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Research Article

Composite Holomorphic Functions and Normal Families

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We study the normality of families of holomorphic functions. We prove the following result. Let $\alpha(z)$, $a_i(z)$, $i = 1, 2, \ldots, p$, be holomorphic functions and $\mathcal{F}$ a family of holomorphic functions in a domain D, $P(z, w) := (w - a_1(z))(w - a_2(z)) \cdots (w - a_p(z))$, $p \geq 2$. If $P_w \circ f(z)$ and $P_w \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z)$, $g(z) \in \mathcal{F}$ and one of the following conditions holds: (1) $P(z_0, z) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$; (2) there exists $z_0 \in D$ such that $P(z_0, z) - \alpha(z_0)$ has only one distinct zero and $\alpha(z)$ is nonconstant. Assume that β_0 is the zero of $P(z_0, z) - \alpha(z_0)$ and that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for all $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D. In particular, the result is a kind of generalization of the famous Montel's criterion. At the same time we fill a gap in the proof of Theorem 1.1 in our original paper (Wu et al., 2010).

1. Introduction and Main Result

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subseteq \mathbf{C}$, and let a be a finite complex value or function. We say that f and g share a CM (or IM) in D provided that $f - a$ and $g - a$ have the same zeros counting (or ignoring) multiplicity in D. It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory ([1, 2] or [3]).

The following theorem was proved by Chang et al. [4] in 2005 and [5] in 2009. It is an extension of a result obtained by Fang and Yuan [6] in 2000.

Theorem A. *Let $\alpha(z)$ be a nonconstant meromorphic function, $\mathcal{F}$ a family of holomorphic functions in a domain D, and $R(z)$ a rational function of degree at least 3. Suppose that $R \circ f(z) \neq \alpha(z)$ for each $f \in \mathcal{F}$ and all $z \in D$. Then $\mathcal{F}$ is normal in D.*

In the case where $\mathcal{F}$ is a family of holomorphic functions and $R(z)$ is a rational function of degree at least 2, the result was proved by Bergweiler [7] in 2004, by Hinchliffe [8] in 2003, and by Clifford [9] in 2005. It extends a result obtained by Fang and Yuan [6] in 2000, in which $R(z)$ is a polynomial of degree at least 2.

Recently, we [10] improved Theorem A in the case of $R(z)$ being polynomial.

Theorem B. *Let $\alpha(z)$ be a holomorphic function, $\mathcal{F}$ a family of meromorphic functions in a domain D, and $P(z)$ a polynomial of degree at least 3. If $P \circ f(z)$ and $P \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z), g(z) \in \mathcal{F}$ and one of the following conditions holds:*

(1) *$P(z) - \alpha(z_0)$ has at least three distinct zeros for any $z_0 \in D$;*

(2) *there exists $z_0 \in D$ such that $P(z) - \alpha(z_0)$ has at most two distinct zeros and $\alpha(z)$ is nonconstant. Assume that β_0 is the zero of $P(z) - \alpha(z_0)$ with multiplicity p and that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for all $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Remark 1.1. $\alpha(z)$ assuming the value $\alpha(z_0)$ with multiplicity k at $z_0 \in D$ means that $\alpha(z) - \alpha(z_0) = (z - z_0)^k \beta(z)$ or $\alpha(z) = (z - z_0)^{-k} \beta(z)$ and $\beta(z_0) \neq 0$.

In this paper, we extend Theorem B in the case of $\mathcal{F}$ being holomorphic and prove Theorem 1.2. In order to state it, we need some notations below. Set

$$P(z, w) := (w - a_1(z))(w - a_2(z)) \cdots (w - a_p(z)), \tag{1.1}$$

where $a_i(z),\ i = 1, 2, \ldots, p$, are holomorphic in D; $P_w \circ f(z) := P(z, f(z))$.

Theorem 1.2. *Let $\alpha(z)$ be a holomorphic function, $\mathcal{F}$ a family of holomorphic functions in a domain D, and $P(z, w)$ a polynomial in variable w as in (1.1) with $p \geq 2$. If $P_w \circ f(z)$ and $P_w \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z),\ g(z) \in \mathcal{F}$ and one of the following conditions holds:*

(1) *$P(z_0, z) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$;*

(2) *there exists $z_0 \in D$ such that $P(z_0, z) - \alpha(z_0)$ has only one distinct zero and $\alpha(z)$ is nonconstant. Assume that β_0 is the zero of $P(z_0, z) - \alpha(z_0)$ and that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for all $f(z) \in \mathcal{F}$,*

then $\mathcal{F}$ is normal in D.

Remark 1.3. Example 1.4 shows that $p = \deg_w P(z, w) \geq 2$ is best possible in Theorem 1.2.

Example 1.4. Let $P(z, w) = w + z^2$, $D = \{|z| < 1\}$, and let $\mathcal{F} := \{f_n\}$, where

$$f_n(z) := nz, \quad n = 1, 2, \ldots. \tag{1.2}$$

If $P_w(f_n(z)) = nz + z^2 = z^2$, then $z = 0$. Hence $P_w \circ f_n(z)$ and $P_w \circ f_m(z)$ share $\alpha(z) := z^2$ IM for each pair $f_n(z),\ f_m(z) \in \mathcal{F}$. Obviously, for $z_0 = 0 \in D$, we have that $P(0, z) - \alpha(0)$ has only one distinct zero $\beta_0 = 0$ and z^2 is nonconstant, noting that the multiplicities $l = 1$ and $k = 2$ of zeros of $f_n(z)$ and $\alpha(z)$ at 0, respectively, satisfy $k \neq lp$, for all $f_n(z) \in \mathcal{F}$. However, clearly, $\mathcal{F}$ is not normal at 0.

Remark 1.5. In Theorem 1.2 setting $\alpha(z)$ constant zero and $P(z,w)$ a polynomial in variable w that vanishes exactly on a finite set of holomorphic functions S, we obtain Corollary 1.6 which generalizes the famous Montel's criterion that a holomorphic family omitting 2 (or more) values is normal.

Corollary 1.6. *Let $\mathcal{F}$ be a family of holomorphic functions on a domain D. Let S be a finite set of holomorphic functions with at least 2 elements. If all functions in $\mathcal{F}$ share the set S ignoring multiplicities, that is, if for all $f(z)$, $g(z) \in \mathcal{F}$ and for all $z \in D$*

$$f(z) \in S \Leftrightarrow g(z) \in S, \tag{1.3}$$

then $\mathcal{F}$ is normal in D.

In 2010, we [11] obtained a normal criterion as follows.

Theorem C. *Let $\alpha(z)$ be an analytic function, $\mathcal{F}$ a family of analytic functions in a domain D, and $H(z)$ a transcendental entire function. If $H \circ f(z)$ and $H \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z), g(z) \in \mathcal{F}$ and one of the following conditions holds:*

(1) *$H(z) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$;*

(2) *$\alpha(z)$ is nonconstant and there exists $z_0 \in D$ such that $H(z) - \alpha(z_0) := (z-\beta_0)^p Q(z)$ has only one distinct zero β_0 and suppose that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for each $f(z) \in \mathcal{F}$, where $Q(\beta_0) \neq 0$;*

(3) *there exists a $z_0 \in D$ such that $H(z) - \alpha(z_0)$ has no zero and $\alpha(z)$ is nonconstant, then $\mathcal{F}$ is normal in D.*

However, there exists a gap in the proof of Theorem C which is Theorem 1.1 in our original paper [11]. We will give the correct proof after the proof of Theorem 1.1 in Section 3.

2. Preliminary Lemmas

In order to prove our result, we need the following lemmas. The first one extends a famous result by Zalcman [12] concerning normal families.

Lemma 2.1 (see [13]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc. Then $\mathcal{F}$ is not normal on the unit disc if and only if there exist*

(a) *a number $0 < r < 1$;*

(b) *points z_n with $|z_n| < r$;*

(c) *functions $f_n \in \mathcal{F}$;*

(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := f_n(z_n + \rho_n \zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$, whose order is at most 2.

Remark 2.2. If $\mathcal{F}$ is a family of holomorphic functions on the unit disc in Lemma 2.1, then $g(\zeta)$ is a nonconstant entire function.

Lemma 2.3 is very useful in the proof of our main theorem. In order to state them, we denote by $U(z_0, r)$ (or $U^0(z_0, r)$) the open (or punctured) disc of radius r around z_0, that is,

$$U(z_0, r) := \{z \in \mathbf{C} : |z - z_0| < r\},$$
$$U^0(z_0, r) := \{z \in \mathbf{C} : 0 < |z - z_0| < r\}. \tag{2.1}$$

Lemma 2.3 (see [9] or [14]). *Let $\{f_n(z)\}$ be a family of analytic functions in $U(z_0, r)$. Suppose that $\{f_n(z)\}$ is not normal at z_0 but is normal in $U^0(z_0, r)$. Then there exists a subsequence $\{f_{n_k}(z)\}$ of $\{f_n(z)\}$ and a sequence of points $\{z_{n_k}\}$ tending to z_0 such that $f_{n_k}(z_{n_k}) = 0$, but $\{f_{n_k}(z)\}$ tending to infinity locally uniformly on $U^0(z_0, r)$.*

3. Proof of the Results

Proof of Theorem 1.1. Without loss of generality, we assume that $D = \{z \in \mathbf{C}, |z| < 1\}$. Then we consider the following two cases.

Case 1. $P(z_0, z) - \alpha(z_0)$ has at least two distinct zeros a and b for any $z_0 \in D$.

Suppose that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z = 0$.

By Lemma 2.1, there exist $z_n \to 0$, $f_n \in \mathcal{F}$, $\rho_n \to 0^+$ such that

$$h_n(\xi) = f_n(z_n + \rho_n \xi) \longrightarrow h(\xi) \tag{3.1}$$

uniformly on any compact subset of $\mathbf{C}$, where $h(\xi)$ is a nonconstant entire function.

Hence

$$P_w \circ f_n(z_n + \rho_n \xi) - \alpha(z_n + \rho_n \xi) \longrightarrow P_w \circ h(\xi) - \alpha(0) \tag{3.2}$$

uniformly on any compact subset of $\mathbf{C}$.

We claim that $P_w \circ h(\xi) - \alpha(0)$ has at least two distinct zeros.

If $h(\xi)$ is a nonconstant polynomial, then both of the two equations of $h(\xi) = a$ and $h(\xi) = b$ have roots. So $P_w \circ h(\xi) - \alpha(0)$ has at least two distinct zeros.

If $h(\xi)$ is a transcendental entire function, then by Picard's theorem [3] at least one of the two equations $h(\xi) = a$ or $h(\xi) = b$ has infinitely many zeros.

Thus, the claim gives that there exist ξ_1 and ξ_2 such that

$$P_w \circ h(\xi_1) - \alpha(0) = 0, \qquad P_w \circ h(\xi_2) - \alpha(0) = 0 \quad (\xi_1 \neq \xi_2). \tag{3.3}$$

We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and $P_w \circ h(\xi) - \alpha(0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_1 and ξ_2, where

$$D_1 = \{\xi \in \mathbf{C}; |\xi - \xi_1| < \delta\}, \qquad D_2 = \{\xi \in \mathbf{C}; |\xi - \xi_2| < \delta\}. \tag{3.4}$$

By (3.2) and Hurwitz's theorem [14], for sufficiently large n there exist points $\xi_{1n} \in D_1$, $\xi_{2n} \in D_2$ such that

$$P_w \circ f_n(z_n + \rho_n \xi_{1n}) - \alpha(z_n + \rho_n \xi_{1n}) = 0,$$
$$P_w \circ f_n(z_n + \rho_n \xi_{2n}) - \alpha(z_n + \rho_n \xi_{2n}) = 0. \quad (3.5)$$

Noting that $P_w \circ f_m(z)$ and $P_w \circ f_n(z)$ share $\alpha(z)$ IM, it follows that

$$P_w \circ f_m(z_n + \rho_n \xi_{1n}) - \alpha(z_n + \rho_n \xi_{1n}) = 0,$$
$$P_w \circ f_m(z_n + \rho_n \xi_{2n}) - \alpha(z_n + \rho_n \xi_{2n}) = 0. \quad (3.6)$$

Taking $n \to \infty$, we obtain

$$P_w \circ f_m(0) - \alpha(0) = 0. \quad (3.7)$$

Since $P(z, w)$ is a polynomial in variable w, we know that the zeros of

$$P_w \circ f_m(\xi) - \alpha(\xi) \quad (3.8)$$

have no accumulation points except for finitely many f_m, and then

$$z_n + \rho_n \xi_{1n} = 0, \qquad z_n + \rho_n \xi_{2n} = 0, \quad (3.9)$$

or equivalently

$$\xi_{1n} = -\frac{z_n}{\rho_n}, \qquad \xi_{2n} = -\frac{z_n}{\rho_n}. \quad (3.10)$$

This contradicts the facts that $\xi_{1n} \in D_1$, $\xi_{2n} \in D_2$, $D_1 \cap D_2 = \emptyset$. So $\mathcal{F}$ is normal in Case 1.

Case 2. There exists $z_0 \in D$ such that $P(z_0, z) - \alpha(z_0)$ has only one distinct zero and $\alpha(z)$ is nonconstant. Assume that β_0 is the zero of $P(z_0, z) - \alpha(z_0)$ and that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for all $f(z) \in \mathcal{F}$.

We will prove that $\mathcal{F}$ is normal at any $z_0 \in D$. Without loss of generality, we can assume that $z_0 = 0$.

We can write $P(z_0, z) - \alpha(0) := (z - \beta_0)^p H(z)$, where $H(z)$ is a polynomial and $H(\beta_0) \neq 0$.

Since $\alpha(z)$ is nonconstant, there exists a punctured neighbourhood $U^0(0, r)$ such that

$$\alpha(z) \neq \alpha(0). \quad (3.11)$$

We claim that $\mathcal{F}$ is normal at $z_0' \in U^0(0, r)$ for small enough r. In fact, $P(z_0, z) - \alpha(z_0')$ has at least three distinct zeros. Then Case 1 tells us that this claim is true.

Next we prove that $\mathcal{F}$ is normal at $z = 0$. For any $\{f_n(z)\} \subset \mathcal{F}$, by the former claim, there exists a subsequence of functions $\{f_{n_m}(z)\}$ such that

$$f_{n_m}(z) - \beta_0 \longrightarrow G(z), \tag{3.12}$$

uniformly on a punctured disc $U^0(0,r)$.

For the sake of simplicity, we denote $\{f_{n_m}(z)\}$ by $\{f_n(z)\}$ in what follows.

Suppose that $\{f_n(z)\}$ is not normal at $z = 0$. We claim that there exists a sequence of points $z_n \in U(0,r)$ $(z_n \to 0)$ such that $P_w \circ f_n(z_n) - \alpha(z_n) = 0$.

By Lemma 2.3, we have that there exists a sequence of points $\{z'_n\}$ tending to 0 such that $f_n(z'_n) = \beta_0$ and $G(z) \equiv \infty$. Thus, $f_n(z) \to \infty$ on $U^0(0,r)$ and $f_n(z_{n'}) = \beta_0$ for a sequence of points $z'_n \to 0$. We know that if n is sufficiently large, then

$$\left|(f_n(z) - \beta) - f_n(z) + \beta_0\right| = \left|\beta - \beta_0\right| \leq \rho < \left|f_n(z) - \beta_0\right| \tag{3.13}$$

for $|z| = \delta$ and $\beta \in U(\beta_0, \rho)$. For large n, we also have $|z'_n| < \delta$, and thus we deduce from Rouché's theorem [15] that $f_n(z)$ takes the value $\beta \in U(\beta_0,\rho)$; that is, we have $f_n(U(0,\delta)) \supset U(\beta_0,\rho)$ for large n. Since also $f_n(\partial U(0,\delta)) \cap U(\beta_0,\rho) = \emptyset$ for large n, we find a component U of $f_n^{-1}(U(\beta_0,\rho))$ contained in $U(0,\delta)$ for such n. Moreover, U is a Jordan domain and $f_n : U \to U(\beta_0,\rho)$ is a proper map.

For $z \in \partial U$, we then have $f_n(z) \in \partial U(\beta_0,\rho)$ and thus $|P_w \circ f_n(z) - \alpha(0)| > \epsilon$. Hence

$$\left|P_w \circ f_n(z) - \alpha(z) - \left(P_w \circ f_n(z) - \alpha(0)\right)\right| = |\alpha(z) - \alpha(0)| < \epsilon < \left|P_w \circ f_n(z) - \alpha(0)\right| \tag{3.14}$$

for $z \in \partial U$. Noting that $P_w \circ f_n(z'_n) - \alpha(0) = 0$, Rouché's theorem [15] now shows that our claim holds.

By the claim and almost the same argument as in Case 1, we obtain that $z_n = 0$ for sufficiently large n. By $P(0,z) - \alpha(0) = (z - \beta_0)^p H(z)$, we have

$$\begin{gathered} P_w \circ f_n(z) - \alpha(z) = \left(f_n(z) - \beta_0\right)^p H\left(f_n(z)\right) - (\alpha(z) - \alpha(0)), \\ \left(f_n(0) - \beta_0\right)^p H\left(f_n(0)\right) = P_w \circ f_n(0) - \alpha(0) = 0. \end{gathered} \tag{3.15}$$

Hence

$$\begin{gathered} P_w \circ f_n(z) - \alpha(z) = z^k \left[z^{lp-k} h_n(z) - \beta(z)\right], \quad \text{if } lp > k, \\ P_w \circ f_n(z) - \alpha(z) = z^{lp} \left[h_n(z) - z^{k-lp}\beta(z)\right], \quad \text{if } lp < k, \end{gathered} \tag{3.16}$$

where $h_n(z)$, $\beta(z)$ are holomorphic functions and $h_n(0) \neq 0, \beta(0) \neq 0$.

Set $H_n(z) := z^{lp-k}h_n(z) - \beta(z)$, if $lp > k$, or $H_n(z) := h_n(z) - z^{k-lp}\beta(z)$, if $lp < k$. Obviously, $H_n(0) = -\beta(0) \neq 0$ or $H_n(0) = h_n(0) \neq 0$.

By the same argument as in the former case, we see that there exists a sequence of points $z_n^* \in U(0,r)$ such that $z_n^* \to 0$ and $H_n(z_n^*) = 0$. Obviously, $z_n^* \neq 0$ and

$$P_w \circ f_n(z_n^*) - \alpha(z_n^*) = 0. \tag{3.17}$$

Noting that $P_w \circ f_n(z)$ and $P_w \circ f_m(z)$ share $\alpha(z)$ IM, we obtain that

$$P_w \circ f_m(z_n^*) - \alpha(z_n^*) = 0 \tag{3.18}$$

for each m. Thus, taking $n \to \infty, P_w \circ f_m(0) - \alpha(0) = 0$. Since the zeros of $P_w \circ f_m(\xi) - \alpha(\xi)$ have no accumulation points except for finitely many f_m, we have $z_n^* = 0$. This contradicts our supposition.

Theorem 1.1 is proved completely. □

Proof of the Gap of Theorem C. In the original proof (3.7) becomes

$$H \circ f_m(0) - \alpha(0) = 0, \tag{3.19}$$

where $H(z)$ is a transcendental entire function.

The zeros of $H \circ f_m(\xi) - \alpha(\xi)$ maybe have accumulation points, for $H \circ f_m(\xi) - \alpha(\xi) \equiv 0$. However, we claim that $H \circ f_m(\xi) - \alpha(\xi) \equiv 0$ does not hold for infinitely many m.

In fact, if it is not true, by using Lemma 2.1 for these infinitely $f_m(z)$, then there exist a sequence of points $z_m \to 0$, $f_m \in \{f_n(z)\}$, and $\rho_m \to 0^+$ such that $g_m(\xi) := f_m(z_m + \rho_m \xi) \to g(\xi)$ uniformly on any compact subset of $\mathbf{C}$, where $g(\xi)$ is a nonconstant entire function. Thus, $H \circ g(\xi) - \alpha(0) \equiv 0$. This is impossible. □

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