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Research Article

On the Inversion of Bessel Ultrahyperbolic Kernel of Marcel Riesz

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We define the Bessel ultrahyperbolic Marcel Riesz operator on the function f by $U^{\alpha}(f) = R_{\alpha}^{B} * f$, where R_{α}^{B} is Bessel ultrahyperbolic kernel of Marcel Riesz, $\alpha \ldots \mathbb{C}$, the symbol $*$ designates as the convolution, and $f \in \mathcal{S}, \mathcal{S}$ is the Schwartz space of functions. Our purpose in this paper is to obtain the operator $E^{\alpha} = (U^{\alpha})^{-1}$ such that, if $U^{\alpha}(f) = \varphi$, then $E^{\alpha}\varphi = f$.

1. Introduction

The n-dimensional ultrahyperbolic operator $\square^k$ iterated k times is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \tag{1.1}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$ and k is a nonnegative integer.

Consider the linear differential equation in the form of

$$\square^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel'fand and Shilov [1] have first introduced the fundamental solution of (1.2), which is a complicated form. Later, Trione [2] has shown that the generalized function $R_{2k}^{H}(x)$, defined by (2.6) with $\gamma = 2k$, is the unique fundamental solution of (1.2) and Téllez [3] has also proved that $R^H_{2k}(x)$ exists only when $n = p + q$ with odd p.

Next, Kananthai [4] has first introduced the operator $\Diamond^k$ called the diamond operator iterated k times, which is defined by

$$\Diamond^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k, \tag{1.3}$$

where $n = p + q$ is the dimension of $\mathbb{R}^n$, for all $x = (x_1, x_2, \dots, x_n)$, and k is a nonnegative integer. The operator $\Diamond^k$ can be expressed in the form

$$\Diamond^k = \triangle^k \square^k = \square^k \triangle^k, \tag{1.4}$$

where $\square^k$ is defined by (1.1), and

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k \tag{1.5}$$

is the Laplace operator iterated k times. On finding the fundamental solution of this product, Kananthai uses the convolution of functions which are fundamental solutions of the operators $\square^k$ and $\triangle^k$. He found that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is the fundamental solution of the operator $\Diamond^k$, that is,

$$\Diamond^k\left((-1)^k R^e_{2k}(x) * R^H_{2k}(x)\right) = \delta(x), \tag{1.6}$$

where $R^H_{2k}(x)$ and $R^e_{2k}(x)$ are defined by (2.6) and (2.11), respectively with $\gamma = 2k$ and $\delta(x)$ is the Dirac delta distribution. The fundamental solution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is called the diamond kernel of Marcel Riesz. A wealth of some effective works on the diamond kernel of Marcel Riesz have been presented by Kananthai [5–10].

In 1978, Domínguez and Trione [11] have introduced the distributional functions $H_\alpha(P \pm i0, n)$ which are causal (anticausal) analogues of the elliptic kernel of Riesz [12]. Next, Cerutti and Trione [13] have defined the causal (anticausal) generalized Marcel Riesz potentials of order α, $\alpha \in \mathbb{C}$, by

$$R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi, \tag{1.7}$$

where $\varphi \in \mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions [14] and $H_\alpha(P \pm i0, n)$ is given by

$$H_\alpha(P \pm i0, n) = \frac{e^{\mp\alpha\pi i/2} e^{\pm q\pi i/2}\Gamma((n-\alpha)/2)(P \pm i0)^{(\alpha-n)/2}}{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}. \tag{1.8}$$

Here, P is defined by

$$P = P(x) = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{1.9}$$

where q is the number of negative terms of the quadratic form P. The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \to 0} \left(P \pm i\epsilon |x|^2 \right)^\lambda, \tag{1.10}$$

where $\epsilon > 0$, $\lambda \in \mathbb{C}$, and $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$; see [1]. They have also studied the inverse operator of R^α, denoted by $(R^\alpha)^{-1}$, such that, if $f = R^\alpha \varphi$, then $(R^\alpha)^{-1} f = \varphi$.

Later, Aguirre [15] has defined the ultrahyperbolic Marcel Riesz operator M^α of the function f by

$$M^\alpha(f) = R_\alpha^H * f, \tag{1.11}$$

where R_α^H is defined by (2.6) and $f \in \mathcal{S}$. He has also studied the operator $N^\alpha = (M^\alpha)^{-1}$ such that, if $M^\alpha(f) = \varphi$, then $N^\alpha \varphi = f$.

Let us consider the diamond kernel of Marcel Riesz $K_{\alpha,\beta}(x)$ introduced by Kananthai in [6], which is given by the convolution

$$K_{\alpha,\beta}(x) = R_\alpha^e * R_\beta^H, \tag{1.12}$$

where R_α^e is elliptic kernel defined by (2.11) and R_β^H is the ultrahyperbolic kernel defined by (2.6). Tellez and Kananthai [16] have proved that $K_{\alpha,\beta}(x)$ exists and is in the space of rapidly decreasing distributions. Moreover, they have also shown that the convolution of the distributional families $K_{\alpha,\beta}(x)$ relates to the diamond operator.

Later, Maneetus and Nonlaopon [17] have defined the diamond Marcel Riesz operator of order (α, β) of the function f by

$$M^{(\alpha,\beta)}(f) = K_{\alpha,\beta} * f, \tag{1.13}$$

where $K_{\alpha,\beta}$ is defined by (1.12), $\alpha, \beta \in \mathbb{C}$, and $f \in \mathcal{S}$. They have also studied the operator $N^{(\alpha,\beta)} = [M^{(\alpha,\beta)}]^{-1}$ such that, if $M^{(\alpha,\beta)}(f) = \varphi$, then $N^{(\alpha,\beta)}\varphi = f$.

In this paper, we define the Bessel ultrahyperbolic Marcel Riesz operator of order α of the function f by

$$U^\alpha(f) = R_\alpha^B * f, \tag{1.14}$$

where $\alpha \in \mathbb{C}$ and $f \in \mathcal{S}$, $\mathcal{S}$ is the Schwartz space of functions. Our aim in this paper is to obtain the operator $E^\alpha = (U^\alpha)^{-1}$ such that, if $U^\alpha(f) = \varphi$, then $E^\alpha \varphi = f$.

Before we proceed to our main theorem, the following definitions and concepts require some clarifications.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point in the n-dimensional Euclidean space $\mathbb{R}^n$. Let

$$u = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.1}$$

be the nondegenerated quadratic form, where $p + q = n$ is the dimension of $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : u > 0 \text{ and } x_i > 0\,(i = 1, 2, \ldots, p)\}$ be the interior of a forward cone, and let $\overline{\Gamma}_+$ denote its closure. For any complex number γ, we define

$$R_\gamma^B(x) = \begin{cases} \dfrac{u^{(\gamma-2|\nu|-n)/2}}{K_n^{|\nu|}(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.2}$$

where

$$K_n^{|\nu|}(\gamma) = \frac{\pi^{(n-1+2|\nu|)/2}\Gamma((2+\gamma-n-2|\nu|)/2)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((2+\gamma-p-2|\nu|)/2)\Gamma((p-\gamma)/2)}, \tag{2.3}$$

$2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$ and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$, see [18–20].

The function $R_\gamma^B(x)$ is called the Bessel ultrahyperbolic kernel and was introduced by Aguirre [21]. It is well known that $R_\gamma^B(x)$ is an ordinary function if $\mathrm{Re}(\gamma - 2|\nu|) \geq n$ and is a distribution of $(\gamma - 2|\nu|)$ if $\mathrm{Re}(\gamma - 2|\nu|) < n$. Let $\mathrm{supp} R_\gamma^B(x)$ denote the support of $R_\gamma^B(x)$ and suppose that $\mathrm{supp} R_\gamma^B(x) \subset \overline{\Gamma}_+$ (i.e., $\mathrm{supp} R_\gamma^B(x)$ is compact).

Letting $\gamma = 2k$ in (2.2) and (2.3), we obtain

$$R_{2k}^B(x) = \frac{u^{(2k-n-2|\nu|)/2}}{K_n(2k)}, \tag{2.4}$$

where

$$K_n(2k) = \frac{\pi^{(n-1+2|\nu|)/2}\Gamma((2+2k-n-2|\nu|)/2)\Gamma((1-2k)/2)\Gamma(2k)}{\Gamma((2+2k-p-2|\nu|)/2)\Gamma((p-2k)/2)}. \tag{2.5}$$

By putting $|\nu| = 0$ in (2.2) and (2.3), then formulae (2.2) and (2.3) reduce to

$$R_\gamma^H(x) = \begin{cases} \dfrac{u^{(\gamma-n)/2}}{K_n(\gamma)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.6}$$

$$K_n(\gamma) = \frac{\pi^{(n-1)/2}\Gamma((\gamma-n)/2+1)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((\gamma-p)/2+1)\Gamma((p-\gamma)/2)}. \tag{2.7}$$

The function $R_\gamma^H(x)$ is called the ultrahyperbolic kernel of Marcel Riesz and was introduced by Nozaki [22]. It is well known that $R_\gamma^H(x)$ is an ordinary function if $\mathrm{Re}(\gamma) \geq n$ and is a distribution of γ if $\mathrm{Re}(\gamma) < n$. Let $\mathrm{supp}R_\gamma^H(x)$ denote the support of $R_\gamma^H(x)$ and suppose that $\mathrm{supp}R_\gamma^H(x) \subset \overline{\Gamma}_+$ (i.e., $\mathrm{supp}R_\gamma^H(x)$ is compact).

By putting $p = 1$ in $R_{2k}^H(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.8}$$

we obtain

$$I_\gamma^H(x) = \frac{v^{(\gamma-n)/2}}{H_n(\gamma)} \tag{2.9}$$

and $v = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, where

$$H_n(\gamma) = \pi^{(n-2)/2}2^{\gamma-1}\Gamma\left(\frac{\gamma+2-n}{2}\right)\Gamma\left(\frac{\gamma}{2}\right). \tag{2.10}$$

The function $I_\gamma^H(x)$ is called the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$. The elliptic kernel of Marcel Riesz is defined by

$$R_\gamma^e(x) = \frac{\omega^{(\gamma-n)/2}}{W_n(\gamma)}, \tag{2.11}$$

where n is the dimension of $\mathbb{R}^n$, $\gamma \in \mathbb{C}$, and

$$W_n(\gamma) = \frac{\pi^{n/2}2^\gamma\Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}. \tag{2.12}$$

Note that $n = p + q$. By putting $q = 0$ (i.e., $n = p$) in (2.6) and (2.7), we can reduce $u^{(\gamma-n)/2}$ to $\omega_p^{(\gamma-p)/2}$, where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$, and reduce $K_n(\gamma)$ to

$$K_p(\gamma) = \frac{\pi^{(p-1)/2}\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((p-\gamma)/2)}. \tag{2.13}$$

Using Legendre's duplication formula

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.14}$$

and

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right) = \pi \sec(\pi z), \tag{2.15}$$

we obtain

$$K_p(\gamma) = \frac{1}{2}\sec\left(\frac{\gamma\pi}{2}\right)W_p(\gamma). \tag{2.16}$$

Thus, for $q = 0$, we have

$$R_\gamma^H(x) = \frac{u^{(\gamma-p)/2}}{K_p(\gamma)} = 2\cos\left(\frac{\gamma\pi}{2}\right)\frac{u^{(\gamma-p)/2}}{W_p(\gamma)} = 2\cos\left(\frac{\gamma\pi}{2}\right)R_\gamma^e(x). \tag{2.17}$$

In addition, if $\gamma = 2k$ for some nonnegative integer k, then

$$R_{2k}^H(x) = 2(-1)^k R_{2k}^e(x). \tag{2.18}$$

The proofs of Lemma 2.3 are given in [2].

Lemma 2.3. *The function $R_\alpha^H(x)$ has the following properties:*

(i) $R_0^H(x) = \delta(x)$;

(ii) $R_{-2k}^H(x) = \Box^k\delta(x)$;

(iii) $\Box^k R_\alpha^H(x) = R_{\alpha-2k}^H(x)$;

(iv) $\Box^k R_{2k}^H(x) = \delta(x)$.

Lemma 2.4. *If $|\nu| \neq 0$, then*

$$R_\gamma^B(x) = h_{\gamma,p,|\nu|}R_{\gamma-2|\nu|}^H(x), \tag{2.19}$$

where $R_\gamma^B(x)$ and $R_{\gamma-2|\nu|}^H(x)$ are defined by (2.2) *and* (2.6), *respectively, and*

$$h_{\gamma,p,|\nu|} = \frac{\Gamma((1-\gamma)/2+|\nu|)\Gamma(\gamma-2|\nu|)\Gamma((p-\gamma)/2)}{\pi^{|\nu|}\Gamma((p-\gamma)/2+|\nu|)\Gamma((1-\gamma)/2)\Gamma(\gamma)}. \tag{2.20}$$

Proof. We get (2.19) by computing directly from definition of $R_\gamma^B(x)$ and $R_{\gamma-2|\nu|}^H(x)$. □

The proof of the following lemma is given in [23].

Lemma 2.5 (the convolutions of $R_\alpha^H(x)$). *(i) If p is odd, then*

$$R_\alpha^H(x) * R_\beta^H(x) = R_{\alpha+\beta}^H(x) + A_{\alpha,\beta}, \tag{2.21}$$

where

$$A_{\alpha,\beta} = -\frac{i}{2}\frac{\sin(\alpha\pi/2)\sin(\beta\pi/2)}{\sin((\alpha+\beta)\pi/2)}\left[H^{+}_{\alpha+\beta} - H^{-}_{\alpha+\beta}\right], \tag{2.22}$$

$$H^{\pm}_{\alpha+\beta} = H_{\alpha+\beta}(P \pm i0, n) \tag{2.23}$$

as defined by (1.8).

(ii) If p is even, then

$$R^{H}_{\alpha}(x) * R^{H}_{\beta}(x) = B_{\alpha,\beta} R^{H}_{\alpha+\beta}(x), \tag{2.24}$$

where

$$B_{\alpha,\beta} = \frac{\cos(\alpha\pi/2)\cos(\beta\pi/2)}{\cos((\alpha+\beta)\pi/2)}. \tag{2.25}$$

Lemma 2.6 (the convolutions of $R^{B}_{\alpha}(x)$). *(i) If p is odd, then*

$$R^{B}_{\alpha}(x) * R^{B}_{\beta}(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|}\left(R^{H}_{\alpha+\beta-2(|\nu|+|\mu|)} + A_{\alpha-2|\nu|,\beta-2|\mu|}\right), \tag{2.26}$$

where $R^{H}_{\alpha}(x)$ and $A_{\alpha-2|\nu|,\beta-2|\mu|}$ are defined by (2.6) *and* (2.22), *respectively.*

(ii) If p is even, then

$$R^{B}_{\alpha}(x) * R^{B}_{\beta}(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|}\left(B_{\alpha-2|\nu|,\beta-2|\mu|} R^{H}_{\alpha+\beta-2(|\nu|+|\mu|)}\right), \tag{2.27}$$

where $B_{\alpha-2|\nu|,\beta-2|\mu|}$ is defined by (2.25).

The proof of this lemma can be easily seen from Lemmas 2.4, 2.5 and [23].

3. The Convolution $R^{B}_{\alpha}(x) * R^{B}_{\beta}(x)$ When $\beta = -\alpha$

We will now consider the property of $R^{B}_{\alpha}(x) * R^{B}_{\beta}(x)$ when $\beta = -\alpha$.

From (2.26) and (2.27), we immediately obtain the following properties.

(1) If p is odd and q is even, then

$$R^{B}_{\alpha}(x) * R^{B}_{\beta}(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|}\left(R^{H}_{\alpha+\beta-2(|\nu|+|\mu|)} + A_{\alpha-2|\nu|,\beta-2|\mu|}\right), \tag{3.1}$$

where $R^{H}_{\alpha}(x)$ and $A_{\alpha-2|\nu|,\beta-2|\mu|}$ are defined by (2.6) and (2.22), respectively.

(2) If p and q are both odd, then

$$R_\alpha^B(x) * R_\beta^B(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|} \left(R^H_{\alpha+\beta-2(|\nu|+|\mu|)} + A_{\alpha-2|\nu|,\beta-2|\mu|} \right). \tag{3.2}$$

(3) If p is even and q is odd, then

$$R_\alpha^B(x) * R_\beta^B(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|} \left(\frac{\cos((\alpha-2|\nu|)\pi/2)\cdot\cos((\beta-2|\mu|)\pi/2)}{\cos((\alpha+\beta-2(|\nu|+|\mu|))\pi/2)} R^H_{\alpha+\beta-2(|\nu|+|\mu|)} \right). \tag{3.3}$$

(4) If p and q are both even, then

$$R_\alpha^B(x) * R_\beta^B(x) = h_{\alpha,p,|\nu|} h_{\beta,p,|\mu|} \left(\frac{\cos((\alpha-2|\nu|)\pi/2)\cdot\cos((\beta-2|\mu|)\pi/2)}{\cos((\alpha+\beta-2(|\nu|+|\mu|))\pi/2)} R^H_{\alpha+\beta-2(|\nu|+|\mu|)} \right). \tag{3.4}$$

Moreover, it follows from (2.22) that

$$\begin{aligned} A_{\alpha-2|\nu|,-(\alpha-2|\nu|)} &= \lim_{\beta-2|\mu|\to-(\alpha-2|\nu|)} A_{\alpha-2|\nu|,\beta-2|\mu|} \\ &= -\frac{i}{2}\lim_{\gamma\to 0} \frac{\sin((\alpha-2|\nu|)\pi/2)\sin((\gamma-(\alpha-2|\nu|))\pi/2)}{\sin(\gamma\pi/2)} \left[H_\gamma^+ - H_\gamma^-\right] \\ &= -\frac{i}{2}\lim_{\gamma\to 0} \frac{\sin((\alpha-2|\nu|)\pi/2)\sin((\gamma-(\alpha-2|\nu|))\pi/2)}{\sin(\gamma\pi/2)} \cdot \lim_{\gamma\to 0}\left[H_\gamma^+ - H_\gamma^-\right], \end{aligned} \tag{3.5}$$

where $\gamma = \alpha+\beta-2(|\nu|+|\mu|)$.

On the other hand, using (2.23) and (1.8), we have

$$\begin{aligned} \lim_{\gamma\to 0}\left[H_\gamma^+ - H_\gamma^-\right] &= \frac{\Gamma(n/2)}{\pi^{n/2}} \left[\lim_{\gamma\to 0} e^{-\gamma\pi i/2} e^{q\pi i/2} \frac{(P+i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \right. \\ &\qquad \left. - \lim_{\gamma\to 0} e^{\gamma\pi i/2} e^{-q\pi i/2} \frac{(P-i0)^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \right] \\ &= \frac{\Gamma(n/2)}{\pi^{n/2}} \left[\lim_{\gamma\to 0} e^{-\gamma\pi i/2} e^{q\pi i/2} \cdot \frac{\operatorname{Res}_{\beta=-n/2}(P+i0)^\beta}{\operatorname{Res}_{\beta=-n/2}\Gamma(\beta+n/2)} \right. \\ &\qquad \left. - \lim_{\gamma\to 0} e^{\gamma\pi i/2} e^{-q\pi i/2} \cdot \frac{\operatorname{Res}_{\beta=-n/2}(P-i0)^\beta}{\operatorname{Res}_{\beta=-n/2}\Gamma(\beta+n/2)} \right]. \end{aligned} \tag{3.6}$$

Now, taking n as an odd integer, we obtain

$$\operatorname*{Res}_{\lambda=-n/2-k}(P \pm i0)^{\lambda} = \frac{e^{\pm q\pi i/2}\pi^{n/2}}{2^{2k}k!\Gamma(n/2+k)}\square^{k}\delta(x), \tag{3.7}$$

where $\square^k$ is defined by (1.1), $p+q=n$, and k is nonnegative integer; see [24, 25]. If p and q are both even, then

$$\operatorname*{Res}_{\lambda=-n/2-k}(P \pm i0)^{\lambda} = \frac{e^{\pm q\pi i/2}\pi^{n/2}}{2^{2k}k!\Gamma(n/2+k)}\square^{k}\delta(x). \tag{3.8}$$

Nevertheless, if p and q are both odd, then

$$\operatorname*{Res}_{\lambda=-n/2-k}(P \pm i0)^{\lambda} = 0. \tag{3.9}$$

Therefore, we have

$$\begin{aligned}\lim_{\gamma\to 0}\left[H_{\gamma}^{+} - H_{\gamma}^{-}\right] &= \frac{\Gamma(n/2)}{\pi^{n/2}}\cdot\frac{\pi^{n/2}}{\Gamma(n/2)}\left[\lim_{\gamma\to 0}e^{-\gamma\pi i/2} - \lim_{\gamma\to 0}e^{\gamma\pi i/2}\right]\delta(x)\\ &= \lim_{\gamma\to 0}\left[-2i\sin(\gamma\pi/2)\right]\delta(x).\end{aligned} \tag{3.10}$$

From (3.6) and (3.9), we have

$$\lim_{\gamma\to 0}\left[H_{\gamma}^{+} - H_{\gamma}^{-}\right] = 0 \tag{3.11}$$

if p and q are both odd (n even).

Applying (3.10) and (3.11) into (3.5), we have

$$\begin{aligned}A_{\alpha-2|\nu|,-\alpha+2|\nu|} &= -\frac{i}{2}\lim_{\gamma\to 0}\frac{\sin((\alpha-2|\nu|)\pi/2)\sin((\gamma-(\alpha-2|\nu|))\pi/2)}{\sin(\gamma\pi/2)}\cdot\lim_{\gamma\to 0}\left[-2i\sin(\gamma\pi/2)\right]\delta(x)\\ &= \sin^{2}((\alpha-2|\nu|)\pi/2)\delta(x)\end{aligned} \tag{3.12}$$

if p is odd and q is even and

$$A_{\alpha-2|\nu|,-\alpha+2|\nu|} = 0 \tag{3.13}$$

if p and q are both odd.

From (3.1)—(3.4) and using Lemmas 2.3, and 2.6 and formulae (3.12) and (3.13), if p is odd and q is even, then we obtain

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left(R_0^H + A_{\alpha-2|\nu|,-\alpha+2|\nu|} \right) \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left[\delta(x) + \sin^2((\alpha-2|\nu|)\pi/2)\delta(x) \right] \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left[1 + \sin^2((\alpha-2|\nu|)\pi/2) \right] \delta(x). \end{aligned} \tag{3.14}$$

If p and q are both odd, then

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \left(R_0^H + A_{\alpha-2|\nu|,-\alpha+2|\nu|} \right) \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \delta(x). \end{aligned} \tag{3.15}$$

If p is even and q is odd, then

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \frac{\cos((\alpha-2|\nu|)\pi/2)\cos((-\alpha+2|\nu|)\pi/2)}{\cos((\alpha-\alpha-2|\nu|+2|\nu|)\pi/2)} R_0^H \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \cos^2((\alpha-2|\nu|)\pi/2)\delta(x). \end{aligned} \tag{3.16}$$

Finally, if p and q are both even, then

$$\begin{aligned} R_\alpha^B(x) * R_{-\alpha}^B(x) &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \frac{\cos((\alpha-2|\nu|)\pi/2)\cos((-\alpha+2|\nu|)\pi/2)}{\cos((\alpha-\alpha-2|\nu|+2|\nu|)\pi/2)} R_0^H \\ &= h_{\alpha,p,|\nu|} h_{-\alpha,p,|\nu|} \cos^2((\alpha-2|\nu|)\pi/2)\delta(x). \end{aligned} \tag{3.17}$$

4. The Main Theorem

Let $M^\alpha(f)$ be the Bessel ultrahyperbolic Marcel Riesz operator of order α of the function f, which is defined by

$$U^\alpha(f) = R_\alpha^B * f, \tag{4.1}$$

where R_α^B is defined by (2.2), $\alpha \in \mathbb{C}$, and $f \in \mathcal{S}$.

Recall that our objective is to obtain the operator $E^\alpha = (U^\alpha)^{-1}$ such that, if $U^\alpha(f) = \varphi$, then $E^\alpha \varphi = f$ for all $\alpha \in \mathbb{C}$.

We are now ready to state our main theorem.

Theorem 4.1. *If* $U^\alpha(f) = \varphi$ *(where* $U^\alpha(f)$ *is defined by* (4.1) *and* $f \in \mathcal{S}$*), then* $E^\alpha\varphi = f$ *such that*

$$E^\alpha = (U^\alpha)^{-1} = \begin{cases} \dfrac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}R^B_{-\alpha} & \text{if } p \text{ is odd and } q \text{ is even,} \\ \dfrac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}R^B_{-\alpha} & \text{if } p \text{ and } q \text{ are both odd,} \\ \dfrac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)R^B_{-\alpha} & \text{if } p \text{ is even with} (\alpha-2|\nu|)/2 \neq 2s+1 \end{cases} \tag{4.2}$$

for any nonnegative integer s.

Proof. By (4.1), we have

$$U^\alpha(f) = R^B_\alpha * f = \varphi, \tag{4.3}$$

where R^B_α is defined by (2.2), $\alpha \in \mathbb{C}$, and $f \in \mathcal{S}$. If p is odd and q is even, then, in view of (3.14), we obtain

$$\begin{aligned} &\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}R^B_{-\alpha} * \left(R^B_\alpha * f\right) \\ &\quad= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}\left(R^B_{-\alpha} * R^B_\alpha\right) * f \\ &\quad= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1} \\ &\qquad\times\left\{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]\delta(x)\right\} * f \\ &\quad= \delta * f = f. \end{aligned} \tag{4.4}$$

Hence,

$$\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left[1+\sin^2((\alpha-2|\nu|)\pi/2)\right]^{-1}R^B_{-\alpha} = (U^\alpha)^{-1} = \left(R^B_\alpha\right)^{-1} \tag{4.5}$$

for all $\alpha \in \mathbb{C}$.

Similarly, if both p and q are odd, then, by (3.15), we obtain

$$\begin{aligned} \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}R^B_{-\alpha} * \left(R^B_\alpha * f\right) &= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\left(R^B_{-\alpha} * R^B_\alpha\right) * f \\ &= \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}\delta(x) * f \\ &= f. \end{aligned} \tag{4.6}$$

Hence,

$$\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}R^B_{-\alpha} = (U^\alpha)^{-1} = \left(R^B_\alpha\right)^{-1} \tag{4.7}$$

for all $\alpha \in \mathbb{C}$.

Finally, if p is even, then, by (3.16) and (3.17), we have

$$\begin{aligned}
&\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)R^B_{-\alpha} * \left(R^B_\alpha * f\right) \\
&\quad = \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)\left(R^B_{-\alpha} * R^B_\alpha\right) * f \\
&\quad = \frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)\left\{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}\cos^2((\alpha-2|\nu|)\pi/2)\delta(x)\right\} * f \\
&\quad = \delta * f = f,
\end{aligned} \tag{4.8}$$

provided that $(\alpha-2|\nu|)/2 \neq 2s+1$ for any nonnegative integer s.

Hence,

$$\frac{1}{h_{\alpha,p,|\nu|}h_{-\alpha,p,|\nu|}}\sec^2((\alpha-2|\nu|)\pi/2)R^B_{-\alpha} = (U^\alpha)^{-1} = \left(R^B_\alpha\right)^{-1} \tag{4.9}$$

for all $\alpha \in \mathbb{C}$ with $(\alpha-2|\nu|)/2 \neq 2s+1$ for any nonnegative integer s.

In this conclusion, formulae (4.5), (4.7), and (4.9) are the desired results, and this completes the proof. □

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