb{T} \times \mathbb{E}^n, \mathbb{E}^n)$ *be almost periodic in* t *uniformly for* $x \in \mathbb{E}^n$. *If* (4.1) *has a minimum norm solution, then, for any* $g(t,x) \in H(f)$, *the following equation:*

$$x^{\Delta} = g(t,x) \tag{4.2}$$

has the same least-value of solutions as that to (4.1).

Proof. Let $\varphi(t)$ be the minimum norm solution to (4.1) and λ is the least-value. Since $g(t,x) \in H(f)$, there exists a sequence $\alpha' \in \Pi$ such that $T_{\alpha'} f(t,x) = g(t,x)$ holds uniformly on $\mathbb{T} \times S$. From Lemma 2.3, there exists $\alpha \subset \alpha'$ such that $T_\alpha \varphi(t) = \psi(t)$ holds uniformly on any compact subset of $\mathbb{T}$. By Lemma 3.5, $\psi(t)$ is a solution to (4.2). For $|\varphi(t)| \le \lambda$, we have $|\psi(t)| \le \lambda$; thus, $\lambda' = \|\psi(t)\| \le \lambda$. Since $\varphi(t) = T_{-\alpha}\psi(t)$ and $|\psi(t)| \le \lambda'$, we have $|\varphi(t)| \le \lambda'$; thus, $\lambda = \|\varphi(t)\| \le \lambda'$. Therefore, $\lambda = \lambda'$. The proof is complete. □

From the process of the proof of Lemma 4.4, one can easily get the following.

Lemma 4.5. *If* $\varphi(t)$ *is a minimum norm solution to* (4.1) *and there exists a sequence* $\alpha' \subseteq \Pi$ *such that* $T_{\alpha'} f(t,x) = g(t,x)$ *exists uniformly on* $\mathbb{T} \times S$, *furthermore, if there exists a subsequence* $\alpha \subset \alpha'$ *such that* $T_\alpha \varphi(t) = \psi(t)$ *holds uniformly on any compact set of* $\mathbb{T}$, *then* $\psi(t)$ *is a minimum norm solution to* (4.2).

Lemma 4.6. *Let* $f(t,x) \in C(\mathbb{T} \times \mathbb{E}^n, \mathbb{E}^n)$ *be almost periodic in* t *uniformly for* $x \in \mathbb{E}^n$ *and, for every* $g(t,x) \in H(f)$, (4.2) *has a unique minimum norm solution; then theses minimum norm solutions are almost periodic on* $\mathbb{T}$.

Proof. For a fixed $g(t,x) \in H(f)$, (4.2) has the unique minimum norm solution $\psi(t)$. Since $g(t,x)$ is almost periodic in t uniformly for $x \in \mathbb{E}^n$, we have that for any sequences $\alpha', \beta' \subseteq \Pi$, there exist common subsequences $\alpha \subset \alpha'$, $\beta \subset \beta'$ such that $T_{\alpha+\beta} g(t,x) = T_\alpha T_\beta g(t,x)$ holds uniformly on $\mathbb{T} \times S$ and $T_\alpha T_\beta \psi(t)$, $T_{\alpha+\beta}\psi(t)$ hold uniformly on $\mathbb{T}$. It follows from Lemmas 4.4 and 4.5 that $T_\alpha T_\beta \psi(t)$ and $T_{\alpha+\beta}\psi(t)$ are minimum norm solutions to the following equation:

$$x^{\Delta} = T_{\alpha+\beta} g(t,x). \tag{4.3}$$

Since the minimum norm solution is unique, we have $T_\alpha T_\beta \psi(t) = T_{\alpha+\beta}\psi(t)$. Therefore, $\psi(t)$ is almost periodic. The proof is complete. □

We will now discuss the linear almost periodic dynamic equation on an almost periodic time scale $\mathbb{T}$:

$$x^{\Delta} = A(t)x + f(t) \tag{4.4}$$

and its associated homogeneous equation

$$x^{\Delta} = A(t)x, \tag{4.5}$$

where $A(t)$ is an almost periodic matrix function and $f(t)$ is an almost periodic vector function.

Definition 4.7. If $B \in H(A)$, thenwe say that

$$y^{\Delta} = B(t)y \tag{4.6}$$

is a homogeneous equation in the hull of (4.4).

Definition 4.8. If $B \in H(A)$ and $g \in H(f)$, then we say that

$$y^{\Delta} = B(t)y + g(t) \tag{4.7}$$

is an equation in the hull of (4.4).

Definition 4.9 (see [31]). Let $A(t)$ be $n \times n$ rd-continuous matrix function on $\mathbb{T}$; the linear system

$$x^{\Delta}(t) = A(t)x(t) \tag{4.8}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants K, α, projection P, and the fundamental solution matrix $X(t)$ of (4.8), satisfying

$$\begin{gathered}\left|X(t)PX^{-1}(s)\right| \le Ke_{\ominus\alpha}(t,s), \quad s,t \in \mathbb{T},\ t \ge s, \\ \left|X(t)(I-P)X^{-1}(s)\right| \le Ke_{\ominus\alpha}(s,t), \quad s,t \in \mathbb{T},\ t \le s.\end{gathered} \tag{4.9}$$

Similar to the proof of Theorem 5.7 (Favard's Theorem) in [19], one can obtain that the following.

Lemma 4.10. *If $A(t)$ is an almost periodic matrix function and $x(t)$ is an almost periodic solution of the homogeneous linear almost periodic dynamic equation $x^{\Delta} = A(t)x$, then $\inf_{t\in\mathbb{T}}|x(t)| > 0$ or $x(t) \equiv 0$.*

Similar to the proof of Theorems 6.3 and 5.8 in [19], one can easily get.

Lemma 4.11. *Suppose that* (4.5) *has an almost periodic solution $x(t)$ and $\inf_{t\in\mathbb{T}}|x(t)| > 0$. If* (4.4) *has bounded solution on $[t_0,\infty) \cap \mathbb{T}$, then* (4.4) *has an almost periodic solution.*

Lemma 4.12. *If every bounded solution of a homogeneous equation in the hull of* (4.4) *is almost periodic, then all bounded solutions of* (4.4) *are almost periodic.*

Proof. According to Lemma 4.10, we know that every nontrivial bounded solution of equations in the hull of (4.4) satisfies $\inf_{t\in\mathbb{T}}|x(t)| > 0$. From Lemma 4.11, it follows that if (4.4) has bounded solutions on $\mathbb{T}$, then (4.4) must have an almost periodic solution $\psi(t)$. If $\varphi(t)$ is an arbitrary bounded solution of (4.4), then $\eta(t) = \psi(t) - \varphi(t)$ is a bounded solution of its associated homogeneous equation (4.5) and it is almost periodic. Thus, $\varphi(t)$ is almost periodic. This completes the proof. □

Lemma 4.13. *If a homogeneous equation in the hull of* (4.4) *has the unique bounded solution $x(t) \equiv 0$, then* (4.4) *has a unique almost periodic solution.*

Proof. Let $\psi(t)$, and $\varphi(t)$ be two bounded solutions to (4.4), then $x(t) = \varphi(t) - \psi(t)$ is a solution of a homogeneous equation in the hull of (4.4), since $x(t) \equiv 0$, we have that $\varphi(t) \equiv \psi(t)$. Thus, by Lemma 4.12, (4.4) has a unique almost periodic solution. This completes the proof. □

Similar to the proof of Lemma 7.4 in [16], one can easily prove that the following.

Lemma 4.14. *Let P be a projection and X a differentiable invertible matrix such that XPX^{-1} is bounded on $\mathbb{T}$. Then, there exists a differentiable matrix S such that $XPX^{-1} = SPS^{-1}$ for all $t \in \mathbb{T}$ and S, S^{-1} are bounded on $\mathbb{T}$. In fact, there is an S of the form $S = XQ^{-1}$, where Q commutes with P.*

Similar to the proof of Lemma 7.5, in [19], one can easily get.

Lemma 4.15. *If* (4.5) *has an exponential dichotomy and $X(t)$ is the fundamental solution matrix of* (4.5), *C non-singular, then $X(t)C$ has an exponential dichotomy with the same projection P if and only if $CP = PC$.*

Similar to the proof of Theorem 7.6 in [19], we can easily obtain.

Lemma 4.16. *Suppose that $A(t)$ is an almost periodic matrix function and* (4.5) *has an exponential dichotomy, then for every $B(t) \in H(A)$,* (4.6) *has an exponential dichotomy with the same projection P and the same constants K, α.*

Lemma 4.17. *If the homogeneous equation* (4.5) *has an exponential dichotomy, then* (4.5) *has only one bounded solution $x(t) \equiv 0$.*

Proof. Let $X(t)$ be the fundamental solution matrix to (4.5). For any sequence $\alpha \subset \Pi$, denote $A_n = A(t+\alpha_n), X_n(t) = X(t+\alpha_n)$. Since the homogeneous equation (4.5) has an exponential dichotomy, it is easy to see that there exists a constant M such that $\|X_n(t)\| \leq M$ and $\|X^{\Delta}(t)\| = \|A_n(t)X_n(t)\| \leq \overline{A}M$, where $\overline{A} = \sup_{t\in\mathbb{T}}\|A(t)\|$. Therefore, by Lemma 2.3, there exists $\{\alpha_{n_k}\} := \alpha' \subset \alpha$ such that $\{X_{n_k}\}$ converges uniformly on any compact subset of $\mathbb{T}$ and $\lim_{n\to+\infty}X(t+\alpha_n)$ exists uniformly on $\mathbb{T}$. So, $X(t)$ is almost periodic. Since the homogeneous equation (4.5) has an exponential dichotomy, $\inf_{t\in\mathbb{T}}x(t) = 0$, from Lemma 4.10, $x(t) \equiv 0$. This completes the proof. □

Lemma 4.18. *If the homogeneous equation* (4.5) *has an exponential dichotomy, then all equations in the hull of* (4.5) *have only one bounded solution $x(t) \equiv 0$.*

Proof. By Lemma 4.16, all equations in the hull of (4.5) have an exponential dichotomy; according to Lemma 4.17, all equations in the hull of (4.5) have only one bounded solution $x(t) \equiv 0$. This completes the proof. □

Similar to the proof of Theorem 7.7 in [19], we have the following.

Theorem 4.19. *Let $A(t)$ be an almost periodic matrix function and $f(t)$ be an almost periodic vector function. If* (4.5) *admits an exponential dichotomy, then* (4.4) *has a unique almost periodic solution:*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{4.10}$$

where $X(t)$ is the fundamental solution matrix of (4.5).

Example 4.20. Consider the following dynamic equation on an almost periodic time scale $\mathbb{T}$

$$x^{\Delta}(t) = Ax(t) + f(t), \tag{4.11}$$

where

$$A = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} \sin\sqrt{3}t \\ \cos\sqrt{2}t \end{pmatrix}, \qquad \mu(t) \neq \frac{1}{6}. \tag{4.12}$$

Obviously, $I+\mu(t)A$ is invertible for all $\mathbb{T}$, so $A \in \mathcal{R}$. We claim that the homogeneous equation of (4.11) admits an exponential dichotomy. In fact, the eigenvalues of the coefficient matrix in (4.11) are $\lambda_1 = \lambda_2 = -6$, according to Theorem 5.35 (Putzer Algorithm) in [1], the P-matrices are given by

$$P_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_1 = (A - \lambda_1 I)P_0 = A + 6I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.13}$$

We want to choose

$$r_1^{\Delta} = -6r_1, \quad r_1(t_0) = 1, \qquad r_2^{\Delta} = r_1 - 6r_2, \quad r_2(t_0) = 0. \tag{4.14}$$

Solving the first IVP for r_1, we get that $r_1(t) = e_{-6}(t,t_0)$. Solving the second IVP, that is,

$$r_2^{\Delta} = -6r_2 + e_{-6}(t,t_0), \;\; r_2(t_0) = 0, \tag{4.15}$$

we obtain

$$r_2 = e_{-6}(t,t_0)\int_{t_0}^{t} \frac{\Delta s}{1-6\mu(s)}. \tag{4.16}$$

Using Theorem 5.35 (Putzer Algorithm) in [1], we get

$$e_A(t,t_0) = r_1(t)P_0 + r_2(t)P_1 = e_{-6}(t,t_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.17}$$

Thus,

$$\left|X(t)P_0X^{-1}(s)\right| = \left\| e_{-6}(t,t_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} e_{\ominus -6}(s,t_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\| \leq \sqrt{2}e_{\ominus 3}(t,s). \tag{4.18}$$

Then, we can take $K = \sqrt{2}$, $\alpha = 3$ such that the homogeneous equation of (4.11) admits an exponential dichotomy. Finally, according to Theorem 4.19, we can get the unique almost periodic solution of (4.11):

$$x(t) = \int_{-\infty}^{t} X(t)P_0X^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P_0)X^{-1}(\sigma(s))f(s)\Delta s$$
$$= \int_{-\infty}^{t} e_{-6}(t,\sigma(s))\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sin\sqrt{3}s\\ \cos\sqrt{2}s\end{pmatrix}\Delta s. \tag{4.19}$$

5. Almost Periodic Solutions to a Nonlinear Dynamic Equation on Time Scales

As an application of our results obtained in the previous sections, in this section, we consider the following dynamic equation with delays on almost periodic time scale $\mathbb{T}$:

$$x^{\Delta}(t) = A(t)x(t) + \sum_{i=1}^{n} f(t, x(t-\tau_i(t))). \tag{5.1}$$

Theorem 5.1. *Suppose that the following hold:*

(H_2) *$A(t)$ is an almost periodic matrix function on $\mathbb{T}$, for every $i = 1, 2, \ldots, n, \tau_i(t)$ is almost periodic on $\mathbb{T}$, $t - \tau_i(t) \in \mathbb{T}$, for $t \in \mathbb{T}$, and $f \in C(T \times \mathbb{R}^n, \mathbb{R}^n)$ is almost periodic uniformly in t for $x \in \mathbb{R}^n$;*

(H_1) *$x^{\Delta}(t) = A(t)x(t)$ admits an exponential dichotomy on $\mathbb{T}$ with positive constants K and α.;*

(H_2) *There exists $M < \alpha/(2+\mu\alpha)Kn$ such that $|f(t,x) - f(t,y)| \le M|x-y|$ for $t \in \mathbb{T}, x, y \in \mathbb{R}^n$.*

Then system (5.1) has a unique almost periodic solution.

Proof. For any $\varphi \in AP(\mathbb{T})$, consider the following equation:

$$x^{\Delta}(t) = A(t)x(t) + \sum_{i=1}^{n} f\big(t, \varphi(t-\tau_i(t))\big). \tag{5.2}$$

According to Theorem 4.19, (5.2) has a unique solution $T\varphi \in AP(\mathbb{T})$ and

$$T\varphi(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))\sum_{i=1}^{n} f\big(s, \varphi(s-\tau_i(s))\big)\Delta s$$
$$- \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))\sum_{i=1}^{n} f\big(s, \varphi(s-\tau_i(s))\big)\Delta s. \tag{5.3}$$

Define a mapping $T : AP(\mathbb{T}) \to AP(\mathbb{T})$ by setting $(T\varphi)(t) = x_{\varphi}(t),$ for all $x \in AP(\mathbb{T})$. From (H_1), we have

$$\begin{aligned} \left|X(t)PX^{-1}(s)\right| &\le Ke_{\ominus\alpha}(t,s), \quad s,t \in \mathbb{T},\ t \ge s, \\ \left|X(t)(I-P)X^{-1}(s)\right| &\le Ke_{\ominus\alpha}(s,t), \quad s,t \in \mathbb{T},\ t \le s. \end{aligned} \tag{5.4}$$

For any φ, $\psi \in AP(\mathbb{T})$, we have

$$\begin{aligned} \|T\varphi - T\psi\| &\le \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))\sum_{i=1}^{n}\left|f\left(s,\varphi(s-\tau_i(s))\right) - f\left(s,\psi(s-\tau_i(s))\right)\right|\Delta s \\ &\quad - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))\sum_{i=1}^{n}\left|f\left(\mathrm{s},\varphi(s-\tau_i(s))\right) - f\left(s,\psi(s-\tau_i(s))\right)\right|\Delta s \\ &\le \left[\int_{-\infty}^{t} K_{\ominus\alpha}(t,\sigma(s))\Delta s + \int_{t}^{+\infty} K_{\ominus\alpha}(\sigma(s),t)\Delta s\right]\sum_{i=1}^{n} M\|\varphi-\psi\| \\ &\le \frac{2+\overline{\mu}\alpha}{\alpha}KnM\|\varphi-\psi\|, \end{aligned} \tag{5.5}$$

where $\overline{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$. Since (H_2), T is a contractive operator. Therefore, (5.1) has a unique almost periodic solution. □

Acknowledgment

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 10971183.

References

[1] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., Boston, Mass, USA, 2001.

[2] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., Boston, Mass, USA, 2003.

[3] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. thesis, Universität Würzburg, 1988.

[4] S. Castillo and M. Pinto, "Asymptotic behavior of functional dynamic equations in time scale," *Dynamic Systems and Applications*, vol. 19, no. 1, pp. 165–177, 2010.

[5] J. Zhou and Y. Li, "Sobolev's spaces on time scales and its applications to a class of second order Hamiltonian systems on time scales," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 5, pp. 1375–1388, 2010.

[6] Y. Li and T. Zhang, "On the existence of solutions for impulsive Duffing dynamic equations on time scales with Dirichlet boundary conditions," *Abstract and Applied Analysis*, vol. 2010, Article ID 152460, 27 pages, 2010.

[7] H. Liu and X. Xiang, "A class of the first order impulsive dynamic equations on time scales," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 9, pp. 2803–2811, 2008.

[8] S. H. Saker, R. P. Agarwal, and D. O'Regan, "Oscillation of second-order damped dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 2, pp. 1317–1337, 2007.
[9] H.-R. Sun and W.-T. Li, "Existence theory for positive solutions to one-dimensional p-Laplacian boundary value problems on time scales," *Journal of Differential Equations*, vol. 240, no. 2, pp. 217–248, 2007.
[10] Y. Li, X. Chen, and L. Zhao, "Stability and existence of periodic solutions to delayed Cohen-Grossberg BAM neural networks with impulses on time scales," *Neurocomputing*, vol. 72, no. 7-9, pp. 1621–1630, 2009.
[11] L. G. Deysach and G. R. Sell, "On the existence of almost periodic motions," *The Michigan Mathematical Journal*, vol. 12, pp. 87–95, 1965.
[12] R. K. Miller, "Almost periodic differential equations as dynamical systems with applications to the existence of A.P. solutions," *Journal of Differential Equations*, vol. 1, pp. 337–345, 1965.
[13] G. Seifert, "Almost periodic solutions and asymptotic stability," *Journal of Mathematical Analysis and Applications*, vol. 21, pp. 136–149, 1968.
[14] G. Seifert, "On uniformly almost periodic sets of functions for almost periodic differential equations," *The Tôhoku Mathematical Journal*, vol. 34, no. 2, pp. 301–309, 1982.
[15] S. Bochner, "A new approach to almost periodicity," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 48, pp. 2039–2043, 1962.
[16] A. M. Fink, *Almost Periodic Differential Equations*, Springer, Berlin, Germany, 1974.
[17] A. M. Fink and G. Seifert, "Liapunov functions and almost periodic solutions for almost periodic systems," *Journal of Differential Equations*, vol. 5, pp. 307–313, 1969.
[18] D. Cheban and C. Mammana, "Invariant manifolds, global attractors and almost periodic solutions of nonautonomous difference equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 56, no. 4, pp. 465–484, 2004.
[19] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, NY, USA, 2nd edition, 1989.
[20] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic/Plenum Publishers, New York, NY, USA, 2001.
[21] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press, Beijing, China, 2003.
[22] C. Y. Zhang, "Pseudo-almost-periodic solutions of some differential equations," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 1, pp. 62–76, 1994.
[23] A. M. Fink and J. A. Gatica, "Positive almost periodic solutions of some delay integral equations," *Journal of Differential Equations*, vol. 83, no. 1, pp. 166–178, 1990.
[24] Y. Hino, T. Naito, Nguyen Van Minh, and J. S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces*, vol. 15, Taylor & Francis, London, UK, 2002.
[25] T. Caraballo and D. Cheban, "Almost periodic and almost automorphic solutions of linear differential/difference equations without Favard's separation condition. I," *Journal of Differential Equations*, vol. 246, no. 1, pp. 108–128, 2009.
[26] N. Boukli-Hacene and K. Ezzinbi, "Weighted pseudo almost periodic solutions for some partial functional differential equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 9, pp. 3612–3621, 2009.
[27] E. H. A. Dads, P. Cieutat, and K. Ezzinbi, "The existence of pseudo-almost periodic solutions for some nonlinear differential equations in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 4, pp. 1325–1342, 2008.
[28] Y. Li and T. Zhang, "Permanence and almost periodic sequence solution for a discrete delay logistic equation with feedback control," *Nonlinear Analysis. Real World Applications*, vol. 12, no. 3, pp. 1850–1864, 2011.
[29] Y. Li and X. Fan, "Existence and globally exponential stability of almost periodic solution for Cohen-Grossberg BAM neural networks with variable coefficients," *Applied Mathematical Modelling*, vol. 33, no. 4, pp. 2114–2120, 2009.
[30] J. B. Conway, *A course in Functional Analysis*, vol. 96 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1985.
[31] J. Zhang, M. Fan, and H. Zhu, "Existence and roughness of exponential dichotomies of linear dynamic equations on time scales," *Computers & Mathematics with Applications*, vol. 59, no. 8, pp. 2658–2675, 2010.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization