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Research Article

Stability in Generalized Functions

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We consider the following additive functional equation with n-independent variables: $f(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1})$ in the spaces of generalized functions. Making use of the heat kernels, we solve the general solutions and the stability problems of the above equation in the spaces of tempered distributions and Fourier hyperfunctions. Moreover, using the mollifiers, we extend these results to the space of distributions.

1. Introduction

The most famous functional equation is the Cauchy equation

$$f(x+y) = f(x) + f(y), \tag{1.1}$$

any solution of which is called additive. It is well known that every measurable solution of (1.1) is of the form $f(x) = ax$ for some constant a. In 1941, Hyers proved the stability theorem for (1.1) as follows.

Theorem 1.1 (see [1]). *Let E_1 be a normed vector space, E_2 a Banach space. Suppose that $f : E_1 \to E_2$ satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| \le \epsilon, \tag{1.2}$$

for all $x, y \in E_1$, then there exists the unique additive mapping $g : E_1 \to E_2$ such that

$$\| f(x) - g(x) \| \le \epsilon, \tag{1.3}$$

for all $x \in E_1$.

The above stability theorem was motivated by Ulam [2]. As noted in the above theorem, the stability problem of the functional equations means how the solution of the inequality differs from the solution of the original equation. Forti [3] noticed that the theorem of Hyers is still true if E_1 is replaced by an arbitrary semigroup. In 1950 Aoki [4] and in 1978 Rassias [5] generalized Hyers' result to the unbounded Cauchy difference. Thereafter, many authors studied the stability problems for (1.1) in various settings (see [6, 7]).

During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [8–17]). Among them, the following additive functional equation with n-independent variables:

$$f\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(x_i - x_{i-1}) \tag{1.4}$$

was proposed by Nakmahachalasint [18], where n is a positive integer with $n > 1$ and $x_0 \equiv x_n$. He solved the general solutions and the stability problems for the above equation. Actually, he proved that (1.4) is equivalent to (1.1).

In this paper, in a similar manner as in [19–23], we solve the general solutions and the stability problems for (1.4) in the spaces of generalized functions such as the space $\mathcal{S}'(\mathbb{R}^m)$ of tempered distributions, the space $\mathcal{F}'(\mathbb{R}^m)$ of Fourier hyperfunctions, and the space $\mathfrak{D}'(\mathbb{R}^m)$ of distributions. Making use of the pullbacks, we first reformulate (1.4) and the related inequality in the spaces of generalized functions as follows:

$$u \circ A = \sum_{i=1}^{n} u \circ P_i + \sum_{i=1}^{n} u \circ B_i, \tag{1.5}$$

$$\left\| u \circ A - \sum_{i=1}^{n} u \circ P_i - \sum_{i=1}^{n} u \circ B_i \right\| \leq \epsilon, \tag{1.6}$$

where A, P_i, and B_i are the functions defined by

$$\begin{gathered} A(x_1, \ldots, x_n) = x_1 + \cdots + x_n, \\ P_i(x_1, \ldots, x_n) = x_i, \quad 1 \leq i \leq n, \\ B_i(x_1, \ldots, x_n) = x_i - x_{i-1}, \quad 1 \leq i \leq n. \end{gathered} \tag{1.7}$$

Here $\circ$ denotes the pullback of generalized functions, and the inequality $\|v\| \leq \epsilon$ in (1.6) means that $|\langle v, \varphi \rangle| \leq \epsilon \|\varphi\|_{L^1}$ for all test functions φ.

In Section 2, we will prove that every solution u in $\mathcal{S}'(\mathbb{R}^m)$ or $\mathcal{F}'(\mathbb{R}^m)$ of (1.5) has the form

$$u = a \cdot x, \tag{1.8}$$

where $a \in \mathbb{C}^m$. Also, we shall figure out that every solution u in $\mathcal{S}'(\mathbb{R}^m)$ or $\mathcal{F}'(\mathbb{R}^m)$ of the inequality (1.6) can be written uniquely in the form

$$u = a \cdot x + \mu(x), \tag{1.9}$$

where μ is a bounded measurable function such that $\|\mu\|_{L^\infty} \le ((10n-3)/(2(2n-1)))\epsilon$. Subsequently, in Section 3, these results are extended to the space $\mathcal{D}'(\mathbb{R}^m)$.

2. Stability in $\mathcal{F}'(\mathbb{R}^m)$

We first introduce the spaces of tempered distributions and Fourier hyperfunctions. Here, we use the m-dimensional notations, $|\alpha| = \alpha_1 + \cdots + \alpha_m$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_m^{\alpha_m}$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$, for $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{R}^m$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\partial_j = \partial/\partial\zeta_j$.

Definition 2.1 (see [24, 25]). We denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^m$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^m} \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty, \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^m$. A linear functional u on $\mathcal{S}(\mathbb{R}^m)$ is said to be tempered distribution if there exist a constant $C \ge 0$ and a nonnegative integer N such that

$$\left|\langle u, \varphi \rangle\right| \le C \sum_{|\alpha|, |\beta| \le N} \sup_{x \in \mathbb{R}^m} \left| x^\alpha \partial^\beta \varphi \right|, \tag{2.2}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^m)$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^m)$.

Note that tempered distributions are generalizations of L^p-functions. These are very useful for the study of Fourier transforms in generality, since all tempered distributions have a Fourier transform. Imposing the growth condition on $\|\cdot\|_{\alpha,\beta}$ in (2.1), a new space of test functions has emerged as follows.

Definition 2.2 (see [26]). We denote by $\mathcal{F}(\mathbb{R}^m)$ the set of all infinitely differentiable functions φ in $\mathbb{R}^m$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{\left| x^\alpha \partial^\beta \varphi(x) \right|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty, \tag{2.3}$$

for some positive constants A, B depending only on φ. The strong dual of $\mathcal{F}(\mathbb{R}^m)$, denoted by $\mathcal{F}'(\mathbb{R}^m)$, is called the Fourier hyperfunction.

It can be verified that the seminorm (2.3) is equivalent to

$$\|\varphi\|_{h,k} = \sup_{x,\alpha} \frac{\left|\partial^{\alpha}\varphi(x)\right| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty \tag{2.4}$$

for some constants $h, k > 0$. It is easy to see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}), \qquad \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{F}'(\mathbb{R}). \tag{2.5}$$

Taking the inclusions (2.5) into account, it suffices to consider the space $\mathcal{F}'(\mathbb{R}^m)$. In order to solve the general solutions and the stability problems for (1.4) in the spaces $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$, we employ the m-dimensional heat kernel, fundamental solution of the heat equation,

$$E_t(x) = E(x,t) = \begin{cases} (4\pi t)^{-m/2} \exp\left(-\dfrac{|x|^2}{4t}\right), & x \in \mathbb{R}^m,\ t > 0, \\ 0, & x \in \mathbb{R}^m,\ t \le 0. \end{cases} \tag{2.6}$$

Since for each $t > 0$, $E(\cdot, t)$ belongs to the space $\mathcal{F}(\mathbb{R}^m)$, the convolution

$$\widetilde{u}(x,t) = (u * E)(x,t) = \langle u_y, E_t(x-y)\rangle, \quad x \in \mathbb{R}^m,\ t > 0 \tag{2.7}$$

is well defined for all u in $\mathcal{F}'(\mathbb{R}^m)$, which is called the Gauss transform of u. Subsequently, the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x), \quad x \in \mathbb{R}^m,\ t, s > 0 \tag{2.8}$$

of the heat kernel is very useful to convert (1.5) into the classical functional equation defined on upper-half plane. We also use the following famous result, so-called heat kernel method, which states as follows.

Theorem 2.3 (see [27]). *Let $u \in \mathcal{S}'(\mathbb{R}^m)$, then its Gauss transform $\widetilde{u}$ is a C^∞-solution of the heat equation*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\widetilde{u}(x,t) = 0, \tag{2.9}$$

satisfying the following:

(i) *there exist positive constants C, M, and N such that*

$$|\widetilde{u}(x,t)| \le Ct^{-M}(1+|x|)^N \quad \text{in } \mathbb{R}^m \times (0,\delta), \tag{2.10}$$

(ii) $\widetilde{u}(x,t) \to u$ *as* $t \to 0^+$ *in the sense that for every* $\varphi \in \mathcal{S}(\mathbb{R}^m)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \widetilde{u}(x,t)\varphi(x)dx. \tag{2.11}$$

Conversely, every C^∞*-solution* $U(x,t)$ *of the heat equation satisfying the growth condition* (2.10) *can be uniquely expressed as* $U(x,t) = \widetilde{u}(x,t)$ *for some* $u \in \mathcal{S}'(\mathbb{R}^m)$.

Similarly, we can represent Fourier hyperfunctions as a special case of the results as in [28]. In this case, the estimate (2.10) is replaced by the following.

For every $\epsilon > 0$, there exists a positive constant C_ϵ such that

$$|\widetilde{u}(x,t)| \leq C_\epsilon \exp\left(\epsilon\left(\frac{|x|}{t} + 1\right)\right) \quad \text{in } \mathbb{R} \times (0,\delta). \tag{2.12}$$

We need the following lemma in order to solve the general solutions for the additive functional equation in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$. In what follows, we denote $x_0 \equiv x_n$ and $t_0 \equiv t_n$.

Lemma 2.4. *Suppose that* $f : \mathbb{R}^m \times (0,\infty) \to \mathbb{C}$ *is a continuous function satisfying*

$$f\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) = \sum_{i=1}^n f(x_i,t_i) + \sum_{i=1}^n f(x_i - x_{i-1}, t_i + t_{i-1}), \tag{2.13}$$

for all $x_1,\ldots,x_n \in \mathbb{R}^m$, $t_1,\ldots,t_n > 0$, *then the solution* f *has the form*

$$f(x,t) = a \cdot x, \tag{2.14}$$

for some $a \in \mathbb{C}^m$.

Proof. Putting $(x_1,\ldots,x_n) = (0,\ldots,0)$ in (2.13) yields

$$f\left(0, \sum_{i=1}^n t_i\right) = \sum_{i=1}^n f(0,t_i) + \sum_{i=1}^n f(0, t_i + t_{i-1}), \tag{2.15}$$

for all $t_1,\ldots,t_n > 0$. In view of (2.15), we see that

$$c := \lim_{t \to 0^+} f(0,t) \tag{2.16}$$

exists. Letting $t_1 = \cdots = t_n \to 0^+$ in (2.15) gives $c = 0$. Setting $(x_1,x_2,\ldots,x_n) = (x,0,\ldots,0)$ and letting $t_1 = t$, $t_2 = \cdots = t_n \to 0^+$ in (2.13), we have

$$f(-x,t) = -f(x,t), \tag{2.17}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Substituting $(x_1, x_2, x_3, \ldots, x_n)$ with $(x, y, 0, \ldots, 0)$ and letting $t_1 = t$, $t_2 = s$, $t_3 = \cdots = t_n \to 0^+$ in (2.13), we obtain from (2.17) that

$$f(x+y, t+s) + f(x-y, t+s) = 2f(x,t), \tag{2.18}$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. Putting $y = 0$ in (2.18) yields

$$f(x, t+s) = f(x,t), \tag{2.19}$$

which shows that $f(x,t)$ is independent with respect to $t > 0$. For that reason, we see from (2.18) that $F(x) := f(x,t)$ satisfies

$$F(x+y) + F(x-y) = 2F(x), \tag{2.20}$$

for all $x, y \in \mathbb{R}^m$. Replacing x by $(x+y)/2$ and y by $(x-y)/2$ in (2.20), we have

$$F(x+y) = F(x) + F(y), \tag{2.21}$$

for all $x, y \in \mathbb{R}^m$. Given the continuity, we obtain

$$f(x,t) = F(x) = a \cdot x, \tag{2.22}$$

for some $a \in \mathbb{C}^m$. □

From the above lemma, we can solve the general solutions for the additive functional equation in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$.

Theorem 2.5. *Every solution u in $\mathcal{F}'(\mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^m)$, resp.) of* (1.5) *has the form*

$$u = a \cdot x, \tag{2.23}$$

for some $a \in \mathbb{C}^m$.

Proof. Convolving the tensor product $E_{t_1}(x_1)\cdots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.5), we have

$$\begin{aligned}
&[(u\circ A)*(E_{t_1}(x_1)\cdots E_{t_n}(x_n))](\xi_1,\dots,\xi_n)\\
&\quad=\langle u\circ A, E_{t_1}(\xi_1-x_1)\cdots E_{t_n}(\xi_n-x_n)\rangle\\
&\quad=\left\langle u,\int\cdots\int E_{t_1}(\xi_1-x_1+x_2+\cdots+x_n)E_{t_2}(\xi_2-x_2)\cdots E_{t_n}(\xi_n-x_n)dx_2\cdots dx_n\right\rangle\\
&\quad=\left\langle u,\int\cdots\int E_{t_1}(\xi_1+\cdots+\xi_n-x_1-\cdots-x_n)E_{t_2}(x_2)\cdots E_{t_n}(x_n)dx_2\cdots dx_n\right\rangle\\
&\quad=\langle u,(E_{t_1}*\cdots*E_{t_n})(\xi_1+\cdots+\xi_n-x_1)\rangle\\
&\quad=\langle u,E_{t_1+\cdots+t_n}(\xi_1+\cdots+\xi_n)\rangle\\
&\quad=\widetilde{u}(\xi_1+\cdots+\xi_n,t_1+\cdots+t_n),\\
&[(u\circ P_i)*(E_{t_1}(x_1)\cdots E_{t_n}(x_n))](\xi_1,\dots,\xi_n)=\widetilde{u}(\xi_i,t_i),\\
&[(u\circ B_i)*(E_{t_1}(x_1)\cdots E_{t_n}(x_n))](\xi_1,\dots,\xi_n)=\widetilde{u}(\xi_i-\xi_{i-1},t_i+t_{i-1}),
\end{aligned}\tag{2.24}$$

where $\widetilde{u}$ is the Gauss transform of u. Thus, (1.5) is converted into the following classical functional equation:

$$\widetilde{u}\left(\sum_{i=1}^{n}x_i,\sum_{i=1}^{n}t_i\right)=\sum_{i=1}^{n}\widetilde{u}(x_i,t_i)+\sum_{i=1}^{n}\widetilde{u}(x_i-x_{i-1},t_i+t_{i-1}),\tag{2.25}$$

for all $x_1,\dots,x_n\in\mathbb{R}^m$, $t_1,\dots,t_n>0$. It follows from Lemma 2.4 that the solution $\widetilde{u}$ of (2.25) has the form

$$\widetilde{u}(x,t)=a\cdot x,\tag{2.26}$$

for some $a\in\mathbb{C}^m$. Letting $t\to 0^+$ in (2.26), we finally obtain the general solution for (1.5). □

We are going to solve the stability problems for the additive functional equation in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$.

Lemma 2.6. *Suppose that $f:\mathbb{R}^m\times(0,\infty)\to\mathbb{C}$ is a continuous function satisfying*

$$\left|f\left(\sum_{i=1}^{n}x_i,\sum_{i=1}^{n}t_i\right)-\sum_{i=1}^{n}f(x_i,t_i)-\sum_{i=1}^{n}f(x_i-x_{i-1},t_i+t_{i-1})\right|\le\epsilon,\tag{2.27}$$

for all $x_1,\dots,x_n\in\mathbb{R}^m$, $t_1,\dots,t_n>0$, then there exists a unique $a\in\mathbb{C}^m$ such that

$$\left|f(x,t)-a\cdot x\right|\le\frac{10n-3}{2(2n-1)}\epsilon,\tag{2.28}$$

for all $x\in\mathbb{R}^m$, $t>0$.

Proof. Putting $(x_1,\ldots,x_n)=(0,\ldots,0)$ in (2.27) yields

$$\left|f\left(0,\sum_{i=1}^{n}t_i\right)-\sum_{i=1}^{n}f(0,t_i)-\sum_{i=1}^{n}f(0,t_i+t_{i-1})\right|\le\epsilon, \tag{2.29}$$

for all $t_1,\ldots,t_n>0$. In view of (2.29), we see that

$$c:=\limsup_{t\to 0^+}f(0,t) \tag{2.30}$$

exists. Letting $t_1=\cdots=t_n\to 0^+$ in (2.29) gives

$$|c|\le\frac{\epsilon}{2n-1}. \tag{2.31}$$

Setting $(x_1,x_2,\ldots,x_n)=(x,0,\ldots,0)$ and letting $t_1=t$, $t_2=\cdots=t_n\to 0^+$ in (2.27), we have

$$\left|f(x,t)+f(-x,t)+(2n-3)c\right|\le\epsilon, \tag{2.32}$$

for all $x\in\mathbb{R}^m$, $t>0$. Substituting $(x_1,x_2,x_3,\ldots,x_n)=(x,x,0,\ldots,0)$ and letting $t_1=t_2=t$, $t_3=\cdots=t_n\to 0^+$ in (2.13), we obtain

$$\left|f(2x,2t)-3f(x,t)-f(-x,t)-f(0,2t)-(2n-5)c\right|\le\epsilon, \tag{2.33}$$

for all $x\in\mathbb{R}^m$, $t>0$. Adding (2.33) to (2.32) yields

$$\left|f(2x,2t)-2f(x,t)-f(0,2t)+2c\right|\le 2\epsilon, \tag{2.34}$$

for all $x\in\mathbb{R}^m$, $t>0$. Letting $t_1=t_2=t$, $t_3=\cdots=t_n\to 0^+$ in (2.29) gives

$$\left|4f(0,t)+(2n-5)c\right|\le\epsilon, \tag{2.35}$$

for all $t>0$. Combining (2.31), (2.34), and (2.35), we have

$$\left|\frac{f(2x,2t)}{2}-f(x,t)\right|\le\frac{10n-3}{4(2n-1)}\epsilon, \tag{2.36}$$

for all $x\in\mathbb{R}^m$, $t>0$. Making use of induction argument in (2.36), we obtain

$$\left|\frac{f(2^kx,2^kt)}{2^k}-f(x,t)\right|\le\frac{10n-3}{2(2n-1)}\epsilon, \tag{2.37}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^m$, $t > 0$. Replacing x, t by $2^l x$, $2^l t$ in (2.37), respectively, and dividing the result by 2^l, we see that for $k \geq l > 0$,

$$\left| \frac{f(2^{k+l}x, 2^{k+l}t)}{2^{k+l}} - \frac{f(2^l x, 2^l t)}{2^l} \right| \leq \frac{10n-3}{2^{l+1}(2n-1)}\epsilon. \tag{2.38}$$

Since the right-hand side of (2.38) tends to 0 as $l \to \infty$, the sequence $\{2^{-k} f(2^k x, 2^k t)\}$ is a Cauchy sequence which converges uniformly. Thus, we may define

$$A(x,t) := \lim_{k\to\infty} \frac{f(2^k x, 2^k t)}{2^k}, \tag{2.39}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Now, we verify from (2.27) that the function A satisfies

$$A\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) = \sum_{i=1}^n A(x_i, t_i) + \sum_{i=1}^n A(x_i - x_{i-1}, t_i + t_{i-1}), \tag{2.40}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$. As observed in Lemma 2.4, the continuous solution of (2.40) has the form

$$A(x,t) = a \cdot x, \tag{2.41}$$

for some $a \in \mathbb{C}^m$. It follows from (2.37) that A is the unique function in $\mathbb{R}^m \times (0,\infty)$ satisfying

$$\left| f(x,t) - A(x,t) \right| \leq \frac{10n-3}{2(2n-1)}\epsilon, \tag{2.42}$$

for all $x \in \mathbb{R}^m$, $t > 0$. □

From the above lemma, we have the following stability theorem for the additive functional equation in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$.

Theorem 2.7. *Suppose that u in $\mathcal{F}'(\mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^m)$, resp.) satisfies the inequality* (1.6)*, then there exists a unique $a \in \mathbb{C}^m$ such that*

$$\|u - a \cdot x\| \leq \frac{10n-3}{2(2n-1)}\epsilon. \tag{2.43}$$

Proof. Convolving the tensor product $E_{t_1}(x_1) \cdots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.6), we have

$$\left| \tilde{u}\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) - \sum_{i=1}^n \tilde{u}(x_i, t_i) - \sum_{i=1}^n \tilde{u}(x_i - x_{i-1}, t_i + t_{i-1}) \right| \leq \epsilon, \tag{2.44}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$, where $\tilde{u}$ is the Gauss transform of u. By Lemma 2.6, we have

$$|\tilde{u}(x,t) - a \cdot x| \le \frac{10n-3}{2(2n-1)}\epsilon, \tag{2.45}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Letting $t \longrightarrow 0^+$ in (2.45), we obtain the conclusion. □

3. Stability in $\mathfrak{D}'(\mathbb{R}^m)$

In this section, we shall extend the previous results to the space of distributions. Recall that a distribution u is a linear functional on $C_c^\infty(\mathbb{R}^m)$ of infinitely differentiable functions on $\mathbb{R}^m$ with compact supports such that for every compact set $K \subset \mathbb{R}^m$, there exist constants $C > 0$ and $N \in \mathbb{N}_0$ satisfying

$$\left|\langle u, \varphi \rangle\right| \le C \sum_{|\alpha| \le N} \sup \left|\partial^\alpha \varphi\right|, \tag{3.1}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^m)$ with supports contained in K. The set of all distributions is denoted by $\mathfrak{D}'(\mathbb{R}^m)$. It is well known that the following topological inclusions hold:

$$C_c^\infty(\mathbb{R}^m) \hookrightarrow \mathcal{S}(\mathbb{R}^m), \qquad \mathcal{S}'(\mathbb{R}^m) \hookrightarrow \mathfrak{D}'(\mathbb{R}^m). \tag{3.2}$$

As we see in [19, 20, 23], by the semigroup property of the heat kernel, (1.5) can be controlled easily in the spaces $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$. But we cannot employ the heat kernel in the space $\mathfrak{D}'(\mathbb{R}^m)$. For that reason, instead of the heat kernel, we use the regularizing functions. We denote by ψ the function on $\mathbb{R}^m$ satisfying

$$\psi(x) = \begin{cases} A \exp\left(-\left(1 - |x|^2\right)^{-1}\right), & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \tag{3.3}$$

where

$$A = \left(\int_{|x|<1} \exp\left(-\left(1 - |x|^2\right)^{-1}\right) dx\right)^{-1}. \tag{3.4}$$

It is easy to see that ψ is an infinitely differentiable function supported in the set $\{x : |x| \le 1\}$ with $\int \psi(x) dx = 1$. For each $t > 0$, we define $\psi_t(x) := t^{-m}\psi(x/t)$, then ψ_t has all the properties of ψ except that the support of ψ_t is contained in the ball of radius t with center at 0. If $u \in \mathfrak{D}'(\mathbb{R}^m)$, then for each $t > 0$, $(u * \psi_t)(x) = \langle u_y, \psi_t(x-y) \rangle$ is a smooth function in $\mathbb{R}^m$ and $(u * \psi_t)(x) \longrightarrow u$ as $t \longrightarrow 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^m)$,

$$\langle u * \psi_t, \varphi \rangle = \int (u * \psi_t)(x) \varphi(x) dx \longrightarrow \langle u, \varphi \rangle \quad \text{as } t \longrightarrow 0^+. \tag{3.5}$$

For each $t > 0$, the function $u * \psi_t$ is called a regularization of u, and the transform which maps u to $u * \psi_t$ is called a mollifier. Making use of the mollifiers, we can solve the general solution for the additive functional equation in the space $\mathfrak{D}'(\mathbb{R}^m)$ as follows.

Theorem 3.1. *Every solution u in $\mathfrak{D}'(\mathbb{R}^m)$ of* (1.5) *has the form*

$$u = a \cdot x, \tag{3.6}$$

for some $a \in \mathbb{C}^m$.

Proof. Convolving the tensor product $\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n)$ of the regularizing functions on both sides of (1.5), we have

$$\begin{aligned}
&\left[(u \circ A) * (\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n))\right](\xi_1,\ldots,\xi_n) \\
&\quad = \langle u \circ A, \psi_{t_1}(\xi_1 - x_1)\cdots\psi_{t_n}(\xi_n - x_n)\rangle \\
&\quad = \left\langle u, \int\cdots\int \psi_{t_1}(\xi_1 - x_1 + x_2 + \cdots + x_n)\psi_{t_2}(\xi_2 - x_2)\cdots\psi_{t_n}(\xi_n - x_n)dx_2\cdots dx_n \right\rangle \\
&\quad = \left\langle u, \int\cdots\int \psi_{t_1}(\xi_1 + \cdots + \xi_n - x_1 - \cdots - x_n)\psi_{t_2}(x_2)\cdots\psi_{t_n}(x_n)dx_2\cdots dx_n \right\rangle \\
&\quad = \langle u, (\psi_{t_1} * \cdots * \psi_{t_n})(\xi_1 + \cdots + \xi_n - x_1)\rangle \\
&\quad = (u * \psi_{t_1} * \cdots * \psi_{t_n})(\xi_1 + \cdots + \xi_n), \\
&\qquad\left[(u \circ P_i) * (\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n))\right](\xi_1,\ldots,\xi_n) = (u * \psi_{t_i})(\xi_i), \\
&\qquad\left[(u \circ B_i) * (\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n))\right](\xi_1,\ldots,\xi_n) = (u * \psi_{t_i} * \psi_{t_{i-1}})(\xi_i - \xi_{i-1}).
\end{aligned} \tag{3.7}$$

Thus, (1.5) is converted into the following functional equation:

$$(u * \psi_{t_1} * \cdots * \psi_{t_n})(x_1 + \cdots + x_n) = \sum_{i=1}^{n}(u * \psi_{t_i})(x_i) + \sum_{i=1}^{n}(u * \psi_{t_i} * \psi_{t_{i-1}})(x_i - x_{i-1}), \tag{3.8}$$

for all $x_1,\ldots,x_n \in \mathbb{R}^m$, $t_1,\ldots,t_n > 0$. In view of (3.8), it is easy to see that, for each fixed $x \in \mathbb{R}^m$,

$$f(x) := \lim_{t\to 0^+}(u * \psi_t)(x) \tag{3.9}$$

exists. Putting $(x_1,\ldots,x_n) = (0,\ldots,0)$ and letting $t_1 = \cdots = t_n \to 0^+$ in (3.8) yield $f(0) = 0$. Setting $(x_1, x_2,\ldots,x_n) = (x, 0,\ldots,0)$ and letting $t_1 = t$, $t_2 = \cdots = t_n \to 0^+$ in (3.8) give

$$(u * \psi_t)(-x) = -(u * \psi_t)(x), \tag{3.10}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Substituting $(x_1, x_2, x_3, \ldots, x_n)$ with $(x, y, 0, \ldots, 0)$ and letting $t_1 = t$, $t_2 = s$, $t_3 = \cdots = t_n \to 0^+$ in (3.8), we obtain from (3.10) that

$$(u * \psi_t * \psi_s)(x+y) + (u * \psi_t * \psi_s)(x-y) = 2(u * \psi_t)(x), \tag{3.11}$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. Letting $t \to 0^+$ in (3.11) yields

$$(u * \psi_s)(x+y) + (u * \psi_s)(x-y) = 2f(x), \tag{3.12}$$

for all $x, y \in \mathbb{R}^m$, $s > 0$. Putting $y = 0$ in (3.12) gives

$$f(x) = (u * \psi_s)(x), \tag{3.13}$$

for all $x \in \mathbb{R}^m$, $s > 0$. Applying (3.13) to (3.11), we see that f satisfies

$$f(x+y) + f(x-y) = 2f(x), \tag{3.14}$$

which is equivalent to the Cauchy equation (1.1) for all $x, y \in \mathbb{R}^m$. Since f is a smooth function in view of (3.13), it follows that $f(x) = a \cdot x$ for some $a \in \mathbb{C}^m$. Letting $s \to 0^+$ in (3.13), we finally obtain the general solution for (1.5). □

Now, we shall extend the stability theorem for the additive equation mentioned in the previous section to the space $\mathfrak{D}'(\mathbb{R}^m)$.

Theorem 3.2. *Suppose that u in $\mathfrak{D}'(\mathbb{R}^m)$ satisfies the inequality* (1.6)*, then there exists a unique $a \in \mathbb{C}^m$ such that*

$$\|u - a \cdot x\| \leq \frac{10n-3}{2(2n-1)}\epsilon. \tag{3.15}$$

Proof. It suffices to show that every distribution satisfying (1.6) belongs to the space $\mathcal{S}'(\mathbb{R}^m)$. Convolving the tensor product $\psi_{t_1}(x_1) \cdots \psi_{t_n}(x_n)$ on both sides of (1.6), we have

$$\left| (u * \psi_{t_1} * \cdots * \psi_{t_n})(x_1 + \cdots + x_n) - \sum_{i=1}^{n} (u * \psi_{t_i})(x_i) - \sum_{i=1}^{n} (u * \psi_{t_i} * \psi_{t_{i-1}})(x_i - x_{i-1}) \right| \leq \epsilon, \tag{3.16}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$. In view of (3.16), it is easy to see that for each fixed x,

$$f(x) := \limsup_{t \to 0^+} (u * \psi_t)(x) \tag{3.17}$$

exists. Putting $(x_1, \ldots, x_n) = (0, \ldots, 0)$ and letting $t_1 = \cdots = t_n \to 0^+$ in (3.16) yield

$$|f(0)| \leq \frac{\epsilon}{2n-1}. \tag{3.18}$$

Setting $(x_1, x_2, \ldots, x_n) = (x, 0, \ldots, 0)$ and letting $t_1 = t$, $t_2 = \cdots = t_n \to 0^+$ in (3.16), we have

$$\left|(u * \psi_t)(x) + (u * \psi_t)(-x) + (2n-3)f(0)\right| \le \epsilon, \tag{3.19}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Substituting $(x_1, x_2, x_3, \ldots, x_n)$ with $(x, y, 0, \ldots, 0)$ and letting $t_1 = t$, $t_2 = s$, $t_3 = \cdots = t_n \to 0^+$ in (3.16), we have

$$\begin{aligned} &\big|(u * \psi_t * \psi_s)(x+y) - (u * \psi_t * \psi_s)(y-x) \\ &\quad -2(u * \psi_t)(x) - (u * \psi_s)(y) - (u * \psi_s)(-y) - (2n-5)f(0)\big| \le \epsilon, \end{aligned} \tag{3.20}$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. It follows from (3.19) that the inequality (3.20) can be rewritten as

$$\left|(u * \psi_t * \psi_s)(x+y) - (u * \psi_t * \psi_s)(y-x) - 2(u * \psi_t)(x) + 2f(0)\right| \le 2\epsilon, \tag{3.21}$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. Letting $t \to 0^+$ in (3.21) yields

$$\left|(u * \psi_s)(x+y) - (u * \psi_s)(y-x) - 2f(x) + 2f(0)\right| \le 2\epsilon, \tag{3.22}$$

for all $x, y \in \mathbb{R}^m$, $s > 0$. Using (3.19) we may write the inequality (3.22) as

$$\left|(u * \psi_s)(x+y) + (u * \psi_s)(x-y) - 2f(x) + (2n-1)f(0)\right| \le 3\epsilon, \tag{3.23}$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. Putting $y = 0$ in (3.23) and dividing the result by 2 give

$$\left|(u * \psi_s)(x) - f(x) + \frac{(2n-1)}{2} f(0)\right| \le \frac{3}{2}\epsilon, \tag{3.24}$$

for all $x \in \mathbb{R}^m, s > 0$. From (3.23) and (3.24), we have

$$\left|f(x+y) + f(x-y) - 2f(x)\right| \le 6\epsilon, \tag{3.25}$$

which is equivalent to

$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \le 6\epsilon, \tag{3.26}$$

for all $x, y \in \mathbb{R}^m$. Thus, by virtue of the result as in [29], there exists a unique function $g : \mathbb{R}^m \to \mathbb{C}$ satisfying

$$g(x+y) = g(x) + g(y) \tag{3.27}$$

such that

$$\left|f(x) - g(x)\right| \le 6\epsilon + \left|g(0)\right|, \tag{3.28}$$

for all $x \in \mathbb{R}^m$. It follows from (3.18), (3.24), and (3.28) that

$$\left|(u * \psi_s)(x) - g(x)\right| \leq 8\epsilon + \left|g(0)\right|, \tag{3.29}$$

for all $x \in \mathbb{R}^m$, $s > 0$. Letting $s \to 0^+$ in (3.29), we obtain

$$\left\|u - g(x)\right\| \leq 8\epsilon + \left|g(0)\right|. \tag{3.30}$$

Inequality (3.30) implies that $h(x) := u - g(x)$ belongs to $(L^1)' = L^\infty$. Thus, we conclude that $u = g(x) + h(x) \in \mathcal{S}'(\mathbb{R}^m)$. □

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