Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 463721, 22 pages
doi:10.1155/2011/463721

Research Article

Stability and Hopf Bifurcation in a Diffusive Predator-Prey System with Beddington-DeAngelis Functional Response and Time Delay

Yuzhen Bai and Xiaopeng Zhang

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

Correspondence should be addressed to Yuzhen Bai, baiyu99@126.com

Received 18 June 2011; Accepted 16 August 2011

Academic Editor: Marcia Federson

This paper is concerned with a diffusive predator-prey system with Beddington-DeAngelis functional response and delay effect. By analyzing the distribution of the eigenvalues, the stability of the positive equilibrium and the existence of spatially homogeneous and spatially inhomogeneous periodic solutions are investigated. Also, it is shown that the small diffusion can affect the Hopf bifurcations. Finally, the direction and stability of Hopf bifurcations are determined by normal form theory and center manifold reduction for partial functional differential equations.

1. Introduction

In this paper, we will study the stability and Hopf bifurcations of a diffusive predator-prey system with Beddington-DeAngelis functional response and delay effect as follows:

$$\begin{gathered} u_t = d_1 \Delta u(t,x) + u(t,x)(1 - u(t-\tau,x)) - sP(u,v), \quad t > 0,\ x \in \Omega, \\ v_t = d_2 \Delta v(t,x) + rP(u,v) - dv(t,x), \quad t > 0,\ x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, \quad t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0, \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where u and v denote the population densities of prey and predator species at time t and space x, respectively; the positive constants d_1 and d_2 represent the diffusion coefficients of prey and predator species, respectively; $s > 0$ (s is called the capturing rate) and $r > 0$ (r is

called the conversion rate) represent the strength of the relative effect of the interaction on the two species; d denotes the death rate of predator species; $P(u,v) = uv/(m+u+nv)$ is the Beddington-DeAngelis functional response function with m and n are positive numbers; $\tau \geq 0$ denotes the generation time of the prey species; Ω is a bounded domain in R^N (N is any positive integer) with a smooth boundary $\partial\Omega$; Δ is the Laplacian operator on Ω; ν is the outward normal to $\partial\Omega$; homogeneous Neumann boundary conditions reflect the situation where the population cannot move across the boundary of the domain.

System (1.1) includes the models which have been discussed by many researchers; for examples, when $\tau = 0$, the models were considered in [1, 2]; if $d_1 = d_2 = 0$ and $\tau = 0$, it was discussed in [3]; if $P(u,v) = 1$, it was discussed in [4]. Moreover, when $\tau = 0$ and $P(u,v) = u^2v/(1+u^2)$, system (1.1) can be transformed into Narcisa Apreutesei's model (see [5]).

There has been an increasing interest in the study of diffusive predator-prey system (see [1, 2, 4, 6–14] and references therein) with functional response. As is known to all, the Beddington-DeAngelis functional response, proposed by Beddington [6] and DeAngelis et al. [8], is more general than those the above authors considered, and it has been studied extensively in the literature [1–3, 7, 14–16]. However, to the authors' best knowledge, few researches have been done on the diffusive predator-prey system with Beddington-DeAngelis functional response and time delay.

The aim of this paper is to extend and develop the work in [1, 2]; that is, we will study the stability and Hopf bifurcation of a diffusive predator-prey system with Beddington-DeAngelis functional response and delay. The system we consider here is more general than the system in [1, 2].

The rest of the paper is organized as follows. In Section 2, we analyze the distribution of the roots of the characteristic equation and give various conditions on the stability of a positive constant steady state and the existence of Hopf bifurcation. In Section 3, we discuss the effect of diffusion on the Hopf bifurcation. In Section 4, by applying the normal form theory and the center manifold reduction of partial functional differential equations by Wu [17], an explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions is given.

2. Analysis of the Characteristic Equations

In this section, by choosing the delay τ as the bifurcation parameter and analyzing the associated characteristic equation of (1.1) at the positive constant steady state, we investigate the stability of the positive constant steady state of (1.1) and obtain the conditions under which (1.1) undergoes Hopf bifurcation.

It can be seen that homogeneous Neumann boundary conditions imposed on (1.1) lead to $E_1(0,0)$ and $E_2(1,0)$, always being two boundary equilibria for any feasible parameters, and (1.1) always having a unique positive constant steady state $E(u^*,v^*)$ provided that the condition

$(A1)$ $0 < d < (ru^*/(u^*+m))$, hold, where

$$u^* = \frac{-(sr-nr-sd)+\sqrt{(sr-nr-sd)^2+4smndr}}{2rn}, \qquad v^* = \frac{(r-d)u^*-dm}{dn}. \tag{2.1}$$

Under $(A1)$, let $\overline{u} = u - u^*$, $\overline{v} = v - v^*$ and drop the bars for simplicity of notations, then (1.1) can be transformed into the following equivalent system:

$$\begin{aligned} u_t &= d_1 \Delta u(t,x) + (u+u^*)(1-(u(t-\tau,x)+u^*)) - \frac{s(u+u^*)(v+v^*)}{m+(u+u^*)+n(v+v^*)}, \\ v_t &= d_2 \Delta v(t,x) + \frac{r(u+u^*)(v+v^*)}{m+(u+u^*)+n(v+v^*)} - d(v+v^*). \end{aligned} \tag{2.2}$$

Let $P(u,v) = uv/(m+u+nv)$. By $u^*(1-u^*) - (su^*v^*/(m+u^*+nv^*)) = 0$ and $-dv^* + (ru^*v^*/(m+u^*+nv^*)) = 0$ (2.2) becomes

$$\begin{aligned} u_t &= d_1 \Delta u(t,x) + (1-u^*-a_1)u(t) - u^*u(t-\tau,x) - a_2 v - uu(t-\tau,x) - f(u,v), \\ v_t &= d_2 \Delta v(t,x) + b_1 u(t) + (b_2 - d)v(t) + g(u,v), \end{aligned} \tag{2.3}$$

where $a_1 = sP_{10}(u^*,v^*)$, $a_2 = sP_{01}(u^*,v^*)$, $b_1 = rP_{10}(u^*,v^*)$, $b_2 = rP_{01}(u^*,v^*)$, and

$$f(u,v) = s\sum_{i+j=2}^{3} \frac{P_{ij}}{i!j!} u^i v^j + \text{h.o.t.}, \qquad g(u,v) = r\sum_{i+j=2}^{3} \frac{P_{ij}}{i!j!} u^i v^j + \text{h.o.t.}, \tag{2.4}$$

where $\partial^{i+j} P(u,v)/\partial u^i \partial v^j|_{(u,v)=(u^*,v^*)}$ denoted by P_{ij} and h.o.t. for shorthand of "higher order terms."

Denote $u_1(t) = u(t,\cdot)$, $u_2(t) = v(t,\cdot)$, and $U = (u_1,u_2)^T$. Then (2.3) can be transformed into an abstract differential equation in the phase space $\ell = C([-\tau,0],X)$,

$$\dot{U}(t) = D\Delta U(t) + L(U_t) + F(U_t), \tag{2.5}$$

with

$$\begin{aligned} D &= \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad L(\phi) = \begin{pmatrix} 1-u^*-a_1 & -a_2 \\ b_1 & b_2-d \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} -u^* & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-\tau) \\ \phi_2(-\tau) \end{pmatrix}, \\ F(\phi) &= \begin{pmatrix} -\phi_1(0)\phi_1(-\tau) - s\displaystyle\sum_{i+j=2}^{3} \frac{P_{ij}}{i!j!}\phi_1^i(0)\phi_2^j(0) \\ r\displaystyle\sum_{i+j=2}^{3} \frac{P_{ij}}{i!j!}\phi_1^i(0)\phi_2^j(0) \end{pmatrix}, \end{aligned} \tag{2.6}$$

where $\phi = (\phi_1,\phi_2)^T \in \ell$.

The linearization of (2.5) is given by

$$\dot{U}(t) = D\Delta U(t) + L(U_t), \tag{2.7}$$

and its characteristic equation is

$$\lambda y - D\Delta y - L\left(e^{\lambda} y\right) = 0, \tag{2.8}$$

where $y \in \mathrm{dom}(\Delta)$ and $y \neq 0$, $\mathrm{dom}(\Delta) \subset X$.

From the properties of the Laplacian operator defined on the bounded domain, the operator Δ on X has the eigenvalues $-k^2$ with the relative eigenfunctions

$$\beta_k^1 = \begin{pmatrix} \gamma_k \\ 0 \end{pmatrix}, \qquad \beta_k^2 = \begin{pmatrix} 0 \\ \gamma_k \end{pmatrix}, \tag{2.9}$$

where $\gamma_k = \cos kx$, $k \in N_0 = \{0, 1, 2, \ldots\}$. Clearly, $(\beta_k^1, \beta_k^2)_{k=0}^{\infty}$ construct a basis of the phase space X and therefore any element y in X can be expanded as Fourier series in the following form:

$$y = \sum_{k=0}^{\infty}\left(\langle y, \beta_k^1\rangle \beta_k^1 + \langle y, \beta_k^2\rangle \beta_k^2\right) = \sum_{k=0}^{\infty}\left(\beta_k^1, \beta_k^2\right)\begin{pmatrix} \langle y, \beta_k^1\rangle \\ \langle y, \beta_k^2\rangle \end{pmatrix}. \tag{2.10}$$

Some simple computations show that

$$L\left(\phi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}\right) = L(\phi)^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix}, \quad k \in N_0. \tag{2.11}$$

From (2.10)-(2.11), (2.8) is equivalent to

$$\sum_{k=0}^{\infty}\left(\langle y, \beta_k^1\rangle, \langle y, \beta_k^2\rangle\right)\left[\left(\lambda I_2 + Dk^2\right) - \begin{pmatrix} -u^* e^{-\lambda\tau} + 1 - u^* - a_1 & -a_2 \\ b_1 & b_2 - d \end{pmatrix}\right]\begin{pmatrix} \beta_k^1 \\ \beta_k^2 \end{pmatrix} = 0. \tag{2.12}$$

Assume that

$(A2)$ $1 - u^* - a_1 < 0$, $b_2 - d < 0$.

Let $-a_3 = 1 - u^* - a_1$, $-b_3 = b_2 - d$, $p = a_3 + b_3$, $r = a_3 b_3 + a_2 b_1$, $s = u^*$, $q = b_3 u^*$, then we conclude that the characteristic equation (2.8) is equivalent to the sequence of the characteristic equations:

$$\lambda^2 + \left(d_1 k^2 + d_2 k^2 + p\right)\lambda + d_1 d_2 k^4 + b_3 d_1 k^2 + a_3 d_2 k^2 + r + \left(s\lambda + d_2 s k^2 + q\right)e^{-\lambda\tau} = 0. \tag{2.13}$$

Obviously, for all $k \in N_0$, $\lambda = 0$ is not a root of (2.13).

Equation (2.13) with $\tau = 0$ is equivalent to the following quadratic equations:

$$\lambda^2 + \left(d_1 k^2 + d_2 k^2 + p + s\right)\lambda + d_1 d_2 k^4 + b_3 d_1 k^2 + a_3 d_2 k^2 + r + d_2 s k^2 + q = 0. \tag{2.14}$$

Let λ_1 and λ_2 be the two roots of (2.14). Then, for all $k \in N_0$,

$$\begin{gathered}\lambda_1 + \lambda_2 = -\left(d_1k^2 + d_2k^2 + p + s\right) < 0,\\ \lambda_1\lambda_2 = d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r + d_2sk^2 + q > 0.\end{gathered} \tag{2.15}$$

Therefore, we have the following Lemma.

Lemma 2.1. *Assume that (A1) and (A2) hold. Then the equilibrium $E(u^*, v^*)$ of* (1.1) *with $\tau = 0$ is asymptotically stable.*

Assume that

(A3) $s^2 - p^2 + 2r < 0,\ r > q$;

(A4) $b_3d_1 + a_3d_2 - d_2s \geq 0$, or $(b_3d_1 + a_3d_2 - d_2s)^2 < 4d_1d_2(r - q)$, if $b_3d_1 + a_3d_2 - d_2s < 0$.

Theorem 2.2. *If (A1)–(A4) hold, then all roots of* (2.13) *have negative real parts for all $\tau \geq 0$. Furthermore, the equilibrium $E(u^*, v^*)$ of the system* (1.1) *is asymptotically stable for all $\tau \geq 0$.*

Proof. Let $\lambda = i\omega$ $(\omega > 0)$ be a root of the characteristic equation (2.13). Then ω satisfies the following equation for some $k \in N_0$:

$$\begin{aligned}&-\omega^2 + \left(d_1k^2 + d_2k^2 + p\right)i\omega + d_1d_2k^4 + b_3d_1k^2\\ &\quad + a_3d_2k^2 + r + \left(s\omega i + d_2sk^2 + q\right)(\cos\omega\tau - i\sin\omega\tau) = 0.\end{aligned} \tag{2.16}$$

Separating the real and imaginary parts of (2.16) leads to

$$\begin{gathered}-\omega^2 + d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r + s\omega\sin\omega\tau + \left(d_2sk^2 + q\right)\cos\omega\tau = 0,\\ \left(d_1k^2 + d_2k^2 + p\right)\omega + s\omega\cos\omega\tau - \left(d_2sk^2 + q\right)\sin\omega\tau = 0,\end{gathered} \tag{2.17}$$

which implies that

$$\begin{aligned}&\omega^4 + \left[\left(d_1^2 + d_2^2\right)k^4 + 2a_3d_1k^2 + 2b_3d_2k^2 - \left(s^2 - p^2 + 2r\right)\right]\omega^2\\ &\quad + \left[\left(d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r\right)^2 - \left(d_2sk^2 + q\right)^2\right] = 0.\end{aligned} \tag{2.18}$$

Let $z = \omega^2$, then (2.18) can be rewritten into the following form:

$$\begin{aligned}&z^2 + \left[\left(d_1^2 + d_2^2\right)k^4 + 2a_3d_1k^2 + 2b_3d_2k^2 - \left(s^2 - p^2 + 2r\right)\right]z\\ &\quad + \left[\left(d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r\right)^2 - \left(d_2sk^2 + q\right)^2\right] = 0.\end{aligned} \tag{2.19}$$

By *(A3)* and *(A4)*, for all $k \in N_0$, we have

$$\begin{gathered} -\left(d_1^2 + d_2^2\right)k^4 - 2a_3d_1k^2 - 2b_3d_2k^2 + \left(s^2 - p^2 + 2r\right) < 0, \\ d_1d_2k^4 + (b_3d_1 + a_3d_2 - d_2s)k^2 + r - q > 0, \end{gathered} \tag{2.20}$$

which imply that (2.19) has no positive roots. Hence, the characteristic equation (2.13) has no purely imaginary roots. By Lemma 2.1 and the theorem proved by Ruan and Wei [18], all roots of (2.13) have negative real parts. □

Notice that (2.13) with $k = 0$ is the characteristic equation of the linearization of (1.1) corresponding system without diffusion (ordinary differential equations, ODEs) at the positive equilibrium. And it has been considered under the condition:

(B1) $r < q$.

It is easy to get that when

$$\tau = \tau_j^0 = \frac{1}{\omega_+^0}\arccos\left\{\frac{\left(q\left(\left(\omega_+^0\right)^2 - r\right) - ps\left(\omega_+^0\right)^2\right)}{s^2\left(\omega_+^0\right)^2 + q^2} + 2j\pi\right\}, \quad (j = 0, 1, 2, \ldots), \tag{2.21}$$

Equation (2.13) with $k = 0$ has simple imaginary roots $\pm i\omega_+^0$, and $\operatorname{Re}\left(d\lambda/d\tau\right)_{\tau=\tau_j^0} > 0$, where $\lambda(\tau)$ is the root of (2.13) with $k = 0$ satisfying $\lambda(\tau_j^0) = i\omega_+^0$, and

$$\omega_+^0 = \frac{\sqrt{2}}{2}\left[s^2 - p^2 + 2r + \sqrt{\left(s^2 - p^2 + 2r\right)^2 - 4\left(r^2 - q^2\right)}\right]^{1/2}, \tag{2.22}$$

and $\tau_0^0 = \min\{\tau_j^0,\ j \in \{0, 1, 2, \ldots\}\}$. Assume that

(B2) $d_1^2 + d_2^2 + 2a_3d_1 + 2b_3d_2 > s^2 - p^2 + 2r,\ d_1d_2 + b_3d_1 + a_3d_2 - d_2s + r - q > 0.$

We have the following result.

Theorem 2.3. *Assume that (A1), (A2), (B1), and (B2) are satisfied. Then for* $\tau = \tau_j^0$ $(j = 0, 1, 2, \ldots)$, (2.13) *has a pair of simple imaginary roots* $\pm i\omega_+^0$, *and all roots of* (2.13), *except* $\pm i\omega_+^0$, *have no zero real parts. Moreover, all the roots of* (2.13) *with* $\tau = \tau_j^0$, *except* $\pm i\omega_+^0$, *have negative real parts.*

Proof. Let $\lambda = i\omega_1$ $(\omega_1 > 0)$ be a root of (2.13) with $k \geq 1$. By the same way in Theorem 2.2, we can obtain

$$\begin{gathered} \omega_1^4 + \left[\left(d_1^2 + d_2^2\right)k^4 + 2a_3d_1k^2 + 2b_3d_2k^2 - \left(s^2 - p^2 + 2r\right)\right]\omega_1^2 \\ + \left[\left(d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r\right)^2 - \left(d_2sk^2 + q\right)^2\right] = 0, \end{gathered} \tag{2.23}$$

for all $k \geq 1$.

Set $z = \omega_1^2$, then

$$\begin{aligned} z^2 &+ \left[\left(d_1^2 + d_2^2\right)k^4 + 2a_3d_1k^2 + 2b_3d_2k^2 - \left(s^2 - p^2 + 2r\right)\right]z \\ &+ \left[\left(d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r\right)^2 - \left(d_2sk^2 + q\right)^2\right] = 0. \end{aligned} \tag{2.24}$$

Let z_1 and z_2 be the roots of (2.24) with $k \geq 1$. We know that if $z_1 + z_2 < 0$ and $z_1z_2 > 0$, then (2.13) with $k \geq 1$ has no purely imaginary roots.

By $(B2)$, it follows that, for $\forall k \geq 1$,

$$\begin{aligned} &-\left(d_1^2 + d_2^2\right)k^4 - 2a_3d_1k^2 - 2b_3d_2k^2 + \left(s^2 - p^2 + 2r\right) \\ &\quad \leq -\left(d_1^2 + d_2^2 + 2a_3d_1 + 2b_3d_2\right) + \left(s^2 - p^2 + 2r\right) < 0, \\ &d_1d_2k^4 + (b_3d_1 + a_3d_2 - d_2s)k^2 + r - q \geq d_1d_2 + b_3d_1 + a_3d_2 - d_2s + r - q > 0. \end{aligned} \tag{2.25}$$

Therefore, (2.13) with $k \geq 1$ have no purely imaginary roots.

Summarizing the above results and combining Theorem 2.3, we have the following theorem on the stability of the positive equilibrium $E(u^*, v^*)$ of system (1.1) and the existence of Hopf bifurcation at $E(u^*, v^*)$. □

Theorem 2.4. *Assume that (A1), (A2), (B1), and (B2) hold. For system* (1.1)*, the following statements are true:*

(I) *If $\tau \in [0, \tau_0^0)$, then the equilibrium point $E(u^*, v^*)$ is asymptotically stable;*

(II) *If $\tau > \tau_0^0$, then the equilibrium point $E(u^*, v^*)$ is unstable;*

(III) *$\tau = \tau_j^0$ $(j = 0, 1, 2, \ldots)$ are Hopf bifurcation values of system* (1.1)*, and these Hopf bifurcations are all spatially homogeneous.*

By the same way in Theorem 2.2, let $\lambda = i\omega$ $(\omega > 0)$ be a root of the characteristic equation (2.13), then ω satisfies the following equation:

$$\begin{aligned} \omega^4 &+ \left[\left(d_1^2 + d_2^2\right)k^4 + 2a_3d_1k^2 + 2b_3d_2k^2 - \left(s^2 - p^2 + 2r\right)\right]\omega^2 \\ &+ \left[\left(d_1d_2k^4 + b_3d_1k^2 + a_3d_2k^2 + r\right)^2 - \left(d_2sk^2 + q\right)^2\right] = 0, \end{aligned} \tag{2.26}$$

for $k \in N_0$.

Now, we make the following assumptions. For a certain $k_0 = \{1, 2, \ldots\}$,

$(C1)$ $d_1d_2 + b_3d_1 + a_3d_2 - d_2s + r - q > 0$;

$(C2)$ $((s^2 - p^2 + 2r)/k_0^2) < d_1^2 + d_2^2 + 2a_3d_1 + 2b_3d_2 < s^2 - p^2 + 2r$;

$(C3)$ $[(d_1^2+d_2^2)k^4 + 2a_3d_1k^2 + 2b_3d_2k^2 - (s^2 - p^2 + 2r)]^2, -4[(d_1d_2^4 + b_3d_1k^2 + a_3d_2k^2 + r)^2 - (d_2s^2 + q)^2] \geq 0,\ k \in N_0 \setminus \{k_0\}$.

Under the assumptions $(C1)$ and $(C2)$, (2.26) with $k = k_0$ has only a positive solution $\omega_+^{k_0}$,

$$\omega_+^{k_0} = \frac{\sqrt{2}}{2}\sqrt{B + \sqrt{B^2 - 4\left[\left(d_1 d_2 k_0^4 + a_3 d_2 k_0^2 + b_3 d_1 k_0^2 + r\right)^2 - \left(d_2 k_0^2 s + q\right)^2\right]}}, \tag{2.27}$$

where $B = s^2 - p^2 + 2r - (d_1^2 + d_2^2)k_0^4 - 2a_3 d_1 k_0^2 - 2b_3 d_2 k_0^2 < 0$.

Set $z = \omega^2$, then (2.26) can be transformed into the following equation:

$$\begin{aligned} z^2 &+ \left[\left(d_1^2 + d_2^2\right)k^4 + 2a_3 d_1 k^2 + 2b_3 d_2 k^2 - \left(s^2 - p^2 + 2r\right)\right]z \\ &+ \left[\left(d_1 d_2 k^4 + b_3 d_1 k^2 + a_3 d_2 k^2 + r\right)^2 - \left(d_2 s k^2 + q\right)^2\right] = 0. \end{aligned} \tag{2.28}$$

Let z_1 and z_2 be the roots of (2.28). If the assumptions *(C1)–(C3)* hold, we have

$$\begin{aligned} z_1 + z_2 &= -\left(d_1^2 + d_2^2\right)k^4 - 2a_3 d_1 k^2 - 2b_3 d_2 k^2 + \left(s^2 - p^2 + 2r\right) \le s^2 - p^2 + 2r < 0, \\ z_1 z_2 &= \left(d_1 d_2 k^4 + a_3 d_2 k^2 + b_3 d_1 k^2 + r\right)^2 - \left(d_2 k^2 s + q\right)^2 > 0, \end{aligned} \tag{2.29}$$

for $k \in N_0 \setminus \{k_0\}$.

Therefore, (2.13) with $k \in N_0 \setminus \{k_0\}$ has no solutions with zero real parts.

In addition, similar to the proof of Theorem 2.2, we have

$$\begin{aligned} -\left(\omega_+^{k_0}\right)^2 + d_1 d_2 k_0^4 + b_3 d_1 k_0^2 + a_3 d_2 k_0^2 + r + s\omega_+^{k_0} \sin \omega_+^{k_0}\tau + \left(d_2 s k_0^2 + q\right) \cos \omega_+^{k_0}\tau = 0, \\ \left(d_1 k_0^2 + d_2 k_0^2 + p\right)\omega_+^{k_0} + s\omega_+^{k_0} \cos \omega_+^{k_0}\tau - \left(d_2 s k_0^2 + q\right) \sin \omega_+^{k_0}\tau = 0 \end{aligned} \tag{2.30}$$

which implies that

$$\begin{aligned} \sin\left(\omega_+^{k_0}\tau\right) &= \frac{\left[\left(\omega_+^{k_0}\right)^2 - d_1 d_2 k_0^4 - b_3 d_1 k_0^2 - a_3 d_2 k_0^2 - r\right] s\omega_+^{k_0}}{\left(d_2 s k_0^2 + q\right)^2 + s^2\left(\omega_+^{k_0}\right)^2} \\ &\quad + \frac{\left(d_1 k_0^2 + d_2 k_0^2 + p\right)\left(d_2 s k_0^2 + q\right)\omega_+^{k_0}}{\left(d_2 s k_0^2 + q\right)^2 + s^2\left(\omega_+^{k_0}\right)^2} \triangleq F\left(\omega_+^{k_0}\right), \end{aligned}$$

$$\cos\left(\omega_+^{k_0}\tau\right) = \frac{\left[\left(\omega_+^{k_0}\right)^2 - d_1d_2k_0^4 - b_3d_1k_0^2 - a_3d_2k_0^2 - r\right]\left(d_2sk_0^2+q\right)}{\left(d_2sk_0^2+q\right)^2 + s^2\left(\omega_+^{k_0}\right)^2}$$
$$- \frac{\left(d_1k_0^2+d_2k_0^2+p\right)s\left(\omega_+^{k_0}\right)^2}{\left(d_2sk_0^2+q\right)^2 + s^2\left(\omega_+^{k_0}\right)^2} \triangleq E\left(\omega_+^{k_0}\right). \tag{2.31}$$

Define

$$\tau_j^{k_0} = \begin{cases} \dfrac{1}{\omega_+^{k_0}}\left(\arccos\left(E\left(\omega_+^{k_0}\right)\right) + 2j\pi\right) & \text{if } F\left(\omega_+^{k_0}\right) \geq 0, \\ \dfrac{1}{\omega_+^{k_0}}\left(2\pi - \arccos\left(E\left(\omega_+^{k_0}\right)\right) + 2j\pi\right) & \text{if } F\left(\omega_+^{k_0}\right) < 0, \end{cases} \quad \text{for } j \in \{0,1,2,\ldots\}. \tag{2.32}$$

From the above analysis, we have the following Theorem.

Theorem 2.5. *Assume that (A1), (A2), and (C1)–(C3) hold. Then for* $\tau = \tau_j^{k_0}$ $(j = 0,1,2,\cdots)$, (2.13) *has a pair of simple imaginary roots* $\pm i\omega_+^{k_0}$, *and all roots of* (2.13), *except* $\pm i\omega_+^{k_0}$, *have no zero real parts. Moreover, all the roots of* (2.13) *with* $\tau = \tau_j^{k_0}$, *except* $\pm i\omega_+^{k_0}$, *have negative real parts.*

Let $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$ be the root of (2.13) near $\tau = \tau_j^{k_0}$ satisfying

$$\alpha\left(\tau_j^{k_0}\right) = 0, \quad \beta\left(\tau_j^{k_0}\right) = \omega_+^{k_0}, \quad j = 0,1,2,\ldots, \tag{2.33}$$

where $\omega_+^{k_0}$ and $\tau_j^{k_0}$ are given by (2.27) and (2.32), respectively. Then we have the following transversality condition.

Lemma 2.6. *Assume that (A1), (A2), and (C1)–(C3) hold. Then*

$$\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\right\}_{\tau=\tau_j^{k_0}} > 0. \tag{2.34}$$

Proof. Differentiating the two sides of (2.13) with respect to τ yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{\left(2\lambda + d_1k_0^2 + d_2k_0^2 + p\right)e^{\lambda\tau} + s}{s\lambda^2 + s\lambda d_2k_0^2 + \lambda q} - \frac{\tau}{\lambda}. \tag{2.35}$$

From (2.30), we have

$$\begin{aligned}
\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_j^{k_0}} &= \operatorname{Re}\left[\frac{(2\lambda + d_1k_0^2 + d_2k_0^2 + p)e^{\lambda\tau} + s}{s\lambda^2 + s\lambda d_2k_0^2 + \lambda q} - \frac{\tau}{\lambda}\right]_{\tau=\tau_j^{k_0}} \\
&= \frac{2\left(\omega_+^{k_0}\right)^2 - s^2 + p^2 - 2r + \left(d_1^2 + d_2^2\right)k_0^4 + 2a_3d_1k_0^2 + 2b_3d_2k_0^2}{s^2\left(\omega_+^{k_0}\right)^2 + \left(sd_2k_0^2 + q\right)^2} \\
&= \frac{2\left(\omega_+^{k_0}\right)^2 - B}{s^2\left(\omega_+^{k_0}\right)^2 + \left(sd_2k_0^2 + q\right)^2},
\end{aligned} \tag{2.36}$$

according to (2.27), and then

$$\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\right\}_{\tau=\tau_j^{k_0}} = \left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_j^{k_0}} > 0. \tag{2.37}$$

Applying Lemma 2.6 and Theorem 2.5, we draw the following conclusions. □

Theorem 2.7. *Assume that (A1), (A2), and (C1)–(C3) hold. For system* (1.1)*, the following statements are true:*

(I) *If $\tau \in [0, \tau_0^{k_0})$, then the equilibrium point $E(u^*, v^*)$ is asymptotically stable;*

(II) *If $\tau > \tau_0^{k_0}$, then the equilibrium point $E(u^*, v^*)$ is unstable;*

(III) *$\tau = \tau_j^{k_0}$ $(j = 0, 1, 2, \ldots)$ are Hopf bifurcation values of system* (1.1)*, and these Hopf bifurcations are all spatially inhomogeneous.*

3. The Effect of Diffusion on Hopf Bifurcations

In the previous section, we have studied the Hopf bifurcations from the positive constant steady-state $E(u^*, v^*)$ of (1.1) when τ crosses through the critical value τ_j^k $(k = 0, k_0;\ j = 1, 2, 3, \ldots)$ and have the following conclusions.

(I) If $(B2)$ holds, then system (1.1) and the corresponding system without diffusion (ODEs) have the same Hopf bifurcations, containing the existence and properties of Hopf bifurcations. In this case, the diffusion has no effect on the Hopf bifurcations of ODEs.

(II) If $(B2)$ does not hold, then system (1.1) and ODEs have the different Hopf bifurcations. In this case, the diffusion has the effect on the Hopf bifurcations of ODEs.

According to Theorems 2.4 and 2.7, system (1.1) undergoes Hopf bifurcations under the different conditions. Comparing the conditions of Theorems 2.4 and 2.7, we have the following conclusions.

(I) When system (1.1) undergoes spatially homogeneous Hopf bifurcation, diffusion coefficients satisfy the condition:

$$d_1^2 + d_2^2 + 2a_3d_1 + 2b_3d_2 > s^2 - p^2 + 2r, \tag{3.1}$$

and in this case, system (1.1) and ODEs have the same properties of Hopf bifurcation.

(II) When system (1.1) undergoes spatially inhomogeneous Hopf bifurcation, diffusion coefficients satisfy the condition

$$d_1^2 + d_2^2 + 2a_3d_1 + 2b_3d_2 < s^2 - p^2 + 2r. \tag{3.2}$$

and in this case, system (1.1) and ODEs have the different properties of Hopf bifurcation.

Summarizing the above results, we can obtain the conclusion. The big diffusion has no effect on the Hopf bifurcation of system (1.1), the small diffusion can make system (1.1) undergo the spatially inhomogeneous Hopf bifurcation.

4. Direction of Hopf Bifurcation and Stability of the Bifurcating Periodic Orbits

In this section, we will study the directions, stability, and the period of bifurcating periodic solutions by using normal formal theory and center manifold theorem of partial functional differential equations presented in [17]. For fixed $j \in \{0, 1, 2, \ldots\}$, we denote τ_j^k by $\tilde{\tau}$. Let

$$\begin{gathered} \tilde{u}(t,x) = u(\tau t, x), \qquad \tilde{v}(t,x) = v(\tau t, x), \qquad \tau = \mu + \tau_k, \\ u_1(t) = u(t,\cdot), \qquad u_2(t) = v(t,\cdot), \qquad U = (u_1, u_2)^T, \end{gathered} \tag{4.1}$$

and drop the tilde for the sake of simplicity. Then system (1.1) can be written as

$$\dot{U}(t) = \tilde{\tau} D \Delta U(t) + L(\tilde{\tau})(U_t) + f(U_t, \mu), \tag{4.2}$$

where $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $L(\mu)(\cdot) : \ell \to X$, and $f : \ell \times R \to X$ are given, respectively, by

$$L(\mu)(\phi) = (\mu + \tilde{\tau})\left(\begin{pmatrix} 1 - u^* - a_1 & -a_2 \\ b_1 & b_2 - d \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + \begin{pmatrix} -u^* & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}\right), \tag{4.3}$$

$$f(\phi, \mu) = (\mu + \tilde{\tau})\begin{pmatrix} -\phi_1(0)\phi_1(-1) - s\displaystyle\sum_{i+j=2}^{3} \frac{P_{ij}}{i!j!}\phi_1^i(0)\phi_2^j(0) \\ r\displaystyle\sum_{i+j=2}^{3} \frac{P_{ij}}{i!j!}\phi_1^i(0)\phi_2^j(0) \end{pmatrix} + \text{h.o.t.}, \tag{4.4}$$

for $\phi = (\phi_1, \phi_2)^T \in \ell$, where h.o.t. denotes *high order terms*.

Consider the linear equation

$$\dot{U}(t) = \tilde{\tau} D \Delta U(t) + L(\tilde{\tau})(U_t). \tag{4.5}$$

From the discussion of Theorems 2.3 and 2.5 in Section 2, we know that the origin (0,0) is an equilibrium of (4.2), and for $\tau = \tilde{\tau}$, the characteristic equation of (4.5) has a pair of simple purely imaginary eigenvalues $\bigwedge_0 = \{i\omega_+^k\tilde{\tau}, -i\omega_+^k\tilde{\tau}\}$, $(k = 0, k_0)$.

Consider the ordinary functional differential equation

$$\dot{X}(t) = -\tilde{\tau} D k^2 X(t) + L(\tilde{\tau})(X_t). \tag{4.6}$$

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \tilde{\tau})$ $(-1 \le \theta \le 0)$, whose entry is of bounded variation such that

$$-\tilde{\tau} D k^2 \phi(0) + L(\tilde{\tau})(\phi) = \int_{-1}^{0} d[\eta(\theta, \tilde{\tau})]\phi(\theta), \tag{4.7}$$

for $\phi \in C([-1,0], R^2)$. In fact, we can choose

$$\eta(\theta, \tilde{\tau}) = \begin{cases} \tilde{\tau}\begin{pmatrix} -d_1k^2 + 1 - u^* - a_1 & -a_2 \\ b_1 & -d_2k^2 + b_2 - d \end{pmatrix}, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ \tilde{\tau}\begin{pmatrix} -u^* & 0 \\ 0 & 0 \end{pmatrix}, & \theta = -1. \end{cases} \tag{4.8}$$

Let $A(\tilde{\tau})$ denote the infinitesimal generators of the semigroup induced by the solutions of (4.6) and A^* be the formal adjoint of $A(\tilde{\tau})$ under the bilinear pairing

$$(\psi, \phi) = \psi(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \psi(\xi - \theta) d[\eta(\theta, \tilde{\tau})]\phi(\xi) d\xi, \tag{4.9}$$

for $\phi \in C([-1,0], R^2)$, $\psi \in (C[0,1], R^2)$. Then $A(\tilde{\tau})$ and A^* are a pair of adjoint operators. From the discussion in Section 2, we know that $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_+^k\tilde{\tau}$, and they are also eigenvalues of A^*. Let P and Q be the center subspaces, that is, the generalized eigenspace of $A(\tilde{\tau})$ and A^* associated with $\bigwedge_0$, respectively. Then Q is the adjoint space of P and $\dim P = \dim Q = 2$, see [17].

Direct computations give the following results.

Lemma 4.1. *Let*

$$C = -\frac{i\omega_+^k + d_1k^2 - 1 + u^* + a_1 + u^*e^{-i\omega_+^k\tilde{\tau}}}{a_2}, \qquad D = \frac{-i\omega_+^k + d_1k^2 - 1 + u^* + a_1 + u^*e^{-i\omega_+^k\tilde{\tau}}}{b_1}. \tag{4.10}$$

Then,

$$p_1(\theta) = e^{i\omega_+^k\tilde{\tau}}(1, C)^T, \quad p_2(\theta) = \overline{p_1(\theta)}, \quad -1 \le \theta \le 0 \tag{4.11}$$

is a basis of P with $\bigwedge_0$, and

$$q_1(s) = (1, D)e^{-i\omega_+^k\tilde{\tau}}, \quad q_2(s) = \overline{q_1(s)}, \quad 0 \le s \le 1 \tag{4.12}$$

is a basis of Q with $\bigwedge_0$.

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with

$$\begin{aligned} \Phi_1(\theta) &= \frac{p_1(\theta) + p_2(\theta)}{2} = \begin{pmatrix} \operatorname{Re}\left\{e^{i\omega_+^k\tilde{\tau}}\theta\right\} \\ \operatorname{Re}\left\{Ce^{i\omega_+^k\tilde{\tau}}\theta\right\} \end{pmatrix}, \\ \Phi_2(\theta) &= \frac{p_1(\theta) - p_2(\theta)}{2i} = \begin{pmatrix} \operatorname{Im}\left\{e^{i\omega_+^k\tilde{\tau}}\theta\right\} \\ \operatorname{Im}\left\{Ce^{i\omega_+^k\tilde{\tau}}\theta\right\} \end{pmatrix}, \end{aligned} \tag{4.13}$$

for $\theta \in [-1, 0]$, and

$$\begin{aligned} \Psi_1^*(s) &= \frac{q_1(s) + q_2(s)}{2} = \begin{pmatrix} \operatorname{Re}\left\{e^{-i\omega_+^k\tau_k}\theta\right\} \\ \operatorname{Re}\left\{De^{-i\omega_+^k\tilde{\tau}}\theta\right\} \end{pmatrix}, \\ \Psi_2^*(s) &= \frac{q_1(s) - q_2(s)}{2i} = \begin{pmatrix} \operatorname{Im}\left\{e^{-i\omega_+^k\tilde{\tau}}\theta\right\} \\ \operatorname{Im}\left\{De^{-i\omega_+^k\tilde{\tau}}\theta\right\} \end{pmatrix}, \end{aligned} \tag{4.14}$$

for $s \in [0, 1]$.

Define

$$\Psi = (\Psi_1, \Psi_2)^T = \begin{pmatrix} (\Psi_1^*, \Phi_1) & (\Psi_1^*, \Phi_2) \\ (\Psi_2^*, \Phi_1) & (\Psi_2^*, \Phi_2) \end{pmatrix}^{-1} \Psi^*, \tag{4.15}$$

then Ψ construct a new basis for Q and $(\Psi, \Phi) = I_2$, see [17]. In addition, $f_k \triangleq (\beta_k^1, \beta_k^2)$, where

$$\beta_k^1 = \begin{pmatrix} \cos kx \\ 0 \end{pmatrix}, \qquad \beta_k^1 = \begin{pmatrix} 0 \\ \cos kx \end{pmatrix}. \tag{4.16}$$

Let $c \cdot f_k$ be defined by $c \cdot f_k = c_1\beta_k^1 + c_2\beta_k^2,\ c = (c_1, c_2)^T \in C([-1,0], X)$.

Then the center subspace of linear equation (4.5) is given by $P_{\mathrm{CN}}\ell$, where

$$P_{\mathrm{CN}}\ell(\phi) = \Phi(\Psi, \langle \phi, f_k \rangle) \cdot f_k, \quad \phi \in \ell, \tag{4.17}$$

and $\ell = P_{\mathrm{CN}}\ell \oplus P_s\ell$, and $P_s\ell$ denotes the complement subspace of $P_{\mathrm{CN}}\ell$ in ℓ.

Let $A_{\tilde{\tau}}$ be the infinitesimal generator induced by the solution of (4.5). Then (4.2) can be rewritten as the abstract form

$$\dot{U}_t = A_{\tilde{\tau}} U_t + R(\mu, U_t), \tag{4.18}$$

where

$$R(\mu, U_t) = \begin{cases} 0, & \theta \in [-1, 0), \\ F(U_t, \mu), & \theta = 0. \end{cases} \tag{4.19}$$

Using the decomposition $\ell = P_{\mathrm{CN}}\ell \oplus P_s\ell$ and (4.17), the solution of (4.2) can be written as

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, \mu), \tag{4.20}$$

where $\left(\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right) = (\Psi, \langle U_t, f_k \rangle)$, and $h(x_1, x_2, \mu) \in P_s\ell,\ h(0,0,0) = 0,\ Dh(0,0,0) = 0$. In particular, the solution of (4.2) on the center manifold is given by

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k + h(x_1, x_2, 0). \tag{4.21}$$

Let $z = x_1 - ix_2$ and $\Psi(0) = (\Psi_1(0), \Psi_2(0))^T$, and notice that $p_1 = \Phi_1 + i\Phi_2$, then

$$\Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \cdot f_k = (\Phi_1, \Phi_2) \begin{pmatrix} \dfrac{z + \overline{z}}{2} \\ \dfrac{i(z - \overline{z})}{2} \end{pmatrix} \cdot f_k = \frac{1}{2}(p_1 z + \overline{p}_1 \overline{z}) \cdot f_k. \tag{4.22}$$

Equation (4.21) can be transformed into

$$U_t = \frac{1}{2}(p_1 z + \overline{p}_1 \overline{z}) \cdot f_k + W(z, \overline{z}) \ , \tag{4.23}$$

where $W(z,\overline{z}) = h(((z+\overline{z})/2), (i(z-\overline{z})/2), 0)$. Moreover, by [17], z satisfies

$$\dot{z} = i\omega_+^k \tilde{\tau} z + g(z, \overline{z}), \tag{4.24}$$

where

$$g(z,\overline{z}) = (\Psi_1(0) - i\Psi_2(0))\langle f(U_t, 0), f_k\rangle. \tag{4.25}$$

Let

$$W(z,\overline{z}) = W_{20}\frac{z^2}{2} + W_{11} z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \text{h.o.t.}, \tag{4.26}$$

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \text{h.o.t.} \tag{4.27}$$

From (4.4) and (4.23), it follows that

$$\begin{aligned}
&\langle f(U_t, 0), f_k\rangle \\
&= \frac{\tilde{\tau}}{2}\begin{pmatrix} -e^{-i\omega_+^k\tilde{\tau}} - sP_{11}C - \dfrac{sP_{20}}{2} - \dfrac{sP_{02}C^2}{2} \\ rP_{11}C + \dfrac{rP_{20}}{2} + \dfrac{rP_{02}C^2}{2} \end{pmatrix} \frac{1}{\pi}\int_0^\pi \cos^3 kx\, dx \frac{z^2}{2} \\
&\quad + \frac{\tilde{\tau}}{4}\begin{pmatrix} -2\cos\omega_+^k\tilde{\tau} - sP_{11}\left(C + \overline{C}\right) - sP_{20} - sP_{02}C\overline{C} \\ rP_{11}\left(C + \overline{C}\right) + rP_{20} + rP_{02}C\overline{C} \end{pmatrix} \frac{1}{\pi}\int_0^\pi \cos^3 kx\, dx z\overline{z} \\
&\quad + \frac{\tilde{\tau}}{2}\begin{pmatrix} -e^{i\omega_+^k\tilde{\tau}} - sP_{11}\overline{C} - \dfrac{sP_{20}}{2} - \dfrac{sP_{02}\overline{C}^2}{2} \\ rP_{11}\overline{C} + \dfrac{rP_{20}}{2} + \dfrac{rP_{02}\overline{C}^2}{2} \end{pmatrix} \frac{1}{\pi}\int_0^\pi \cos^3 kx\, dx \frac{\overline{z}^2}{2}
\end{aligned}$$

$$
+\tilde{\tau}\left(\begin{array}{c}
-\left\langle\left(\frac{W_{20}^{(1)}(0)}{2}e^{i\omega_+^k\tilde{\tau}}+W_{11}^{(1)}(0)e^{-i\omega_+^k\tilde{\tau}}+\frac{W_{20}^{(1)}(-1)}{2}+W_{11}^{(1)}(-1)\right)\cos kx,\cos kx\right\rangle \\
-sP_{11}\left\langle\left(\frac{W_{20}^{(1)}(0)}{2}\overline{C}+W_{11}^{(1)}(0)C+W_{20}^{(2)}(0)2+W_{11}^{(2)}(0)\right)\cos kx,\cos kx\right\rangle \\
-s\left\langle\left[P_{20}\left(\frac{W_{20}^{(1)}(0)}{2}+W_{11}^{(1)}(0)\right)+P_{02}\left(\frac{W_{20}^{(2)}(0)}{2}\overline{C}+W_{11}^{(2)}(0)C\right)\right]\cos kx,\cos kx\right\rangle \\
-s\left\langle\left[P_{21}\left(\frac{\overline{C}}{8}+\frac{C}{4}\right)+P_{12}\left(\frac{C^2}{8}+\frac{C\overline{C}}{4}\right)\right]\cos^3 kx,\cos kx\right\rangle \\
-s\left\langle\left(\frac{P_{30}}{8}+\frac{P_{03}C^2\overline{C}}{8}\right)\cos^3 kx,\cos kx\right\rangle \\
\\
rP_{11}\left\langle\left(\frac{W_{20}^{(1)}(0)}{2}\overline{C}+W_{11}^{(1)}(0)C+W_{20}^{(2)}(0)2+W_{11}^{(2)}(0)\right)\cos kx,\cos kx\right\rangle \\
+r\left\langle\left[P_{20}\left(\frac{W_{20}^{(1)}(0)}{2}+W_{11}^{(1)}(0)\right)+P_{02}\left(\frac{W_{20}^{(2)}(0)}{2}\overline{C}+W_{11}^{(2)}(0)C\right)\right]\cos kx,\cos kx\right\rangle \\
+r\left\langle\left[P_{21}\left(\frac{\overline{C}}{8}+\frac{C}{4}\right)+P_{12}\left(\frac{C^2}{8}+\frac{C\overline{C}}{4}\right)\right]\cos^3 kx,\cos kx\right\rangle \\
+r\left\langle\left(\frac{P_{30}}{8}+\frac{P_{03}C^2\overline{C}}{8}\right)\cos^3 kx,\cos kx\right\rangle
\end{array}\right)
$$

$$
\times\frac{z^2\overline{z}}{2}+\text{h.o.t.},
\tag{4.28}
$$

where $\langle W_{ij}^{(n)}(\theta),\cos kx\rangle = 1/\pi\int_0^\pi W_{ij}^{(n)}(\theta)(x)\cos kx\,dx,\ i+j=2,\ n=1,2.$

Notice that $\int_0^\pi \cos^3 kx\,dx = 0$, for all $k\in N=\{1,2,\ldots\}$.

Let $(\Psi_1,\Psi_2)=\Psi_1(0)-i\Psi_2(0)$. Then we can obtain the following quantities:

$$
g_{20}=\begin{cases}0, & k\in N,\\ \frac{\tilde{\tau}}{2}\left[\left(-e^{-i\omega_+^k\tilde{\tau}}-sP_{11}C-\frac{sP_{20}}{2}-\frac{sP_{02}C^2}{2}\right)\Psi_1\right. & \\ \quad\left.+\left(rP_{11}C+\frac{rP_{20}}{2}+\frac{rP_{02}C^2}{2}\right)\Psi_2\right], & k=0.\end{cases}
$$

$$
g_{11}=\begin{cases}0, & k\in N,\\ \frac{\tilde{\tau}}{4}\left[\left(-2\cos\omega_+^k\tilde{\tau}-sP_{11}\left(C+\overline{C}\right)-sP_{20}-sP_{02}C\overline{C}\right)\Psi_1\right. & \\ \quad\left.+\left(rP_{11}\left(C+\overline{C}\right)+rP_{20}+rP_{02}C\overline{C}\right)\Psi_2\right], & k=0.\end{cases}
$$

$$
g_{02}=\overline{g_{20}},
$$

$$
\begin{aligned}
g_{21} = \tilde{\tau}\Bigg[& -\left\langle \left(\frac{W_{20}^{(1)}(0)}{2} e^{i\omega_+^k \tilde{\tau}} + W_{11}^{(1)}(0) e^{-i\omega_+^k \tilde{\tau}} + \frac{W_{20}^{(1)}(-1)}{2} + W_{11}^{(1)}(-1) \right) \cos kx, \cos kx \right\rangle \\
& - sP_{11} \left\langle \left(\frac{W_{20}^{(1)}(0)}{2} \overline{C} + W_{11}^{(1)}(0) C + W_{20}^{(2)}(0) 2 + W_{11}^{(2)}(0) \right) \cos kx, \cos kx \right\rangle \\
& - s \left\langle \left[P_{20} \left(\frac{W_{20}^{(1)}(0)}{2} + W_{11}^{(1)}(0) \right) + P_{02} \left(\frac{W_{20}^{(2)}(0)}{2} \overline{C} + W_{11}^{(2)}(0) C \right) \right] \cos kx, \cos kx \right\rangle \\
& - s \left\langle \left[P_{21} \left(\frac{\overline{C}}{8} + \frac{C}{4} \right) + P_{12} \left(\frac{C^2}{8} + \frac{C\overline{C}}{4} \right) \right] \cos^3 kx, \cos kx \right\rangle \\
& - s \left\langle \left(\frac{P_{30}}{8} + \frac{P_{03} C^2 \overline{C}}{8} \right) \cos^3 kx, \cos kx \right\rangle \Bigg] \Psi_1 \\
+ \tilde{\tau} \Bigg[& r P_{11} \left\langle \left(\frac{W_{20}^{(1)}(0)}{2} \overline{C} + W_{11}^{(1)}(0) C + W_{20}^{(2)}(0) 2 + W_{11}^{(2)}(0) \right) \cos kx, \cos kx \right\rangle \\
& + r \left\langle \left[P_{20} \left(\frac{W_{20}^{(1)}(0)}{2} + W_{11}^{(1)}(0) \right) + P_{02} \left(\frac{W_{20}^{(2)}(0)}{2} \overline{C} + W_{11}^{(2)}(0) C \right) \right] \cos kx, \cos kx \right\rangle \\
& + r \left\langle \left[P_{21} \left(\frac{\overline{C}}{8} + \frac{C}{4} \right) + P_{12} \left(\frac{C^2}{8} + \frac{C\overline{C}}{4} \right) \right] \cos^3 kx, \cos kx \right\rangle \\
& + r \left\langle \left(\frac{P_{30}}{8} + \frac{P_{03} C^2 \overline{C}}{8} \right) \cos^3 kx, \cos kx \right\rangle \Bigg] \Psi_2.
\end{aligned}
\tag{4.29}
$$

Since $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1, 0]$ appear in g_{21}, we need to compute them. It follows from (4.26) that

$$
\dot{W}(z, \overline{z}) = W_{20} z \dot{z} + W_{11} \dot{z} \overline{z} + W_{11} \overline{z} \dot{\overline{z}} + W_{02} \overline{z} \dot{\overline{z}} + \text{h.o.t.}, \tag{4.30}
$$

$$
A_{\tilde{\tau}} W = A_{\tilde{\tau}} W_{20} \frac{z^2}{2} + A_{\tilde{\tau}} W_{11} z \overline{z} + A_{\tilde{\tau}} W_{02} \frac{\overline{z}^2}{2} + \text{h.o.t.} \tag{4.31}
$$

In addition, by [17], $W(z, \overline{z})$ satisfies

$$
\dot{W} = A_{\tilde{\tau}} W + H(z, \overline{z}), \tag{4.32}
$$

where

$$
H(z, \overline{z}) = H_{20} \frac{z^2}{2} + H_{11} z \overline{z} + H_{02} \frac{\overline{z}^2}{2} + \text{h.o.t.} = X_0 f(U_t, 0) - \Phi\left(\Psi, \langle X_0 f(U_t, 0), f_k \rangle\right) \cdot f_k. \tag{4.33}
$$

Thus, from (4.24), (4.27)–(4.33), we can obtain that

$$\begin{gathered}\left(2i\omega_+^k\tilde{\tau} - A_{\tilde{\tau}}\right)W_{20} = H_{20},\\ -A_{\tilde{\tau}}W_{11} = H_{11},\\ \left(-2i\omega_+^k\tilde{\tau} - A_{\tilde{\tau}}\right)W_{02} = H_{02}.\end{gathered} \tag{4.34}$$

Noticing that $A_{\tilde{\tau}}$ has only two eigenvalues $\pm i\omega_+^k\tilde{\tau}$; therefore, (4.34) has a unique solution W_{ij} in Q given by

$$\begin{gathered}W_{20} = \left(2i\omega_+^k\tilde{\tau} - A_{\tilde{\tau}}\right)^{-1}H_{20},\\ W_{11} = -A_{\tilde{\tau}}^{-1}H_{11},\\ W_{02} = \left(-2i\omega_+^k\tilde{\tau} - A_{\tilde{\tau}}\right)^{-1}H_{02}.\end{gathered} \tag{4.35}$$

From (4.33), we know that for $-1 \le \theta < 0$,

$$\begin{aligned}H(z,\overline{z}) &= -\Phi(\theta)\Psi(0)\langle f(\mathrm{U}_t,0), f_k\rangle \cdot f_k\\ &= -\left(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2i}\right)(\Psi_1(0),\Psi_2(0))\langle f(U_t,0), f_k\rangle \cdot f_k\\ &= -\frac{1}{2}\left(p_1(\theta)(\Psi_1(0)-i\Psi_2(0))\right) + p_2(\theta)(\Psi_1(0)+i\Psi_2(0))\langle f(U_t,0), f_k\rangle \cdot f_k\\ &= -\frac{1}{2}\left(p_1(\theta)g_{20}+p_2(\theta)\overline{g_{02}}\right)\cdot f_k\frac{z^2}{2} - \frac{1}{2}\left(p_1(\theta)g_{11}+p_2(\theta)\overline{g_{11}}\right)\cdot f_k z\overline{z} + \text{h.o.t.}\end{aligned} \tag{4.36}$$

Therefore, for $-1 \le \theta < 0$,

$$H_{20}(\theta) = \begin{cases} 0, & k \in N,\\ -\dfrac{1}{2}\left[p_1(\theta)g_{20}+p_2(\theta)\overline{g_{02}}\right]\cdot f_0, & k = 0.\end{cases} \tag{4.37}$$

$$H_{11}(\theta) = \begin{cases} 0, & k \in N,\\ -\dfrac{1}{2}\left[p_1(\theta)g_{11}+p_2(\theta)\overline{g_{11}}\right]\cdot f_0, & k = 0.\end{cases} \tag{4.38}$$

$$H(z,\overline{z})(0) = f(U_t,0) - \Phi\left(\Psi, \left\langle f(U_t,0), f_k\right\rangle\right) \cdot f_k,$$

$$H_{20}(0) = \begin{cases} \dfrac{\widetilde{\tau}}{2}\begin{pmatrix} -e^{-i\omega_+^k\widetilde{\tau}} - sP_{11}C - \dfrac{sP_{20}}{2} - \dfrac{sP_{02}C^2}{2} \\ rP_{11}C + \dfrac{rP_{20}}{2} + \dfrac{rP_{02}C^2}{2} \end{pmatrix}\cos^2 kx, & k \in N, \\ \dfrac{\widetilde{\tau}}{2}\begin{pmatrix} -e^{-i\omega_+^k\widetilde{\tau}} - sP_{11}C - \dfrac{sP_{20}}{2} - \dfrac{sP_{02}C^2}{2} \\ rP_{11}C + \dfrac{rP_{20}}{2} + \dfrac{rP_{02}C^2}{2} \end{pmatrix} \\ \quad -\dfrac{1}{2}\left[p_1(\theta)g_{20} + p_2(\theta)\overline{g_{02}}\right] \cdot f_0, & k = 0. \end{cases}$$

$$H_{11}(0) = \begin{cases} \dfrac{\widetilde{\tau}}{4}\begin{pmatrix} -2\cos\omega_+^k\widetilde{\tau} - sP_{11}\left(C + \overline{C}\right) - sP_{20} - sP_{02}C\overline{C} \\ rP_{11}\left(C + \overline{C}\right) + rP_{20} + rP_{02}C\overline{C} \end{pmatrix}\cos^2 kx, \\ & k \in N, \\ \dfrac{\widetilde{\tau}}{4}\begin{pmatrix} -2\cos\omega_+^k\widetilde{\tau} - sP_{11}\left(C + \overline{C}\right) - sP_{20} - sP_{02}C\overline{C} \\ rP_{11}\left(C + \overline{C}\right) + rP_{20} + rP_{02}C\overline{C} \end{pmatrix} \\ \quad -\dfrac{1}{2}\left[p_1(\theta)g_{11} + p_2(\theta)\overline{g_{11}}\right] \cdot f_0, & k = 0. \end{cases} \tag{4.39}$$

By the definition of $A_{\widetilde{\tau}}$, we have from (4.34),

$$\dot{W}_{20}(\theta) = 2i\omega_+^k\widetilde{\tau}W_{20}(\theta) + \frac{1}{2}\left[p_1(\theta)g_{20} + p_2(\theta)\overline{g_{02}}\right] \cdot f_k, \ -1 \le \theta \le 0. \tag{4.40}$$

Note that $p_1(\theta) = p_1(0)e^{i\omega_+^k\widetilde{\tau}\theta}$, $-1 \le \theta \le 0$. Hence,

$$W_{20}(\theta) = \frac{1}{2}\left[\frac{ig_{20}p_1(\theta)}{\omega_+^k\widetilde{\tau}} + \frac{i\overline{g_{02}}p_2(\theta)}{3\omega_+^k\widetilde{\tau}}\right] \cdot f_k + E_1 e^{2i\omega_+^k\widetilde{\tau}\theta}, \tag{4.41}$$

$$E_1 = \begin{cases} W_{20}, & k \in N, \\ W_{20}(0) - \dfrac{1}{2}\left[\dfrac{ig_{20}p_1(0)}{\omega_+^0\widetilde{\tau}} + \dfrac{i\overline{g_{02}}p_2(0)}{3\omega_+^0\widetilde{\tau}}\right] \cdot f_0, & k = 0. \end{cases} \tag{4.42}$$

Using the definition of $A_{\tilde{\tau}}$, and combining (4.34) and (4.41), we have

$$\begin{aligned}
&2i\omega_+^0\tilde{\tau}\left[-\frac{1}{2}\left(\frac{ig_{20}p_1(0)}{\omega_+^0\tilde{\tau}}+\frac{i\overline{g_{02}}p_2(0)}{3\omega_+^0\tilde{\tau}}\cdot f_0\right)+E_1\right]-\tilde{\tau}D\Delta\left[-\frac{1}{2}\left(\frac{ig_{20}p_1(0)}{\omega_+^0\tilde{\tau}}+\frac{i\overline{g_{02}}p_2(0)}{3\omega_+^0\tilde{\tau}}\cdot f_0\right)+E_1\right]\\
&\qquad -L(\tilde{\tau})\left[-\frac{1}{2}\left(\frac{ig_{20}p_1(\theta)}{\omega_+^0\tilde{\tau}}+\frac{i\overline{g_{02}}p_2(\theta)}{3\omega_+^0\tilde{\tau}}\cdot f_0\right)+E_1e^{2i\omega_+^0\tilde{\tau}\theta}\right]\\
&\qquad =\frac{\tilde{\tau}}{2}\begin{pmatrix}-e^{-i\omega_+^k\tilde{\tau}}-sP_{11}C-\dfrac{sP_{20}}{2}-\dfrac{sP_{02}C^2}{2}\\ rP_{11}C+\dfrac{rP_{20}}{2}+\dfrac{rP_{02}C^2}{2}\end{pmatrix}-\frac{1}{2}\left[p_1(\theta)g_{20}+p_2(\theta)\overline{g_{02}}\right]\cdot f_0.
\end{aligned}\tag{4.43}$$

Notice that

$$\begin{aligned}
\tilde{\tau}D\Delta\left[p_1(0)\cdot f_0\right]+L(\tilde{\tau})\left[p_1(\theta)\cdot f_0\right]&=i\omega_+^0\tilde{\tau}p_1(0)\cdot f_0,\\
\tilde{\tau}D\Delta\left[p_2(0)\cdot f_0\right]+L(\tilde{\tau})\left[p_2(\theta)\cdot f_0\right]&=-i\omega_+^0\tilde{\tau}p_2(0)\cdot f_0.
\end{aligned}\tag{4.44}$$

Then, for $k\in N_0$,

$$2i\omega_+^0\tilde{\tau}E_1-\tilde{\tau}D\Delta E_1-L(\tilde{\tau})(E_1)e^{2i\omega_+^0\tilde{\tau}\theta}=\frac{\tilde{\tau}}{2}\begin{pmatrix}-e^{-i\omega_+^k\tilde{\tau}}-sP_{11}C-\dfrac{sP_{20}}{2}-\dfrac{sP_{02}C^2}{2}\\ rP_{11}C+\dfrac{rP_{20}}{2}+\dfrac{rP_{02}C^2}{2}\end{pmatrix}\cos^2kx.\tag{4.45}$$

From the above expression, we obtain that

$$E_1=\frac{1}{2}E\begin{pmatrix}-e^{-i\omega_+^k\tilde{\tau}}-sP_{11}C-\dfrac{sP_{20}}{2}-\dfrac{sP_{02}C^2}{2}\\ rP_{11}C+\dfrac{rP_{20}}{2}+\dfrac{rP_{02}C^2}{2}\end{pmatrix}\cos^2kx,\tag{4.46}$$

where

$$E=\begin{pmatrix}2i\omega_+^k+d_1k^2-1+u^*+a_1+u^*e^{-2i\omega_+^k\tilde{\tau}} & a_2\\ -b_1 & 2i\omega_+^k+d_2k^2-b_2+d\end{pmatrix}^{-1}.\tag{4.47}$$

By the same way, we have

$$\begin{aligned}
-\dot{W}_{11}(\theta)&=-\frac{1}{2}\left[p_1(\theta)g_{11}+p_2(\theta)\overline{g_{11}}\right]\cdot f_k,\quad -1\le\theta<0,\\
W_{11}(\theta)&=\frac{1}{2}\left[\frac{-ig_{11}p_1(\theta)}{\omega_+^k\tilde{\tau}}+\frac{i\overline{g_{11}}p_2(\theta)}{\omega_+^k\tilde{\tau}}\right]+E_2.
\end{aligned}\tag{4.48}$$

Similar to the above, we can obtain that

$$E_2 = \frac{1}{4}E_0\begin{pmatrix} -2\cos\omega_+^k\tilde{\tau} - sP_{11}\left(C+\overline{C}\right) - sP_{20} - sP_{02}C\overline{C} \\ rP_{11}\left(C+\overline{C}\right) + rP_{20} + rP_{02}C\overline{C} \end{pmatrix}\cos^2 kx, \tag{4.49}$$

where

$$E_0 = \begin{pmatrix} d_1k^2 - 1 + 2u^* + a_1 & a_2 \\ -b_1 & d_2k^2 - b_2 + d \end{pmatrix}^{-1}. \tag{4.50}$$

So far, $W_{20}(\theta)$ and $W_{11}(\theta)$ have been expressed by the parameters of the system (1.1). And, hence, g_{21} can be expressed also. Thus, we can compute the following values:

$$\begin{gathered} c_1(0) = \frac{i}{2\omega_+^k\tilde{\tau}}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{1}{3}\left|g_{02}\right|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\left\{\lambda'\left(\tau_j^k\right)\right\}}, \\ \beta_2 = 2\operatorname{Re}\{c_1(0)\}, \\ T_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2\operatorname{Im}\left\{\lambda'\left(\tau_j^k\right)\right\}}{\omega_+^k\tilde{\tau}}, \end{gathered} \tag{4.51}$$

where μ_2 determines the direction of Hopf bifurcation, β_2 determines the stability of bifurcating periodic solution, and T_2 determines the period of the bifurcating periodic solution. Hence, we have the following result.

Theorem 4.2. *The signs of μ_2, β_2, T_2 determine the properties of Hopf bifurcation described in Theorems 2.4 and 2.7. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical), and the bifurcating periodic solutions exist (nonexist) for $\tau > \tau_k$ ($\tau < \tau_k$). If $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable). If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions of system* (1.1) *increases (decreases).*

Remark 4.3. From the previous computable results, the expressions of E_1 and E_2 contain the diffusion coefficient. According to the definition of g_{21}, the values of g_{21} have related on E_1 and E_2. Therefore, the signs of β_2 and μ_2 which determine the stability and direction of spatially inhomogeneous periodic solutions strictly depend on the diffusion coefficient of d_1 and d_2.

Acknowledgments

The authors are grateful to the referees for their valuable comments and suggestions on the paper. Project supported by the Natural Science Fopundation of Shandong Province of China

(ZR2011AQ006, ZR2011AM008), and the STPF of University in Shandong Province of China (J10LA13).

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