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Research Article

Unbounded Solutions of the Difference Equation $x_n = x_{n-l}x_{n-k} - 1$

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The following difference equation $x_n = x_{n-l}x_{n-k} - 1$, $n \in \mathbb{N}_0$, where $k, l \in \mathbb{N}$, $k < l$, $\gcd(k, l) = 1$, and the initial values $x_{-l}, \ldots, x_{-2}, x_{-1}$ are real numbers, has been investigated so far only for some particular values of k and l. To get any general result on the equation is turned out as a not so easy problem. In this paper, we give the first result on the behaviour of solutions of the difference equation of general character, by describing the long-term behavior of the solutions of the equation for all values of parameters k and l, where the initial values satisfy the following condition $\min\{x_{-l}, \ldots, x_{-2}, x_{-1}\}$.

1. Introduction and Preliminaries

Studying nonlinear difference equations which do not stem from differential equations attracted considerable attention recently (see, e.g., [1–23] and the references therein).

Some particular cases of the following simple-look polynomial-type difference equation

$$x_n = x_{n-l}x_{n-k} - 1, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k, l \in \mathbb{N}$, $k < l$, $\gcd(k, l) = 1$, and the initial values $x_{-l}, \ldots, x_{-2}, x_{-1}$ are real numbers, have been investigated recently in papers [9–11]. More precisely, in [9] case $k = 1$, $l = 2$, was investigated, in [10] case $k = 2$, $l = 3$ was investigated, while in [11] case $k = 1$, $l = 3$ was investigated. Studying (1.1) turned out to be much more interesting than we had expected. Beside this, the behaviour of solutions of the equation is quite different for different values of k and l, and the methods we have used so far have been very different. Hence, it is of some interest to obtain some results which hold for all values of k and l. The problem turned out to be a tough task, however, we managed to obtain a result of general character.

In this paper, we completely describe the long-term behavior of solutions of difference equation (1.1) where $k, l \in \mathbb{N}$, $k < l$, $\gcd(k, l) = 1$, and the initial values $x_{-l}, \dots, x_{-2}, x_{-1}$ satisfy the condition

$$\min\{x_{-l}, \dots, x_{-2}, x_{-1}\} > \frac{1+\sqrt{5}}{2}. \tag{1.2}$$

Note that if $\overline{x}$ is an equilibrium of (1.1), then it satisfies the following equation:

$$\overline{x}^2 - \overline{x} - 1 = 0. \tag{1.3}$$

Hence (1.1) has exactly two equilibria, one positive and one negative, which we denote by $\overline{x}_1$ and $\overline{x}_2$, respectively:

$$\overline{x}_1 =: \frac{1-\sqrt{5}}{2}, \qquad \overline{x}_2 =: \frac{1+\sqrt{5}}{2}. \tag{1.4}$$

2. An Auxiliary Result

In this section, we prove an auxiliary result on the periodicity which will be used in the proof of the main result in this paper. The result is incorporated in the following lemma. Before we formulate and prove the lemma recall that a solution of difference equation (1.1) is called trivial if it is eventually equal to one of the equilibria in (1.4).

Lemma 2.1. *Assume that $k, l \in \mathbb{N}$, $k < l$, and $\gcd(k, l) = 1$, then difference equation* (1.1) *does not have any nontrivial periodic solution of period l.*

Proof. First, note that any solution of (1.1) with nonzero terms can be prolonged for all negative indices by the equation

$$x_{n-l} = \frac{x_n + 1}{x_{n-k}}. \tag{2.1}$$

Assume that $p_0, p_1, \dots, p_{l-1}$ is a nontrivial solution of difference equation (1.1) with period l.

The solution obviously satisfies the following nonlinear system of l (algebraic) equations:

$$p_{i+k} = p_{i+k}\, p_i - 1, \quad i = 0, 1, \dots, l-1, \tag{2.2}$$

where if an index $i + k$ is outside the set $\{0, 1, \dots, l-1\}$, we regard that

$$p_{i+k} = p_{i+k(\bmod\ l)}. \tag{2.3}$$

If some of p_is were equal to zero, we would get $0 = -1$, which would be a contradiction. Hence, we have that

$$p_i \neq 0, \quad \text{for each } i \in \{0, 1, \dots, l-1\}. \tag{2.4}$$

Moreover, according to the above-mentioned comment related to negative indices the solution can be regarded as a two-sided periodic solution, that is, the solution is of the form $(x_n)_{n=-\infty}^{\infty}$.

System (2.2) is equivalent to

$$p_i = \frac{p_{i+k}+1}{p_{i+k}} = f(p_{i+k}), \quad i = 0,1,\ldots,l-1. \tag{2.5}$$

From this and l periodicity of the sequence (p_i), it follows that

$$p_i = f^{[l]}(p_{i+lk}) = f^{[l]}(p_i), \quad i = 0,1,\ldots,l-1, \tag{2.6}$$

that is, p_i, $i = 0,1,\ldots,l-1$, are solutions of the equation

$$x = f^{[l]}(x). \tag{2.7}$$

It is clear that the equation can be written in the form

$$x = \frac{a_l x + b_l}{c_l x + d_l}, \tag{2.8}$$

for some real numbers $a_l,\ b_l,\ c_l$, and d_l and that they are obtained in the following way:

$$\begin{bmatrix} a_l & b_l \\ c_l & d_l \end{bmatrix} = \begin{bmatrix} a_{l-1} & b_{l-1} \\ c_{l-1} & d_{l-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^l. \tag{2.9}$$

Hence,

$$a_l = a_{l-1} + b_{l-1}, \quad b_l = a_{l-1}, \quad l \geq 2, \tag{2.10}$$

$$c_l = c_{l-1} + d_{l-1}, \quad d_l = c_{l-1}, \quad l \geq 2. \tag{2.11}$$

We now prove that

$$c_l = a_{l-1}, \quad d_l = b_{l-1}, \quad l \geq 2. \tag{2.12}$$

For $l = 2$, the equality in (2.12) is obvious. If (2.12) is true for $l-1$, then from the inductive hypothesis and the equalities in (2.10) and (2.11), we obtain the following relations:

$$\begin{gathered} c_l = c_{l-1} + d_{l-1} = a_{l-2} + b_{l-2} = a_{l-1}, \\ d_l = c_{l-1} = a_{l-2} = b_{l-1}, \quad l \geq 3, \end{gathered} \tag{2.13}$$

finishing the inductive proof of the claim.

If x is a solution to (2.8), then a simple calculation shows that the following equality holds:

$$c_l x^2 + (d_l - a_l)x - b_l = 0, \tag{2.14}$$

which is equivalent to

$$a_{l-1}\left(x^2 - x - 1\right) = 0. \tag{2.15}$$

From this, and since from (2.10) we have $a_l \geq a_{l-1} \geq \cdots \geq a_1 = 1$, for $l \geq 2$, it follows that $x = \overline{x}_1$ or $x = \overline{x}_2$. Hence, each p_i is equal to one of these two numbers.

Assume that $p_i = \overline{x}_1$ for some $i \in \{0, 1, \ldots, l-1\}$, then

$$p_i = f^{[j]}(p_i) = p_{i-jk}, \tag{2.16}$$

for each $j \in \{0, 1, \ldots, l-1\}$. If it were $i - j_1 k = i - j_2 k \ (\operatorname{mod} l)$ for some $j_1, j_2 \in \{0, 1, \ldots, l-1\}$, such that $j_1 \neq j_2$, we would have $(j_1 - j_2)k = 0 \ (\operatorname{mod} l)$, which is impossible due to the assumption $\gcd(k, l) = 1$. Hence, $p_i = \overline{x}_1$, for each $i \in \{0, 1, \ldots, l-1\}$.

Similarly, it is proved that if $p_i = \overline{x}_2$ for some $i \in \{0, 1, \ldots, l-1\}$, then $p_i = \overline{x}_2$, for each $i \in \{0, 1, \ldots, l-1\}$. From all, the above mentioned, the lemma follows. □

3. Unbounded Solutions of (1.1)

The following general theorem shows the existence of unbounded solutions of (1.1) relative to the set of initial conditions of the equation. The existence of various type of solutions of difference equations, such as monotonous, nontrivial, or periodic, has attracted also some attention recently (see, for example, [2–6, 8, 12–17, 24, 25] and the related references therein).

Theorem 3.1. *Assume that $k, l \in \mathbb{N}$, $k < l$, and $(x_n)_{n \geq -l}$ is a solution of (1.1), then the following statements hold true:*

(a) *if*

$$\min\{|x_{-l}|, \ldots, |x_{-2}|, |x_{-1}|\} > \frac{1 + \sqrt{5}}{2} = \overline{x}_2, \tag{3.1}$$

then the subsequences

$$\left(\left|x_{(m-1)l-i}\right|\right)_{m \in \mathbb{N}}, \quad i = 1, 2, \ldots, l \tag{3.2}$$

are strictly increasing,

(b) *if condition (3.1) holds and there is an $i_0 \in \{1, 2, \ldots, l\}$ such that the subsequence $(|x_{(m-1)l-i_0}|)_{m \in \mathbb{N}}$ is bounded, then the sequence $(|x_n|)_{n \geq -l}$ is bounded too,*

(c) *if $\gcd(k, l) = 1$ and*

$$\min\{x_{-l}, \ldots, x_{-2}, x_{-1}\} > \frac{1 + \sqrt{5}}{2} = \overline{x}_2, \tag{3.3}$$

then the sequence $(x_n)_{n \geq -l}$ tends to $+\infty$ as $n \to +\infty$.

Proof. (a) From the hypothesis, we have that

$$\left|x_{j-k}\right| - 1 > \overline{x}_2 - 1, \quad j = 0, 1, \ldots, k-1, \tag{3.4}$$

and so

$$\left|x_{j-l}\right|\left(\left|x_{j-k}\right| - 1\right) > \overline{x}_2\left(\overline{x}_2 - 1\right) = 1, \quad j = 0, 1, \ldots, k-1. \tag{3.5}$$

Hence,

$$\left|x_{j-l}\right|\left|x_{j-k}\right| - \left|x_{j-l}\right| > 1, \quad j = 0, 1, \ldots, k-1, \tag{3.6}$$

or equivalently

$$\left|x_{j-l}\right|\left|x_{j-k}\right| - 1 > \left|x_{j-l}\right|, \quad j = 0, 1, \ldots, k-1. \tag{3.7}$$

On the other hand, we have

$$\left|x_j\right| = \left|x_{j-l}x_{j-k} - 1\right| > \left|x_{j-l}\right|\left|x_{j-k}\right| - 1, \quad j = 0, 1, \ldots, k-1. \tag{3.8}$$

Combining (3.7) and (3.8), we have that

$$\left|x_j\right| > \left|x_{j-l}\right| > \overline{x}_2, \quad j = 0, 1, \ldots, k-1. \tag{3.9}$$

Assume that we have proved

$$\overline{x}_2 < \left|x_{j-l}\right| < \left|x_j\right|, \tag{3.10}$$

for $0 \le j \le j_0 < l-1$.

Since $j_0 + 1 - k \le j_0$, we can apply (3.10) and get

$$\left|x_{j_0+1-l}\right|\left(\left|x_{j_0+1-k}\right| - 1\right) > \overline{x}_2\left(\overline{x}_2 - 1\right) = 1. \tag{3.11}$$

Hence, from (1.1), the triangle inequality, (3.11), and hypothesis (3.10), we obtain

$$\left|x_{j_0+1}\right| = \left|x_{j_0+1-l}x_{j_0+1-k} - 1\right| > \left|x_{j_0+1-l}\right|\left|x_{j_0+1-k}\right| - 1 > \left|x_{j_0+1-l}\right| > \overline{x}_2. \tag{3.12}$$

Hence, by the induction, we get

$$\overline{x}_2 < \left|x_{j-l}\right| < \left|x_j\right|, \quad j = 0, 1, \ldots, l-1. \tag{3.13}$$

Now assume that we have proved

$$\overline{x}_2 < \left|x_{j-l}\right| < \left|x_j\right| < \left|x_{l+j}\right| < \cdots < \left|x_{ml+j}\right|, \quad j = 0, 1, \dots, l-1, \tag{3.14}$$

for some $m \in \mathbb{N}_0$.

Since (1.1) is autonomous, the sequence $y_n = x_{n+(m+1)l}$, $n \geq -l$ is the solution of the equation with initial conditions

$$y_{-l} = x_{ml}, \dots, y_{-1} = x_{ml+l-1}. \tag{3.15}$$

By what we have proved, it follows that

$$\overline{x}_2 < \left|y_{j-l}\right| < \left|y_j\right|, \quad j = 0, 1, \dots, l-1, \tag{3.16}$$

or equivalently

$$\overline{x}_2 < \left|x_{ml+j}\right| < \left|x_{(m+1)l+j}\right|, \quad j = 0, 1, \dots, l-1, \tag{3.17}$$

from which the inductive proof of the statement follows.

(b) Without loss of generality, we may assume that the subsequence $(|x_{(m-1)l}|)_{m\in\mathbb{N}_0}$ is bounded, since the other cases are obtained by shifting indices. Since by (a) the subsequence is increasing, then it increasingly converges, say to $a > \overline{x}_2$.

From this, and since

$$x_{ml-k} = \frac{1 + x_{ml}}{x_{(m-1)l}}, \quad m \in \mathbb{N}_0, \tag{3.18}$$

we get

$$|x_{ml-k}| = \frac{|1 + x_{ml}|}{\left|x_{(m-1)l}\right|} < \frac{1+a}{\overline{x}_2}, \quad m \in \mathbb{N}_0. \tag{3.19}$$

Hence, the subsequence $(|x_{ml-k}|)_{m\in\mathbb{N}_0}$ is bounded too, which along with the statement in (a) implies that it is convergent. Inductively, we get that the subsequences $(|x_{ml-sk}|)$ are bounded and consequently convergent for each $s \in \{0, 1, \dots, l-1\}$. Now, note that these l subsequences have disjoint sets of indices. Indeed, if we had

$$m_1 l - s_1 k = m_2 l - s_2 k, \tag{3.20}$$

for some $m_1, m_2 \in \mathbb{N}$ and $s_1, s_2 \in \{0, 1, \dots, l-1\}$, then we would have $(s_1 - s_2)k \equiv 0 \pmod{l}$. Since $|s_1 - s_2| < l$, it would mean that $\gcd(k, l) > 1$, which would be a contradiction.

This implies that all the subsequences $(|x_{(m-1)l-i}|)_{m\in\mathbb{N}}$, $i = 1, 2, \dots, l$ are convergent, from which the boundedness of $(|x_n|)_{n\geq -l}$ follows.

(c) Since in this case solution $(x_n)_{n\geq -l}$ of (1.1) is positive, by the proof of (b), we have that the subsequences $(x_{(m-1)l-i})_{m\in\mathbb{N}}$, $i = 1, 2, \dots, l$ are convergent. Hence, the solution either converges to a period l solution or to an equilibrium of (1.1). However, according to Lemma 2.1, (1.1) does not have any nontrivial solution of period l. Therefore, it must converge to an equilibrium, but, this is not possible because the largest equilibrium point is smaller than $\min\{x_{-l}, \ldots, x_{-2}, x_{-1}\}$, which is a contradiction, finishing the proof of the theorem. □

Question. An interesting problem is to investigate whether condition (3.1) guarantees that $|x_n| \to \infty$ as $n \to \infty$.

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