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*Research Article*

# Solutions of a Class of Deviated-Advanced Nonlocal Problems for the Differential Inclusion $x^1(t) \in F(t, x(t))$

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We study the existence of solutions for deviated-advanced nonlocal and integral condition problems for the differential inclusion $x^1(t) \in F(t, x(t))$.

## 1. Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1–12] and references therein. Consider the deviated-advanced nonlocal problem

$$\frac{dx(t)}{dt} \in F(t, x(t)), \quad \text{a.e. } t \in (0,1), \tag{1.1}$$

$$\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \alpha \sum_{j=1}^{n} b_j x(\psi(\eta_j)), \quad a_k, b_j > 0, \tag{1.2}$$

where $\tau_k, \eta_j \in (0,1)$, $\alpha > 0$ is a parameter, and $\psi$ and $\phi$ are, respectively, deviated and advanced given functions.

Our aim here is to study the existence of at least one absolutely continuous solution $x \in AC[0,1]$ for the problem (1.1)-(1.2) when the set-valued function $F : R \to P(R)$ is $L^1$-Carathéodory.

As an application, we deduce the existence of a solution for the nonlocal problem of the differential inclusion (1.1) with the deviated-advanced integral condition

$$\int_0^1 x(\phi(s))ds = \alpha \int_0^1 x(\psi(s))ds. \tag{1.3}$$

It must be noticed that the following nonlocal and integral conditions are special cases of our nonlocal and integral conditions

$$\begin{gathered} x(\phi(\tau)) = \alpha x(\psi(\eta)), \quad \tau, \eta \in (0,1), \\ \sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \alpha x(\psi(\eta)), \quad \tau_k, \eta \in (0,1), \\ \sum_{k=1}^{m} a_k x(\phi(\tau_k)) = 0, \quad \tau_k \in (0,1), \\ \int_0^1 x(\phi(s))ds = \alpha x(\psi(\eta)), \quad \eta \in (0,1), \\ \alpha \int_0^1 x(\psi(s))ds = x(\phi(\tau)), \quad \tau \in (0,1), \\ \int_0^1 x(\phi(s))ds = 0, \\ \int_0^1 x(\psi(s))ds = 0. \end{gathered} \tag{1.4}$$

As an example of the deviated function $\phi : (0,1) \to (0,1)$, we have $\phi(t) = \beta t, \beta \in (0,1)$. As an example of the advanced function $\psi : (0,1) \to (0,1)$, we have $\psi(t) = t^{\beta}, \beta \in (0,1)$.

## 2. Preliminaries

The following preliminaries are needed.

*Definition 2.1.* A set-valued function $F : [0,1] \times R \to P(R)$ is called $L^1$-Carathéodory if

(a) $t \to F(t,x)$ is measurable for each $x \in R$,

(b) $x \to F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$,

(c) there exists $m \in L^1([0,1], D)$, $D \subset R$ such that

$$|F(t,x)| = \sup\{|v| : v \in F(t,x)\} \leq m(t), \quad \text{for almost all } t \in [0,1]. \tag{2.1}$$

*Definition 2.2.* A single-valued function $f : [0,1] \times R \to R$ is called $L^1$-Carathéodory if

(i) $t \to f(t,x)$ is measurable for each $x \in R$,

(ii) $x \to f(t,x)$ is continuous for almost all $t \in [0,1]$,

(iii) there exists $m \in L^1([0,1],D)$, $D \subset R$ such that $|f| \le m$.

*Definition 2.3.* The set

$$S^1_{F(\cdot,x(t))} = \{f \in ([0,1],R) : f(t,x) \in F(t,x(t)) \text{ for a.e. } t \in [0,1]\} \tag{2.2}$$

is called the set of selections of the set-valued function $F$.

**Theorem 2.4.** *For any $L^1$-Carathéodory set-valued function $F$, the set $S^1_{F(\cdot,x(t))}$ is nonempty [1, 13].*

**Theorem 2.5** (Carathéodory, [14]). *Let $f : [0,1] \times R \to R$ be $L^1$-Carathéodory. Then the problem*

$$\frac{dx(t)}{dt} = f(t,x(t)), \quad \text{for a.e. } t > 0,\ x(0) = x_0, \tag{2.3}$$

*has at least one solution $x \in \mathrm{AC}[0,T]$.*

## 3. Existence of Solution

Consider the following assumptions.

(i) $F : [0,1] \times R \to P(R^+)$ is $L^1$-Carathéodory.

(ii)

$$\alpha \sum_{j=1}^{n} b_j \neq \sum_{k=1}^{m} a_k. \tag{3.1}$$

(iii) $\phi : (0,1) \to (0,1), \phi(t) \le t$ is a deviated continuous function.

(iv) $\psi : (0,1) \to (0,1), \psi(t) \ge t$ is an advanced continuous function.

Now we have the following lemma.

**Lemma 3.1.** *Let assumptions (i)-(ii) be satisfied. The solution of the nonlocal problem* (1.1)-(1.2) *can be expressed by the integral equation*

$$x(t) = A\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds\right) + \int_0^t f(s,x(s))ds, \tag{3.2}$$

*where $f(t,x) \in F(t,x)$, for all $x \in R$, and $A = (\alpha \sum_{j=1}^{n} b_j - \sum_{k=1}^{m} a_k)^{-1}$.*

*Proof.* From the assumption that the set-valued function $F : [0,1] \times R \to P(R^+)$ is $L^1$-Carathéodory, then (Theorem 2.4) there exists a single-valued selection $f : [0,1] \times R \to R^+$ such that

$$\frac{d}{dt}x(t) = f(t,x) \in F(t,x), \quad \forall x \in R. \tag{3.3}$$

This selection $f(t,x)$ is $L^1$-Carathéodory.

Integrating (3.3), we get

$$x(t) = x(0) + \int_0^t f(s,x(s))ds. \tag{3.4}$$

Let $t = \phi(\tau_k)$. Then

$$\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \sum_{k=1}^{m} a_k x(0) + \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds. \tag{3.5}$$

Let $t = \psi(\eta_j)$. Then

$$\alpha\sum_{j=1}^{n} b_j x(\psi(\eta_j)) = \alpha\sum_{j=1}^{n} b_j x(0) + \alpha\sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$x(0) = A\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha\sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds\right), \tag{3.7}$$

where $A = (\alpha\sum_{j=1}^{n} b_j - \sum_{k=1}^{m} a_k)^{-1}$, $\alpha\sum_{j=1}^{n} b_j \neq \sum_{k=1}^{m} a_k$.

Substituting (3.7) into (3.4), we obtain

$$x(t) = A\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha\sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds\right) + \int_0^t f(s,x(s))ds. \tag{3.8}$$

This proves that the solution of the nonlocal problem (1.1)-(1.2) can be expressed by the integral equation (3.2). □

For the existence of the solution, we have the following theorem.

**Theorem 3.2.** *Assume that (i)–(iv) are satisfied. Then the integral equation* (3.2) *has at least one continuous solution* $x \in C[0,1]$.

*Proof.* Define a subset $Q_r \subset C[0,1]$ by

$$Q_r = \left\{ x \in C[0,1] : |x(t)| \le r,\ r = AM\left(1 + \sum_{k=1}^{m} a_k + \alpha \sum_{j=1}^{n} b_j\right)\right\}. \tag{3.9}$$

Clearly, the set $Q_r$ is nonempty, closed, and convex.

Let $H$ be an operator defined by

$$(Hx)(t) = A\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds\right) + \int_0^t f(s,x(s))ds. \tag{3.10}$$

Let $x \in Q_r$. Let $\{x_n(t)\}$ be a sequence in $Q_r$ converging to $x(t)$, $x_n(t) \to x(t)$, for all $t \in I$. Then

$$\begin{aligned} \lim_{n\to\infty} (Hx_n)(t) = {} & A\left(\sum_{k=1}^{m} a_k \lim_{n\to\infty} \int_0^{\phi(\tau_k)} f(s,x_n(s))ds - \alpha \sum_{j=1}^{n} b_j \lim_{n\to\infty} \int_0^{\psi(\eta_j)} f(s,x_n(s))ds\right) \\ & + \lim_{n\to\infty} \int_0^t f(s,x_n(s))ds, \end{aligned} \tag{3.11}$$

By assumptions (i)-(ii) and the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} (Hx_n)(t) = (Hx)(t). \tag{3.12}$$

Then $H$ is continuous.

Now, letting $x \in Q_r$, (then $\phi(t) \le t$ and $\psi(t) \ge t$), we obtain

$$\begin{aligned} (Hx)(t) \le {} & A\left(\sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s,x(s))ds - \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} f(s,x(s))ds\right) \\ & + \int_0^t f(s,x(s))ds, \\ |(Hx)(t)| \le {} & A\left(\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|f(s,x(s))\right| ds + \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} \left|f(s,x(s))\right| ds\right) \\ & + \int_0^t \left|f(s,x(s))\right| ds \\ \le {} & A\left(\sum_{k=1}^{m} a_k \int_0^{\tau_k} m(s)ds + \alpha \sum_{j=1}^{n} b_j \int_0^{\eta_j} m(s)ds\right) + \int_0^t m(s)ds \end{aligned}$$

$$\le A\left(\sum_{k=1}^{m} a_k M + \alpha\sum_{j=1}^{n} b_j M\right) + M$$
$$\le AM\left(1 + \sum_{k=1}^{m} a_k + \alpha\sum_{j=1}^{n} b_j\right) \le r. \tag{3.13}$$

Then $\{Hx(t)\}$ is uniformly bounded in $Q_r$.

Also for $t_1, t_2 \in (0,1), t_1 < t_2$ such that $|t_2 - t_1| < \delta$, we have

$$\begin{aligned}(Hx)(t_2) - (Hx)(t_1) &= \int_0^{t_2} f(s,x(s))ds - \int_0^{t_1} f(s,x(s))ds,\\ |(Hx)(t_2) - (Hx)(t_1)| &\le \int_{t_1}^{t_2} \left|f(s,x(s))\right|ds\\ &\le \int_{t_1}^{t_2} m(s)ds,\\ |(Hx)(t_2) - (Hx)(t_1)| &\le \varepsilon.\end{aligned} \tag{3.14}$$

Hence the class of functions $\{Hx(t)\}$ is equicontinuous. By Arzela-Ascoli's theorem, $\{Hx(t)\}$ is relatively compact. Since all conditions of Schauder's theorem hold, then $H$ has a fixed point in $Q_r$.

Therefore the integral equation (3.2) has at least one continuous solution $x \in C(0,1)$. Now,

$$\begin{aligned}\lim_{t\to 0} x(t) &= A\lim_{t\to 0}\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha\sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds\right)\\ &\quad + \lim_{t\to 0}\int_0^{t} f(s,x(s))ds\\ &= A\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha\sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s,x(s))ds\right) = x(0).\end{aligned} \tag{3.15}$$

Also

$$\begin{aligned}x(1) = \lim_{t\to 1} x(t) &= A\left(\sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s,x(s))ds - \alpha\sum_{j=1}^{n} b_j \int_0^{\phi(\eta_j)} f(s,x(s))ds\right)\\ &\quad + \int_0^{1} f(s,x(s))ds.\end{aligned} \tag{3.16}$$

Then the integral equation (3.2) has at least one continuous solution $x \in C[0,1]$. □

The following theorem proves the existence of at least one solution for the nonlocal problem(1.1)-(1.2).

**Theorem 3.3.** *Let (i)–(iv) be satisfied. Then the nonlocal problem* (1.1)-(1.2) *has at least one solution* $x \in \mathrm{AC}[0,1]$.

*Proof.* From Theorem 3.2 and the integral equation (3.2), we deduce that there exists at least one solution, $x \in \mathrm{AC}[0,1]$, of the integral equation (3.2).

To complete the proof, we prove that the integral equation (3.2) satisfies nonlocal problem (1.1)-(1.2).

Differentiating (3.2), we get

$$\frac{dx}{dt} = f(t, x(t)) \in F(t, x(t)), \quad \text{a.e. } t \in (0,1). \tag{3.17}$$

Letting $t = \phi(\tau_k)$ in (3.2), we obtain

$$\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \sum_{k=1}^{m} a_k \left( A \sum_{k=1}^{m} a_k + 1 \right) \int_0^{\phi(\tau_k)} f(s, x(s)) ds - \alpha A \sum_{k=1}^{m} a_k \sum_{j=1}^{n} b_j \int_0^{\psi(\eta_j)} f(s, x(s)) ds. \tag{3.18}$$

Also, letting $t = \psi(\eta_j)$ in (3.2), we obtain

$$\begin{aligned} \alpha \sum_{j=1}^{n} b_j x(\psi(\eta_j)) &= \alpha A \sum_{j=1}^{n} b_j \sum_{k=1}^{m} a_k \int_0^{\phi(\tau_k)} f(s, x(s)) ds \\ &\quad + \alpha \sum_{j=1}^{n} b_j \left( 1 - \alpha A \sum_{j=1}^{n} b_j \right) \int_0^{\psi(\eta_j)} f(s, x(s)) ds. \end{aligned} \tag{3.19}$$

And from (3.19) from (3.18), we obtain

$$\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \alpha \sum_{j=1}^{n} b_j x(\psi(\eta_j)). \tag{3.20}$$

This complete the proof of the equivalence between the nonlocal problem (1.1)-(1.2) and the integral equation (3.2).

This implies that there exists at least one absolutely continuous solution $x \in \mathrm{AC}[0,1]$ of the nonlocal problem (1.1)-(1.2). □

## 4. Nonlocal Integral Condition

Let $x \in [0,1]$ be a solution of the nonlocal problem (1.1)-(1.2). Let $a_k = t_k - t_{k-1}$, $\tau_k \in (t_{k-1}, t_k) \subset (0,1)$. Also, let $b_j = t_j - t_{j-1}$, $\eta_j \in (t_{j-1}, t_j) \subset (0,1)$. Then the nonlocal condition (1.2) will be

$$\sum_{k=1}^{m}(t_k - t_{k-1})x(\phi(\tau_k)) = \alpha \sum_{j=1}^{n}(t_j - t_{j-1})x(\psi(\eta_j)). \tag{4.1}$$

From the continuity of the solution $x$ of the nonlocal condition (1.2) we obtain

$$\lim_{m\to\infty}\sum_{k=1}^{m}(t_k - t_{k-1})x(\phi(\tau_k)) = \lim_{n\to\infty}\alpha \sum_{j=1}^{n}(t_j - t_{j-1})x(\psi(\eta_j)). \tag{4.2}$$

That is, the nonlocal condition (1.2) is transformed to the integral condition

$$\int_0^1 x(\phi(s))ds = \alpha \int_0^1 x(\psi(s))ds, \tag{4.3}$$

and the solution of the integral equation (3.2) will be

$$\begin{aligned} x(t) = A^*\left(\int_0^1\int_0^{\phi(s)} f(\theta, x(\theta))d\theta ds - \alpha\int_0^1\int_0^{\psi(s)} f(\theta, x(\theta))d\theta ds\right) \\ + \int_0^t f(s, x(s))ds, \quad A^* = (\alpha - 1)^{-1}. \end{aligned} \tag{4.4}$$

Now, we have the following theorem.

**Theorem 4.1.** *Let assumptions (i)–(iv) of Theorem 3.2 be satisfied. Then the nonlocal problem with the integral condition*

$$\begin{gathered} \frac{dx(t)}{dt} = f(t, x(t)) \in F(t, x(t)), \quad \text{for a.e. } t \in (0,1), \\ \int_0^1 x(\phi(s))ds = \alpha \int_0^1 x(\psi(s))ds \end{gathered} \tag{4.5}$$

*has at least one solution $x \in AC[0,1]$ represented by* (4.4).

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