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Research Article

Continuous g-Frame in Hilbert C^*-Modules

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We give a generalization of g-frame in Hilbert C^*-modules that was introduced by Khosravies then investigated some properties of it by Xiao and Zeng. This generalization is a natural generalization of continuous and discrete g-frames and frame in Hilbert space too. We characterize continuous g-frame g-Riesz in Hilbert C^*-modules and give some equality and inequality of these frames.

1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [1] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies et al. [2] and popularized from then on. The theory of frames plays an important role in signal processing because of their importance to quantization [3], importance to additive noise [4], as well their numerical stability of reconstruction [4] and greater freedom to capture signal characteristics [5, 6]. See also [7–9]. Frames have been used in sampling theory [10, 11], to oversampled perfect reconstruction filter banks [12], system modelling [13], neural networks [14] and quantum measurements [15]. New applications in image processing [16], robust transmission over the Internet and wireless [17–19], coding and communication [20, 21] were given. For basic results on frames, see [4, 12, 22, 23].

let H be a Hilbert space, and I a set which is finite or countable. A system $\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1.1}$$

for all $f \in H$. The constants A and B are called frame bounds. If $A = B$ we call this frame a tight frame and if $A = B = 1$ it is called a Parseval frame.

In [24] Sun introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

Let U and V be two Hilbert spaces, and $\{V_j : j \in J\}$ is a sequence of subspaces of V, where J is a subset of $\mathbf{Z}$. $L(U, V_j)$ is the collection of all bounded linear operators from U into V_j. We call a sequence $\{\Lambda_j \in L(U, V_j) : j \in J\}$ a generalized frame, or simply a g-frame, for U with respect to $\{V_j : j \in J\}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in J}\|\Lambda_j f\|^2 \leq B\|f\|^2 \tag{1.2}$$

for all $f \in U$. The constants A and B are called g-frame bounds. If $A = B$ we call this g-frame a tight g-frame, and if $A = B = 1$ it is called a Parseval g-frame.

On the other hand, the concept of frames especially the g-frames was introduced in Hilbert C^*-modules, and some of their properties were investigated in [25–27]. Frank and Larson [25] defined the standard frames in Hilbert C^*-modules in 1998 and got a series of result for standard frames in finitely or countably generated Hilbert C^*-modules over unital C^*-algebras. As for Hilbert C^*-module, it is a generalization of Hilbert spaces in that it allows the inner product to take values in a C^*-algebra rather than the field of complex numbers. There are many differences between Hilbert C^*-modules and Hilbert spaces. For example, we know that any closed subspace in a Hilbert space has an orthogonal complement, but it is not true for Hilbert C^*-module. And we cannot get the analogue of the Riesz representation theorem of continuous functionals in Hilbert C^*-modules generally. Thus it is more difficult to make a discussion of the theory of Hilbert C^*-modules than that of Hilbert spaces in general. We refer the readers to papers [28, 29] for more details on Hilbert C^*-modules. In [27, 30], the authors made a discussion of some properties of g-frame in Hilbert C^*-module in some aspects.

The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by Kaiser [23] and independently by Ali at al. [31]. These frames are known as continuous frames. Let H be a Hilbert space, and let $(M; \mathcal{S}; \mu)$ be a measure space. A continuous frame in H indexed by M is a family $h = \{h_m \in H : m \in M\}$ such that

(a) for any $f \in H$, the function $\tilde{f} : M \to C$ defined by $\tilde{f}(m) = \langle h_m, f\rangle$ is measurable;

(b) there is a pair of constants $0 < A, B$ such that, for any $f \in H$,

$$A\|f\|_H^2 \leq \left\|\tilde{f}\right\|_{L^2(\mu)}^2 \leq B\|f\|_H^2. \tag{1.3}$$

If $M = \mathbf{N}$ and μ is the counting measure, the continuous frame is a frame.

The paper is organized as follows. In Sections 2 and 3 we recall the basic definitions and some notations about continuous g-frames in Hilbert C^*-module; we also give some basic properties of g-frames which we will use in the later sections. In Section 4, we give some characterization for continuous g-frames in Hilbert C^*-modules. In Section 5, we extend some important equalities and inequalities of frame in Hilbert spaces to continuous frames and continuous g-frames in Hilbert C^*-modules.

2. Preliminaries

In the following we review some definitions and basic properties of Hilbert C^*-modules and g-frames in Hilbert C^*-module; we first introduce the definition of Hilbert C^*-modules.

Definition 2.1. Let A be a C^*-algebra with involution *. An inner product A-module (or pre-Hilbert A-module) is a complex linear space H which is a left A-module with map $\langle\cdot,\cdot\rangle : H \times H \to A$ which satisfies the following properties:

(1) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$ for all $f, g, h \in H$ and $\alpha, \beta \in \mathbf{C}$;

(2) $\langle af, g\rangle = a\langle f, g\rangle$ for all $f, g \in H$ and $a \in A$;

(3) $\langle f, g\rangle = \langle g, f\rangle^*$ for all $f, g \in H$;

(4) $\langle f, f\rangle \geq 0$ for all $f \in H$ and $\langle f, f\rangle = 0$ if and only if $f = 0$.

For $f \in H$, we define a norm on H by $\|f\|_H = \|\langle f, f\rangle\|_A^{1/2}$. Let H is complete with this norm, it is called a Hilbert C^*-module over A or a Hilbert A-module.

An element a of a C^*-algebra A is positive if $a^* = a$ and spectrum of a is a subset of positive real number. We write $a \geq 0$ to mean that a is positive. It is easy to see that $\langle f, f\rangle \geq 0$ for every $f \in H$, hence we define $|f| = \langle f, f\rangle^{1/2}$.

Frank and Larson [25] defined the standard frames in Hilbert C^*-modules. If H be a Hilbert C^*-module, and I a set which is finite or countable. A system $\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\langle f, f\rangle \leq \sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle \leq B\langle f, f\rangle \tag{2.1}$$

for all $f \in H$. The constants A and B are called frame bounds.

A. Khosravi and B. Khosravi [27] defined g-frame in Hilbert C^*-module. Let U and V be two Hilbert C^*-module, and $\{V_i : i \in I\}$ is a sequence of subspaces of V, where I is a subset of Z and $\mathrm{End}_A^*(U, V_i)$ is the collection of all adjointable A-linear maps from U into V_i that is, $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in H$ and $T \in \mathrm{End}_A^*(U, V_i)$. We call a sequence $\{\Lambda_i \in \mathrm{End}_A^*(U, V_i) : i \in I\}$ a generalized frame, or simply a g-frame, for Hilbert C^*-module U with respect to $\{V_i : i \in I\}$ if there are two positive constants A and B such that

$$A\langle f, f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle \leq B\langle f, f\rangle \tag{2.2}$$

for all $f \in U$. The constants A and B are called g-frame bounds. If $A = B$ we call this g-frame a tight g-frame, and if $A = B = 1$ it is called a Parseval g-frame.

Let $(M; \mathcal{S}; \mu)$ be a measure space, let U and V be two Hilbert C^*-modules, $\{V_m : m \in M\}$ is a sequence of subspaces of V, and $\mathrm{End}_A^*(U, V_m)$ is the collection of all adjointable A-linear maps from U into V_m.

Definition 2.2. We call a net $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ a continuous generalized frame, or simply a continuous g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ if

(a) for any $f \in U$, the function $\widetilde{f} : M \to V_m$ defined by $\widetilde{f}(m) = \Lambda_m f$ is measurable;

(b) there is a pair of constants $0 < A, B$ such that, for any $f \in U$,

$$A\langle f, f\rangle \leq \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \leq B\langle f, f\rangle. \tag{2.3}$$

The constants A and B are called continuous g-frame bounds. If $A = B$ we call this continuous g-frame a continuous tight g-frame, and if $A = B = 1$ it is called a continuous Parseval g-frame. If only the right-hand inequality of (2.3) is satisfied, we call $\{\Lambda_m : m \in M\}$ the continuous g-Bessel for U with respect to $\{V_m : m \in M\}$ with Bessel bound B.

If $M = \mathbf{N}$ and μ is the counting measure, the continuous g-frame for U with respect to $\{V_m : m \in M\}$ is a g-frame for U with respect to $\{V_m : m \in M\}$.

Let X be a Banach space, (Ω, μ) a measure space, and function $f : \Omega \to X$ a measurable function. Integral of the Banach-valued function f has defined Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions for example triangle inequality. The reader is referred to [32, 33] for more details. Because every C^*-algebra and Hilbert C^*-module is a Banach space thus we can use this integral and its properties.

Example 2.3. Let U be a Hilbert C^*-module on C^*-algebra A, and let $\{f_m : m \in M\}$ be a frame for U. Let Λ_m be the functional induced by

$$\Lambda_m f = \langle f, f_m\rangle, \quad \forall f \in U. \tag{2.4}$$

Then $\{\Lambda_m : m \in M\}$ is a g-frame for Hilbert C^*-module U with respect to $V = V_m = A$.

Example 2.4. If $\psi \in L^2(\mathbf{R})$ is admissible, that is,

$$C_\psi := \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\gamma)|^2}{|\gamma|} d\gamma < \infty. \tag{2.5}$$

and, for $a, b \in \mathbf{R}$ and $a \neq 0$

$$\psi_{a,b}(x) = T_b D_a \psi(x) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right), \quad \forall x \in \mathbf{R}, \tag{2.6}$$

then $\{\psi_{a,b}\}_{a\neq 0, b\in\mathbf{R}}$ is a continuous frame for $L^2(\mathbf{R})$ with respect to $\mathbf{R} \setminus \{0\} \times \mathbf{R}$ equipped with the measure $da\, db/a^2$ and, for all $f \in L^2(\mathbf{R})$,

$$f(x) = \frac{1}{C_\psi} \iint_{-\infty}^{\infty} W_\psi(f)(a, b) \psi_{a,b}(x) \frac{da\, db}{a^2}, \tag{2.7}$$

where W_ψ is the continuous wavelet transform defined by

$$W_\psi(f)(a,b) = \langle f, \psi_{a,b}\rangle = \int_{-\infty}^{\infty} f(x)\frac{1}{\sqrt{|a|}}\overline{\psi\left(\frac{x-b}{a}\right)}dx. \tag{2.8}$$

For details, see [12, Proposition 11.1.1 and Corollary 11.1.2].

3. Continuous g-Frame Operator and Dual Continuous g-Frame on Hilbert C^*-Algebra

Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$. Define the continuous g-frame operator S on U by

$$Sf = \int_M \Lambda_m^* \Lambda_m f d\mu(m). \tag{3.1}$$

Lemma 3.1 (see [33]). *Let (Ω, μ) be a measure space, X and Y are two Banach spaces, $\lambda : X \to Y$ be a bounded linear operator and $f : \Omega \to X$ measurable function; then*

$$\lambda\left(\int_\Omega f d\mu\right) = \int_\Omega (\lambda f) d\mu. \tag{3.2}$$

Proposition 3.2. *The frame operator S is a bounded, positive, selfadjoint, and invertible.*

Proof. First we show, S is a selfadjoint operator. By Lemma 3.1 and property (3) of Definition 2.1 for any $f, g \in U$ we have

$$\begin{aligned}\langle Sf, g\rangle &= \left\langle \int_M \Lambda_m^* \Lambda_m f d\mu(m), g\right\rangle = \int_M \langle \Lambda_m^* \Lambda_m f, g\rangle d\mu(m)\\ &= \int_M \langle f, \Lambda_m^* \Lambda_m g\rangle d\mu(m) = \left\langle f, \int_M \Lambda_m^* \Lambda_m g d\mu(m)\right\rangle\\ &= \langle f, Sg\rangle.\end{aligned} \tag{3.3}$$

It is clear that we have

$$\langle Sf, f\rangle = \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m). \tag{3.4}$$

Now we show that S is a bounded operator

$$\|S\| = \sup_{\|f\|\le 1} \|\langle Sf, f\rangle\| = \sup_{\|f\|\le 1} \left\| \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| \le B. \tag{3.5}$$

Inequality (2.3) and equality (3.4) mean that

$$A\langle f,f\rangle \le \langle Sf,f\rangle \le B\langle f,f\rangle \tag{3.6}$$

or in the notation from operator theory $AI \le S \le BI$; thus S is a positive operator. Furthermore, $0 \le A^{-1}S - I \le ((B-A)/A)I$, and consequently $\|A^{-1}S - I\| \le 1$; this shows that S is invertible. □

Proposition 3.3. *Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$ with continuous g-frame operator S with bounds A and B. Then $\{\widetilde{\Lambda}_m : m \in M\}$ defined by $\widetilde{\Lambda}_m = \Lambda_m S^{-1}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$ with continuous g-frame operator S^{-1} with bounds $1/B$ and $1/A$. That is called continuous canonical dual g-frame of $\{\Lambda_m : m \in M\}$.*

Proof. Let $\widetilde{S}$ be the continuous g-frame operator associated with $\{\widetilde{\Lambda}_m : m \in M\}$ that is $\widetilde{S}f = \int_M \widetilde{\Lambda_m}^* \widetilde{\Lambda_m} f\, d\mu(m)$. Then for $f \in U$,

$$\begin{aligned} S\widetilde{S}f &= \int_M S\widetilde{\Lambda_m}^* \widetilde{\Lambda_m} f\, d\mu(m) = \int_M SS^{-1}\Lambda_m^*\Lambda_m S^{-1} f\, d\mu(m) \\ &= \int_M \Lambda_m^*\Lambda_m S^{-1} f\, d\mu(m) = SS^{-1}f = f. \end{aligned} \tag{3.7}$$

Hence $\widetilde{S} = S^{-1}$.

Since $\{\Lambda_m : m \in M\}$ is a continuous g-frame for H, then $AI \le S \le BI$. On other hand since I and S are selfadjoint and S^{-1} commutative with I and S, $AIS^{-1} \le SS^{-1} \le BIS^{-1}$, and hence $B^{-1}I \le S^{-1} \le A^{-1}I$. □

Remark 3.4. We have $\widetilde{\Lambda}_m \widetilde{S}^{-1} = \Lambda_m S^{-1} S = \Lambda_m$. In other words $\{\Lambda_m : m \in M\}$ and $\{\widetilde{\Lambda}_m : m \in M\}$ are dual continuous g-frame with respect to each other.

4. Some Characterizations of Continuous g-Frames in Hilbert C^*-Module

In this section, we will characterize the equivalencies of continuous g-frame in Hilbert C^*-module from several aspects. As for Theorem 4.2, we show that the continuous g-frame is equivalent to which the middle of (2.3) is norm bounded. As for Theorems 4.3 and 4.6, the characterization of g-frame is equivalent to the characterization of bounded operator T.

Lemma 4.1 (see [34]). *Let A be a C^*-algebra, U and V two Hilbert A-modules, and $T \in \mathrm{End}_A^*(U,V)$. Then the following statements are equivalent:*

(1) *T is surjective;*

(2) *T^* is bounded below with respect to norm, that is, there is $m > 0$ such that $\|T^* f\| \ge m\|f\|$ for all $f \in U$;*

(3) *T^* is bounded below with respect to the inner product, that is, there is $m' > 0$ such that $\langle T^* f, T^* f\rangle \ge m'\langle f,f\rangle$.*

Theorem 4.2. *Let $\Lambda_m \in \mathrm{End}_A^*(U,V_m)$ for any $m \in M$. Then $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$ if and only if there exist constants $A, B > 0$ such that*

for any $f \in U$

$$A\|f\|^2 \leq \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| \leq B\|f\|^2. \tag{4.1}$$

Proof. Let $\{\Lambda_m : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$. Then inequality (4.1) is an immediate result of C^*-algebra theory.

If inequality (4.1) holds, then by Proposition 3.2, $\langle S^{1/2}f, S^{1/2}f \rangle = \langle Sf, f \rangle = \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m)$, hence $\sqrt{A}\|f\| \leq \|S^{1/2}f\| \leq \sqrt{B}\|f\|$ for any $f \in U$. Now by use of Lemma 4.1, there are constants $C, D > 0$ such that

$$C\langle f, f \rangle \leq \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \leq D\langle f, f \rangle, \tag{4.2}$$

which implies that $\{\Lambda_m : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$. □

We define

$$\bigoplus_{m \in M} V_m = \left\{ g = \{g_m\} : g_m \in V_m, \left\| \int_M |g_m|^2 d\mu(m) \right\| < \infty \right\}. \tag{4.3}$$

For any $f = \{f_m : m \in M\}$ and $g = \{g_m : m \in M\}$, if the A-valued inner product is defined by $\langle f, g \rangle = \int_M \langle f_m, g_m \rangle d\mu(m)$, the norm is defined by $\|f\| = \|\langle f, f \rangle\|^{1/2}$, then $\bigoplus_{m \in M} V_m$ is a Hilbert A-module (see [28]).

Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$, we define synthesis operator $T : \bigoplus_{m \in M} V_m \to U$ by; $T(g) = \int_M \Lambda_m^* g_m d\mu(m)$ for all $g = \{g_m : m \in M\} \in \bigoplus_{m \in M} V_m$. So analysis operator is defined for map $F : U \to \bigoplus_{m \in M} V_m$ by $F(f) = \{\Lambda_m : m \in M\}$ for any $f \in U$.

Theorem 4.3. *A net* $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ *is a continuous g-frame for* U *with respect to* $\{V_m : m \in M\}$ *if and only if synthesis operator* T *is well defined and surjective.*

Proof. Let $\{\Lambda_m : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$; then operator T is well defined and $\|T\| \leq \sqrt{B}$ because

$$\begin{aligned}
\|Tg\|^2 &= \left\| \int_M \Lambda_m^* g_m d\mu(m) \right\|^2 = \sup_{f \in U, \|f\|=1} \left\| \left\langle \int_M \Lambda_m^* g_m d\mu(m), f \right\rangle \right\| \\
&= \sup_{\substack{f \in U, \\ \|f\|=1}} \left\| \int_M \langle g_m, \Lambda_m f \rangle d\mu(m) \right\| \\
&\leq \sup_{\substack{f \in U, \\ \|f\|=1}} \left\| \int_M \langle g_m, g_m \rangle d\mu(m) \right\| \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| \\
&\leq B \left\| \int_M \langle g_m, g_m \rangle d\mu(m) \right\|.
\end{aligned} \tag{4.4}$$

For any $f \in U$, by that S is invertible, there exist $g \in U$ such that $f = Sg = \int_M \lambda_m^* \lambda_m g d\mu(m)$. Since $\{\Lambda_m : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$, so $\{\lambda_m g : m \in M\} \in \bigoplus_{m\in M} V_m$ and $T(\{\lambda_m g\}_m \in M) = \int_M \lambda_m^* \lambda_m g d\mu(m) = f$, which implies that T is surjective.

Now let T be a well-defined operator. Then for any $f \in U$ we have

$$\begin{aligned}
\left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| &= \left\| \int_M \langle f, \Lambda_m^* \Lambda_m f \rangle d\mu(m) \right\| = \left\| \left\langle f, \int_M \Lambda_m^* \Lambda_m f d\mu(m) \right\rangle \right\| \\
&\le \|f\| \left\| \int_M \Lambda_m^* \Lambda_m f d\mu(m) \right\| \\
&= \|f\| \|T(\{\Lambda_m f\}_{m\in M})\| \le \|f\| \|T\| \|\{\Lambda_m f\}_{m\in M}\| \\
&\le \|f\| \|T\| \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\|^{1/2}.
\end{aligned} \tag{4.5}$$

It follow that $\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \| \le \|f\|^2 \|T\|^2$.

On the other hand, since T is surjective, by Lemma 4.1, T^* is bounded below, so $T^*|_{R(T^*)}$ is invertible. Then for any $f \in U$, we have $(T^*|_{R(T^*)})^{-1} T^* f = f$, so $\|f\|^2 \le \|T^* f\|^2 \|(T^*|_{R(T^*)})^{-1}\|^2$. It is easy to check that

$$\begin{gathered}
T^* : U \longrightarrow \bigoplus_{m\in M} V_m, \quad T^*(f) = \{\Lambda_m : m \in M\} \\
\|T^* f\|^2 = \|\{\Lambda_m\}_{m\in M}\|^2 = \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\|.
\end{gathered} \tag{4.6}$$

Hence $\|f\|^2 \le \| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \| \|(T^*|_{R(T^*)})^{-1}\|^2$. □

Corollary 4.4. *A net $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Bessel net for U with respect to $\{V_m : m \in M\}$ if and only if synthesis operator T is well defined and $\|T\| \le \sqrt{D}$.*

Definition 4.5. A continuous g-frame $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ in Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ is said to be a continuous g-Riesz basis if it satisfies that

(1) $\lambda_m \neq 0$ for any $m \in M$;

(2) if $\int_K \Lambda_m^* g_m d\mu(m) = 0$, then every summand $\Lambda_m^* g_m$ is equal to zero, where $\{g_m\}_{m\in K} \in \bigoplus_{m\in K} V_m$ and K is a measurable subset of M.

Theorem 4.6. *A net $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Riesz for U with respect to $\{V_m : m \in M\}$ if and only if synthesis operator T is homeomorphism.*

Proof. We firstly suppose that $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Bessel net for U with respect to $\{V_m : m \in M\}$. By Theorem 4.3 and that it is g-frame, T is surjective. If $Tf = \int_M \Lambda_m^* f_m d\mu(m) = 0$ for some $f = \{f_m : m \in M\} \in \bigoplus_{m\in M} V_m$, according to the definition of continuous g-Riesz basis we have $\Lambda_m^* f_m = 0$ for any $m \in M$, and $\Lambda_m \neq 0$, so $f_m = 0$ for any $m \in M$, namely $f = 0$. Hence T is injective.

Now we let the synthesis operator T be homeomorphism. By Theorem 4.3 $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$. It is obviouse that $\Lambda_m \neq 0$ for any $m \in M$. Since T is injective, so if $Tf = \int_M \Lambda_m^* f_m d\mu(m) = 0$, then $f = \{f_m : m \in M\} = 0$, so $\Lambda_m^* f_m = 0$. Therefore $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Riesz for U with respect to $\{V_m : m \in M\}$. □

Theorem 4.7. *Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$, with g-frame bounds $A_1, B_1 \geq 0$. Let $\Gamma_m \in \mathrm{End}_A^*(U, V_m)$ for any $m \in M$. Then the following are equivalent:*

(1) *$\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$;*

(2) *there exists a constant $N > 0$, such that for any $f \in U$, one has*

$$\begin{aligned} &\left\| \int_M \langle (\Lambda_m - \Gamma_m) f, (\Lambda_m - \Gamma_m) f \rangle d\mu(m) \right\| \\ &\quad \leq N \min\left(\left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\|, \left\| \int_M \langle \Gamma_m f, \Gamma_m f \rangle d\mu(m) \right\| \right). \end{aligned} \tag{4.7}$$

Proof. First we let $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$ with bounds $A_2, B_2 > 0$. Then for any $f \in U$, we have

$$\begin{aligned} &\left\| \int_M \langle (\Lambda_m - \Gamma_m) f, (\Lambda_m - \Gamma_m) f \rangle d\mu(m) \right\| \\ &\quad = \left\| \{(\Lambda_m - \Gamma_m) f\}_{m \in M} \right\|^2 \\ &\quad \leq \left\| \{\Lambda_m f\}_{m \in M} \right\|^2 + \left\| \{\Gamma_m f\}_{m \in M} \right\|^2 \\ &\quad = \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| + \left\| \int_M \langle \Gamma_m f, \Gamma_m f \rangle d\mu(m) \right\| \\ &\quad \leq \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| + B_2 \|f\|^2 \\ &\quad \leq \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| + \frac{B_2}{A_1} \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| \\ &\quad = \left(1 + \frac{B_2}{A_1} \right) \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\|. \end{aligned} \tag{4.8}$$

Similarly we can obtain

$$\left\| \int_M \langle (\Lambda_m - \Gamma_m) f, (\Lambda_m - \Gamma_m) f \rangle d\mu(m) \right\| \leq \left(1 + \frac{B_1}{A_2} \right) \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\|. \tag{4.9}$$

Let $N = \min\{1 + B_2/A_1, 1 + B_1/A_2\}$; then inequality (4.7) holds.

Next we suppose that inequality (4.7) holds. For any $f \in U$, we have

$$\begin{aligned}
A_1\|f\|^2 &\leq \left\|\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| = \|\{\Lambda_m f\}_{m\in M}\|^2 \\
&\leq \|\{(\Lambda_m - \Gamma_m)f\}_{m\in M}\|^2 + \|\{\Gamma_m f\}_{m\in M}\|^2 \\
&= \left\|\int_M \langle (\Lambda_m - \Gamma_m)f, (\Lambda_m - \Gamma_m)f\rangle d\mu(m)\right\| + \left\|\int_M \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m)\right\| \\
&\leq N\left\|\int_M \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m)\right\| + \left\|\int_M \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m)\right\| \\
&= (N+1)\left\|\int_M \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m)\right\|.
\end{aligned} \tag{4.10}$$

Also we can obtain

$$\begin{aligned}
\left\|\int_M \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m)\right\| &\leq \|\{\Lambda_m f\}_{m\in M}\|^2 + \|\{(\Lambda_m - \Gamma_m)f\}_{m\in M}\|^2 \\
&= \left\|\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| + \left\|\int_M \langle (\Lambda_m - \Gamma_m)f, (\Lambda_m - \Gamma_m)f\rangle d\mu(m)\right\| \\
&\leq N\left\|\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| + \left\|\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| \\
&= (N+1)\left\|\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| \leq B_1(N+1)\|f\|^2.
\end{aligned} \tag{4.11}$$

□

Next we will introduce a bounded operator L about two g-Bessel sequences in Hilbert C^*-module. The idea is derived from the operator S_{VW} which was considered for fusion frames by Găvruţa in [35]. In this paper, we will use the operator L to characterize the g-frames of Hilbert C^*-module further. Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ and $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be two g-Bessel sequences for U with respect to $\{V_m : m \in M\}$, with Bessel bounds $B_1, B_2 > 0$, respectively. Then there exists a well-defined operator

$$L : U \longrightarrow U, \quad Lf = \int_M \Gamma_m^* \Lambda_m f \, d\mu(m) \quad \forall f \in U. \tag{4.12}$$

As a matter of fact, for any $f \in U$, we have

$$\begin{aligned}
\left\| \int_M \Gamma_m^* \Lambda_m f \, d\mu(m) \right\|^2 &= \sup_{\substack{g \in U, \\ \|g\|=1}} \left\| \left\langle \int_M \Gamma_m^* \Lambda_m f \, d\mu(m), g \right\rangle \right\|^2 \\
&= \sup_{\substack{g \in U, \\ \|g\|=1}} \left\| \int_M \langle \Lambda_m f, \Gamma_m g \rangle d\mu(m) \right\|^2 \\
&\leq \sup_{\substack{g \in U, \\ \|g\|=1}} \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| \left\| \int_M \langle \Gamma_m g, \Gamma_m g \rangle d\mu(m) \right\| \\
&\leq B_2 \left\| \int_M \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\|.
\end{aligned} \tag{4.13}$$

It is easy to know that $L^* f = \int_M \Lambda_m^* \Gamma_m f \, d\mu(m)$ and $\|L\| \leq \sqrt{B_1 B_2}$.

Theorem 4.8. *Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$, with g-frame bounds $A_1, B_1 \geq 0$. Suppose that $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Bessel net for U with respect to $\{V_m : m \in M\}$. If L is surjective, then $\{\Gamma_m : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$.*

On the contrary, if $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Riesz basis for U with respect to $\{V_m : m \in M\}$, then L is surjective.

Proof. Suppose that $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$. By Theorem 4.3, we can define the synthesis operator T of (4.12). It is easy to check that the adjoint operator of T is analysis operator as follows:

$$T^* : U \longrightarrow \bigoplus_{m \in M} V_m \quad \text{by } T^*(f) = \{\Lambda_m : m \in M\} \tag{4.14}$$

for any $f \in U$.

On the other hand, since $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Bessel net for U with respect to $\{V_m : m \in M\}$, by Corollary 4.4 we also can define the corresponding operator $Q : \bigoplus_{m \in M} V_m \to U$, $Q(g) = \int_M \Gamma_m^*(g_m) d\mu(m)$.

Hence we have $Lf = \int_M \Gamma_m^* \Lambda f \, d\mu(m) = QT^* f$ for any $f \in U$, namely, $L = QT^*$. Since L is surjective, then for any $f \in U$, there exists $g \in U$ such that $f = Lg = QT^* g$, and $T^* g \in \bigoplus_{m \in M} V_m$, it follows that Q is surjective. By Theorem 4.3 we know that $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$.

On the contrary, suppose that $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-Riesz basis and $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$. By Theorem 4.6, T is homeomorphous, so is T^*. By Theorem 4.3 Q is surjective, therefore $L = QT^*$ is surjective. □

Theorem 4.9. *Let* $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ *be a continuous g-Riesz basis,* $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ *is a continuous g-Bessel net for* U *with respect to* $\{V_m : m \in M\}$. *Then* $\{\Gamma_m : m \in M\}$ *a continuous g-Riesz basis for* U *with respect to* $\{V_m : m \in M\}$ *if and only if* L *is invertible.*

Proof. We first suppose that L is invertible. Since $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a g-Riesz basis, $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a g-Bessel sequence for U with respect to $\{V_m : m \in M\}$, by Theorem 4.3 and Corollary 4.4 we can define the operators T, Q mentioned before and T is homeomorphous, hence T^* is also invertible. From the proof of Theorem 4.7 we know that $L = QT^*$. Since L is invertible, so is Q. By Theorem 4.6 we have that $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a g-Riesz basis for U with respect to $\{V_m : m \in M\}$.

Now we let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ and $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be two g-Riesz basis for U with respect to $\{V_m : m \in M\}$. By Theorem 4.6 both T, Q are invertible, so $L = QT^*$ is invertible too. □

5. Some Equalities for Continuous g-frames in Hilbert C^*-Modules

Some equalities for frames involving the real parts of some complex numbers have been established in [36]. These equalities generalized in [30] for g-frames in Hilbert C^*-modules. In this section, we generalize the equalities to a more general form which generalized before equalities and we deduce some equalities for g-frames in Hilbert C^*-modules to alternate dual g-frame.

In [37], the authors verified a longstanding conjecture of the signal processing community: a signal can be reconstructed without information about the phase. While working on efficient algorithms for signal reconstruction, the authors of [38] established the remarkable Parseval frame equality given below.

Theorem 5.1. *If* $\{f_j : j \in J\}$ *is a Parseval frame for Hilbert space* H, *then for any* $K \subset J$ *and* $f \in H$, *one has*

$$\sum_{j\in K} |\langle f, f_j\rangle|^2 - \left\| \sum_{j\in K} \langle f, f_j\rangle f_j \right\|^2 = \sum_{j\in K^c} |\langle f, f_j\rangle|^2 - \left\| \sum_{j\in K^c} \langle f, f_j\rangle f_j \right\|^2. \tag{5.1}$$

Theorem 5.1 was generalized to alternate dual frames [36]. If $\{f_j : j \in J\}$ is a frame, then frame $\{g_j : j \in J\}$ is called alternate dual frame of $\{f_j : j \in J\}$ if for any $f \in H$, $f = \sum_{j\in J} \langle f, g_j\rangle f_j$.

Theorem 5.2. *If* $\{f_j : j \in J\}$ *is a frame for Hilbert space* H *and* $\{g_j : j \in J\}$ *is an alternate dual frame of* $\{f_j : j \in J\}$, *then for any* $K \subset J$ *and* $f \in H$, *one has*

$$\begin{aligned} &\mathrm{Re}\left(\sum_{j\in K} \langle f, g_j\rangle \overline{\langle f, f_j\rangle} \right) - \left\| \sum_{j\in K} \langle f, g_j\rangle f_j \right\|^2 \\ &\quad = \mathrm{Re}\left(\sum_{j\in K^c} \langle f, g_j\rangle \overline{\langle f, f_j\rangle} \right) - \left\| \sum_{j\in K^c} \langle f, g_j\rangle f_j \right\|^2. \end{aligned} \tag{5.2}$$

Recently, Zhu and Wu in [39] generalized equality (5.2) to a more general form which does not involve the real parts of the complex numbers.

Theorem 5.3. *If $\{f_j : j \in J\}$ is a frame for Hilbert space H and $\{g_j : j \in J\}$ is an alternate dual frame of $\{f_j : j \in J\}$, then for any $K \subset J$ and $f \in H$, one has*

$$\left(\sum_{j\in K}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right) - \left\|\sum_{j\in K}\langle f, g_j\rangle f_j\right\|^2 = \overline{\left(\sum_{j\in K^c}\langle f, g_j\rangle\overline{\langle f, f_j\rangle}\right)} - \left\|\sum_{j\in K^c}\langle f, g_j\rangle f_j\right\|^2. \tag{5.3}$$

Now, we extended this equality to continuous g-frames and g-frames in Hilbert C^*-modules and Hilbert spaces. Let H be a Hilbert C^*-module. If $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous g-frame for U with respect to $\{V_m : m \in M\}$, then continuous g-frame $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is called alternate dual continuous g-frame of $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ if for any $f \in H$, $f = \int_M \langle f, \Gamma_m f\rangle \Lambda_m f d\mu(m)$.

Lemma 5.4 (see [30]). *Let H be a Hilbert C^*-module. If $P, Q \in \mathrm{End}_A^*(H, H)$ are two bounded A-linear operators in H and $P + Q = I_H$, then one has*

$$P - P^*P = Q^* - Q^*Q. \tag{5.4}$$

Now, we present main theorem of this section. In following, some result of this theorem for the discrete case will be present.

Theorem 5.5. *Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ and continuous g-frame $\{\Gamma_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is alternate dual continuous g-frame of $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in H$, one has*

$$\begin{aligned}&\int_K \langle f, \Gamma_m f\rangle\langle f, \Lambda_m f\rangle^* d\mu(m) - \left|\int_K \langle f, \Gamma_m f\rangle \Lambda_m f d\mu(m)\right|^2 \\ &\quad = \left(\int_{K^c} \langle f, \Gamma_m f\rangle\langle f, \Lambda_m f\rangle^* d\mu(m)\right)^* - \left|\sum_{j\in K^c} \langle f, \Gamma_m f\rangle \Lambda_m f d\mu(m)\right|^2.\end{aligned} \tag{5.5}$$

Proof. For any measurable subset $K \subset M$, let the operator U_K be defined for any $f \in H$ by $U_K f = \int_K \langle f, \Gamma_m f\rangle \Lambda_m f d\mu(m)$.

Then it is easy to prove that the operator U_K is well defined and the integral $\int_K \langle f, \Gamma_m f\rangle \Lambda_m f d\mu(m)$ it is finite. By definition alternate dual continuous g-frame $U_K + U_{K^c} = 1$.

Thus, by Lemma 5.4 we have

$$\begin{aligned}
&\int_K \langle f,\Gamma_m f\rangle\langle f,\Lambda_m f\rangle^* d\mu(m) - \left|\int_K \langle f,\Gamma_m f\rangle\Lambda_m f\, d\mu(m)\right|^2 \\
&\quad = \int_K \langle\langle f,\Gamma_m f\rangle\Lambda_m f, f\rangle d\mu(m) - \langle U_K f, U_K f\rangle \\
&\quad = \langle U_K f, f\rangle - \langle U_K^* U_K f, f\rangle = \langle U_{K^c}^* f, f\rangle - \langle U_{K^c}^* U_{K^c} f, f\rangle \\
&\quad = \langle f, U_{K^c} f\rangle - \langle U_{K^c} f, U_{K^c} f\rangle \\
&\quad = \left(\int_{K^c} \langle f,\Gamma_m f\rangle\langle f,\Lambda_m f\rangle^* d\mu(m)\right)^* - \left|\sum_{j\in K^c} \langle f,\Gamma_m f\rangle\Lambda_m f\, d\mu(m)\right|^2.
\end{aligned} \tag{5.6}$$

Hence the theorem holds. The proof is completed. □

Corollary 5.6. *Let $\{\Lambda_j \in \mathrm{End}_A^*(U,V_j) : j\in J\}$ be a discrete g-frame, for Hilbert C^*-module U with respect to $\{V_j : j\in J\}$, and discrete g-frame $\{\Gamma_j \in End_A^*(U,V_j) : j\in J\}$ is alternate dual discrete g-frame of $\{\Lambda_j \in End_A^*(U,V_j) : j\in J\}$, then for any subset $K\subset J$ and $f\in H$, one has*

$$\begin{aligned}
&\sum_{j\in K}\langle f,\Gamma_j f\rangle\langle f,\Lambda_j f\rangle^* - \left|\sum_{j\in K}\langle f,\Gamma_j f\rangle\Lambda_j f\right|^2 \\
&\quad = \left(\sum_{j\in K^c}\langle f,\Gamma_j f\rangle\langle f,\Lambda_j f\rangle^*\right)^* - \left|\sum_{j\in K^c}\langle f,\Gamma_j f\rangle\Lambda_j f\right|^2.
\end{aligned} \tag{5.7}$$

Corollary 5.7. *Let $\{\Lambda_j \in L(U,V_j) : j\in J\}$ be a g-frame, for Hilbert space U with respect to $\{V_j : j\in J\}$ and g-frame $\{\Gamma_j \in L(U,V_j) : j\in J\}$ is alternate dual g-frame of $\{\Lambda_j \in L(U,V_j) : j\in J\}$, then for any measurable subset $K\subset J$ and $f\in H$, one has*

$$\begin{aligned}
&\sum_{j\in K}\langle f,\Gamma_j f\rangle\overline{\langle f,\Lambda_j f\rangle} - \left|\sum_{j\in K}\langle f,\Gamma_j f\rangle\Lambda_j f\right|^2 \\
&\quad = \overline{\left(\sum_{j\in K^c}\langle f,\Gamma_j f\rangle\overline{\langle f,\Lambda_j f\rangle}\right)} - \left|\sum_{j\in K^c}\langle f,\Gamma_j f\rangle\Lambda_j f\right|^2.
\end{aligned} \tag{5.8}$$

The following results generalize the results in [30] in the case of continuous g-frames.

Lemma 5.8 (see [30]). *Let H be a Hilbert C^*-module. If T is a bounded, selfadjoint linear operator and satisfy $\langle Tf, f\rangle = 0$, for all $f\in H$, then $T = 0$.*

Lemma 5.9 (see [30]). *Let H be a Hilbert C^*-module. If $P, Q \in \mathrm{End}_A^*(H,H)$ are two bounded, selfadjoint A-linear operators in H and $P+Q=I_H$, then one has*

$$\langle Pf, f\rangle + |Qf|^2 = \langle Qf, f\rangle + |Pf|^2 \geq \frac{3}{4}\langle f, f\rangle. \tag{5.9}$$

Theorem 5.10. *Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$ and let $\{\widetilde{\Lambda}_m : m \in M\}$ be the canonical dual continuous g-frame of $\{\Lambda_m : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in U$, one has*

$$\begin{aligned}
&\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \int_M \left\langle \widetilde{\Lambda}_m S_{K^c} f, \widetilde{\Lambda}_m S_{K^c} f \right\rangle d\mu(m) \\
&\quad = \int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \int_M \left\langle \widetilde{\Lambda}_m S_K f, \widetilde{\Lambda}_m S_K f \right\rangle d\mu(m) \\
&\quad \geq \frac{3}{4}\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m).
\end{aligned} \tag{5.10}$$

Proof. Since S is an invertible, positive operator on U, and $S_K + S_{K^c} = S$, then $S^{-1/2}S_K S^{-1/2} + S^{-1/2}S_{K^c}S^{-1/2} = I_U$. Let $P = S^{-1/2}S_K S^{-1/2}$, $Q = S^{-1/2}S_{K^c}S^{-1/2}$. By Lemma 5.9, we obtain

$$\left\langle S^{-1/2}S_K S^{-1/2} f, f\right\rangle + \left|S^{-1/2}S_{K^c}S^{-1/2} f\right|^2 = \left\langle S^{-1/2}S_{K^c}S^{-1/2} f, f\right\rangle + \left|S^{-1/2}S_K S^{-1/2} f\right|^2 \geq \frac{3}{4}\langle f, f\rangle. \tag{5.11}$$

Replacing f by $S^{1/2}f$, then one has

$$\langle S_K f, f\rangle + \left\langle S^{-1}S_{K^c} f, S_{K^c} f\right\rangle = \langle S_{K^c} f, f\rangle + \left\langle S^{-1}S_K f, S_K f\right\rangle \geq \frac{3}{4}\langle Sf, f\rangle. \tag{5.12}$$

On the other hand, we have

$$\begin{aligned}
\langle S_K f, f\rangle &= \left\langle \int_K \Lambda_m^* \Lambda_m f\, d\mu(m), f\right\rangle = \int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m), \\
\int_K \left\langle \widetilde{\Lambda}_m f, \widetilde{\Lambda}_m f\right\rangle d\mu(m) &= \int_K \left\langle \Lambda_m S^{-1} f, \Lambda_m S^{-1} f\right\rangle d\mu(m) = \int_K \left\langle \Lambda_m^* \Lambda_m S^{-1} f, \Lambda_m S^{-1} f\right\rangle d\mu(m) \\
&= \left\langle \int_K \Lambda_m^* \Lambda_m S^{-1} f\, d\mu(m), \Lambda_m S^{-1} f\right\rangle = \left\langle SS^{-1} f, S^{-1} f\right\rangle \\
&= \left\langle f, S^{-1} f\right\rangle = \left\langle S^{-1} f, f\right\rangle.
\end{aligned} \tag{5.13}$$

Associating with (5.12) the proof is finished. □

Corollary 5.11. *Let $\{f_m \in H : m \in M\}$ be a continuous frame for Hilbert C^*-module H with canonical dual frame $\{\tilde{f}_m \in H : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in H$, one has*

$$\begin{aligned}\int_K \langle f, f_m\rangle\langle f_m, f\rangle d\mu(m) + \int_M \left\langle S_{K^c} f, \tilde{f}_j\right\rangle\left\langle \tilde{f}_j, S_{K^c} f\right\rangle d\mu(m) \\ = \int_{K^c} \langle f, f_m\rangle\langle f_m, f\rangle d\mu(m) + \int_M \left\langle S_K f, \tilde{f}_j\right\rangle\left\langle \tilde{f}_j, S_K f\right\rangle d\mu(m).\end{aligned} \tag{5.14}$$

Proof. For any $f \in U$, if we let $\Lambda_m f = \langle f, f_j\rangle$ in Theorem 5.10, then we get the conclusion. □

Theorem 5.12. *Let $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ be a continuous Parseval g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in U$, one has*

$$\begin{aligned}\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \left|\int_{K^c} \Lambda_m^* \Lambda_m f\, d\mu(m)\right|^2 \\ = \int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \left|\int_K \Lambda_m^* \Lambda_m f\, d\mu(m)\right|^2 \geq \frac{3}{4}\langle f, f\rangle.\end{aligned} \tag{5.15}$$

Proof. Since $\{\Lambda_m \in \mathrm{End}_A^*(U, V_m) : m \in M\}$ is a continuous Parseval g-frame in Hilbert C^*-module U with respect to $\{V_m : m \in M\}$, then for any $f \in U$, we have

$$\int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) = \langle f, f\rangle. \tag{5.16}$$

So

$$\langle Sf, f\rangle = \left\langle \int_M \Lambda_m^* \Lambda_m f\, d\mu(m), f\right\rangle = \int_M \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) = \langle f, f\rangle. \tag{5.17}$$

Hence for any $f \in U$, we have $\langle (S - I_U) f, f\rangle = 0$. Let $T = S - I_U$. Since S is bounded, selfadjoint, then $T^* = (S - I_U)^* = S^* - I_U^* = S - I_U = T$, so T is also bounded, selfadjoint. By Lemma 5.8, we have $T = 0$, namely, $S = I_U$, so $\tilde{\Lambda}_m = \Lambda_m S^{-1} = \Lambda_m$. From (5.16), then we have that: for any

measurable subset $K \subset M$ and $f \in U$,

$$\begin{aligned}\int_M \left\langle \tilde{\Lambda}_m S_K f, \tilde{\Lambda}_m S_K f \right\rangle d\mu(m) &= \int_M \langle \Lambda_m S_K f, \Lambda_m S_K f \rangle d\mu(m) \\ &= \langle S_K f, S_K f \rangle = \left| \int_K \Lambda_m^* \Lambda_m f d\mu(m) \right|^2, \\ \int_M \left\langle \tilde{\Lambda}_m S_{K^c} f, \tilde{\Lambda}_m S_{K^c} f \right\rangle d\mu(m) &= \int_M \langle \Lambda_m S_{K^c} f, \Lambda_m S_{K^c} f \rangle d\mu(m) \\ &= \langle S_{K^c} f, S_{K^c} f \rangle = \left| \int_{K^c} \Lambda_m^* \Lambda_m f d\mu(m) \right|^2.\end{aligned} \tag{5.18}$$

Combining (5.16) and Theorem 5.10, we get the result. □

Corollary 5.13. *Let $\{f_m \in H : m \in M\}$ be a continuous Parseval frame for Hilbert C^*-module H, then for any measurable subset $K \subset M$ and $f \in H$, one has*

$$\begin{aligned}&\int_K \langle f, f_m \rangle \langle f_m, f \rangle d\mu(m) + \left| \int_{K^c} \langle f, f_j \rangle f_j d\mu(m) \right|^2 \\ &\quad = \int_{K^c} \langle f, f_m \rangle \langle f_m, f \rangle d\mu(m) + \left| \int_K \langle f, f_j \rangle f_j d\mu(m) \right|^2.\end{aligned} \tag{5.19}$$

Corollary 5.14. *Let $\{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ be a continuous λ-tight g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in U$, one has*

$$\begin{aligned}&\lambda \int_K \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) + \left| \int_{K^c} \Lambda_m^* \Lambda_m f d\mu(m) \right|^2 \\ &\quad = \lambda \int_{K^c} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) + \left| \int_K \Lambda_m^* \Lambda_m f d\mu(m) \right|^2.\end{aligned} \tag{5.20}$$

Proof. Since $\{\Lambda_m : m \in M\}$ is a continuous λ-tight g-frame, then $\{\Lambda_m : m \in M\}$ is a continuous g-Parseval frame, by Theorem 5.12 we know that the conclusion holds. □

Corollary 5.15. *Let $\{f_m \in H : m \in M\}$ be a continuous λ-tight frame for Hilbert C^*-module H then for any measurable subset $K \subset M$ and $f \in H$, one has*

$$\begin{aligned}&\lambda \int_K \langle f, f_m \rangle \langle f_m, f \rangle d\mu(m) + \left| \int_{K^c} \langle f, f_j \rangle f_j d\mu(m) \right|^2 \\ &\quad = \lambda \int_{K^c} \langle f, f_m \rangle \langle f_m, f \rangle d\mu(m) + \left| \int_K \langle f, f_j \rangle f_j d\mu(m) \right|^2.\end{aligned} \tag{5.21}$$

Corollary 5.16. *Let $\{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ be a continuous λ-tight g-frame, for Hilbert C^*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $K, L \subset M, K \bigcap L = \phi$ and $f \in U$, one has*

$$\left|S_{K\bigcup L}f\right|^2 - \left|S_{K^c\setminus L}f\right|^2 = \left|S_K f\right|^2 - \left|S_{K^c} f\right|^2 + 2\lambda \int_L \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m). \tag{5.22}$$

Proof. Since for any $f \in U$, by Corollary 5.14, we get

$$\begin{aligned}
\left|S_{K\bigcup L}f\right|^2 - \left|S_{K^c\setminus L}f\right|^2 &= \lambda \int_{K\bigcup L} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) - \lambda \int_{K^c\setminus L} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \\
&= \lambda \int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \lambda \int_L \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \\
&\quad - \lambda \int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \lambda \int_L \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \\
&= \left\|S_K f\right\|^2 - \left\|S_{K^c} f\right\|^2 + 2\lambda \int_L \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m).
\end{aligned} \tag{5.23}$$

□

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