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Research Article

The Centre of the Spaces of Banach Lattice-Valued Continuous Functions on the Generalized Alexandroff Duplicate

Faruk Polat

Department of Mathematics, Firat University, 23119 Elazig, Turkey

Correspondence should be addressed to Faruk Polat, faruk.polat@gmail.com

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We characterize the centre of the Banach lattice of Banach lattice E-valued continuous functions on the Alexandroff duplicate of a compact Hausdorff space K in terms of the centre of $C(K, E)$, the space of E-valued continuous functions on K. We also identify the centre of $CD_0(Q, E) = C(Q, E) + c_0(Q, E)$ whose elements are the sums of E-valued continuous and discrete functions defined on a compact Hausdorff space Q without isolated points, which was given by Alpay and Ercan (2000).

1. Preliminaries and Definitions

Throughout the paper, our terminology is mainly standard and a background on Riesz spaces and Banach lattices may be obtained from [1] or [2]. In order to avoid trivial cases, we assume that all topological spaces are nonempty and all Banach lattices are nonzero.

The *centre* of a Banach lattice E, denoted by $Z(E)$, is the lattice of the linear operators, $T : E \to E$ for which there exists a real number $\lambda > 0$ such that $|Tx| \le \lambda |x|$ for all $x \in E$. The operator norm of a central operator T is the minimum of those λ with this property. It is well known that $Z(E)$ equipped with the operator norm is an AM-space with order unit. The order unit is identity operator I.

For a given locally compact Hausdorff space K and a Banach lattice E, $C_0(K, E)$ denotes the space of all continuous functions f from K into E which *vanish at infinity*; that is, there exists a compact set $A \subset K$ such that $\|f(k)\| < \varepsilon$ for each $\varepsilon > 0$ and $k \in K \setminus A$. We consider this space to be normed by

$$\|f\| = \sup\{\|f(k)\| : k \in K\}, \tag{1.1}$$

and ordered by

$$f \geq g \Longleftrightarrow f(k) \geq g(k), \quad \forall k \in K. \tag{1.2}$$

One can show that $C_0(K, E)$ is a Banach lattice with these definitions.

Ercan and Wickstead [3] showed that the centre of $C_0(K, E)$ is isometrically Riesz isomorhic to $C^b(K, Z(E)_s)$ the space of all functions f from K into $Z(E)$ such that f is norm bounded, continuous, and $f(k_\alpha)(e) \to f(k)(e)$ in E for each $e \in E$ whenever $k_\alpha \to k$ in K. Here, $Z(E)$ is given the strong operator topology.

If K is a compact Hausdorff space, then $C_0(K, E) = C(K, E)$, where $C(K, E)$ is the space of continuous functions $f : K \to E$. Hence, the centre of $C(K, E)$ can also be identified with $C^b(K, Z(E)_s)$. We will use this identification in the sequel.

If K is a discrete topological space, then $C_0(K, E)$ is the space of E-valued bounded functions f on K such that the set

$$\{k \in K : \varepsilon < \|f(k)\|\} \tag{1.3}$$

is finite for each $\varepsilon > 0$, and we will write $c_0(K, E)$ in this case.

Let Σ and Γ be compact Hausdorff and locally compact Hausdorff topologies on a nonempty set K, respectively, such that Σ is *coarser* than Γ. These topologies on K will be denoted by K_Σ and K_Γ. The compact Hausdorff topology on $K \times \{0, 1\}$ generated by the open base $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\begin{gathered} \mathcal{A}_1 = \{H \times \{1\} : H \text{ is } \Gamma\text{-open}\}, \\ \mathcal{A}_2 = \{G \times \{0, 1\} \setminus M \times \{1\} : G \text{ is } \Sigma\text{-open}, \ M \text{ is } \Gamma\text{-compact}\} \end{gathered} \tag{1.4}$$

is called *generalized Alexandroff duplicate* of K and denoted by $K_{\Sigma,\Gamma} \otimes \{0, 1\}$ (see [4]). When Γ is discrete topology on K, the compact Hausdorff topological space $K_{\Sigma,\Gamma} \otimes \{0, 1\}$ will be denoted by $A(K)$. The space $A(K)$ was first considered by Engelking [5]. For $K = [0, 1]$ under the usual metric topology, $A(K)$ was constructed by Alexandroff and Urysohn [6] as an example of a compact Hausdorff space containing a discrete dense subspace. This space is called *the Alexandroff duplicate*.

Note that $K \times \{0\}$ is a closed subspace of $K_{\Sigma,\Gamma} \otimes \{0, 1\}$ and the map $k \to (k, 0)$ is a homeomorphism between K_Σ and $K \times \{0\}$.

In [4], it is not proved that $K_{\Sigma,\Gamma} \otimes \{0, 1\}$ is a compact Hausdorff space. We give the proof here for the benefit of the reader.

Theorem 1.1. *$K_{\Sigma,\Gamma} \otimes \{0, 1\}$ is a compact Hausdorff space.*

Proof. Consider an open cover $\{O_i\}_{i \in I}$ of $K_{\Sigma,\Gamma} \otimes \{0, 1\}$. By replacing each set in the cover by a union of basic open neighborhoods of all points in the set, we can assume that the cover is formed by basic open neighborhoods of the form

$$\{H_\alpha \times \{1\}\}_{\alpha \in I} \cup \{G_\gamma \times \{0, 1\} \setminus M_\gamma \times \{1\}\}_{\gamma \in \Omega}, \tag{1.5}$$

where H_α is a Γ-open set, G_γ is a Σ-open set, and M_γ is a Γ-compact set. It is easy to see that $\{G_\gamma \times \{0\}\}_{\gamma\in\Omega}$ is an open cover of $K\times\{0\}$, thus there is a finite subcover $G_{\gamma_1}\times\{0\},\dots,G_{\gamma_n}\times\{0\}$. Then,

$$G_{\gamma_1} \times \{0,1\} \setminus M_{\gamma_1} \times \{1\} \cup \cdots \cup G_{\gamma_n} \times \{0,1\} \setminus M_{\gamma_n} \times \{1\} \tag{1.6}$$

misses only finitely many Γ-compact sets $M_{\gamma_1}\times\{1\},\dots,M_{\gamma_n}\times\{1\}$.

As M_{γ_j} $(j=1,2,\dots,n)$ is compact, we have that $M_{\gamma_j}\times\{1\}\subset \cup H_\alpha \times\{1\}$. So, $M_{\gamma_j}\times\{1\}\subset \cup_{p=1}^{n} H_{p^j}\times\{1\}$. Hence, if we add the corresponding open sets from the cover, then we obtain a finite cover of the entire space $K_{\Sigma,\Gamma}\otimes\{0,1\}$. Therefore, $K_{\Sigma,\Gamma}\otimes\{0,1\}$ is compact.

To show that $K_{\Sigma,\Gamma}\otimes\{0,1\}$ is Hausdorff, it is enough to show that $(k,0)$ and $(k,1)$ can be separated. Let V be a Γ-open neighborhood of k such that $cl_\Gamma(V)$ (closure of V in K_Γ) is compact. Then, $K_{\Sigma,\Gamma}\otimes\{0,1\}\setminus(cl_\Gamma(V)\times\{1\})$ and $V\times\{1\}$ are the separating open sets of $(k,0)$ and $(k,1)$, respectively. This completes the proof. □

If K_Σ is a compact Hausdorff space without isolated points and K_Γ is a discrete topological space, then $C(K_\Sigma,E)\cap c_0(K_\Gamma,E)=\{0\}$ and $CD_0(K_\Sigma,E)=C(K_\Sigma,E)\oplus c_0(K_\Gamma,E)$ is a Banach lattice under the pointwise ordering and supremum norm of the sums $f+d$, where $f\in C(K_\Sigma,E)$ and $d\in c_0(K_\Gamma,E)$. We refer to [7–9] for more detailed information on these spaces. In [4], it is showed that $CD_0(K_\Sigma,E)$ is isometrically Riesz isomorphic to $C(A(K),E)$, where $A(K)$ is the Alexandroff duplicate of K. We will use this identification in the sequel to characterize the centre of the space $CD_0(K_\Sigma,E)$.

2. Main Results

Let Σ and Γ be compact Hausdorff and locally compact Hausdorff topologies on K, respectively, such that Σ is coarser than Γ, and let E be a Banach lattice. Then $C^{b_*}(K_\Sigma,Z(E)_s)$ denotes the set of all norm bounded and continuous functions f from K into $Z(E)$ such that $r_\alpha f(k_\alpha)(e)\to rf(k)(e)$ in E for each $e\in E$ whenever $(k_\alpha,r_\alpha)\to(k,r)$ in $K_{\Sigma,\Gamma}\otimes\{0,1\}$.

We consider the vector space $C^{b}(K_\Sigma,Z(E)_s)\times C^{b_*}(K_\Sigma,Z(E)_s)$ equipped with coordinatewise algebraic operations, the order

$$0\le(f,d)\Longleftrightarrow 0\le f(k)(e),\quad 0\le f(k)(e)+d(k)(e)\quad \text{for each } k\in K, \tag{2.1}$$

and the norm

$$\|(f,d)\|=\max\{\|f(k)+rd(k)\|:(k,r)\in K\times\{0,1\}\}. \tag{2.2}$$

The norm defined on $C^{b}(K_\Sigma,Z(E)_s)\times C^{b_*}(K_\Sigma,Z(E)_s)$ makes it a Banach space. This is clear, as this norm is equivalent to standard products norms (we have, e.g., $(1/2)\max\{\|f\|,\|d\|\}\le\|(f,d)\|\le(\|f\|+\|d\|)$). This has no relation to Banach lattices, but it is just a property of Banach spaces. The space $C^{b}(K_\Sigma,Z(E)_s)\times C^{b_*}(K_\Sigma,Z(E)_s)$ is a lattice. This is proved by computing $|(f,d)|=(|f|,|f+d|-|f|)$, where the absolute values on the right-hand side are pointwise. The norm defined on $C^{b}(K_\Sigma,Z(E)_s)\times C^{b_*}(K_\Sigma,Z(E)_s)$ is a Riesz norm. This is obvious from definitions. Therefore, the space $C^{b}(K_\Sigma,Z(E)_s)\times C^{b_*}(K_\Sigma,Z(E)_s)$ is a Banach lattice.

Actually, the space $C^b(K_\Sigma, Z(E)_s) \times C^{b_*}(K_\Sigma, Z(E)_s)$ is isometrically Riesz isomorphic to $C^b(K_{\Sigma,\Gamma} \otimes \{0,1\}, Z(E)_s)$ the space of norm bounded, continuous functions f from $K \times \{0,1\}$ into $Z(E)$ such that $f(k_\alpha, r_\alpha)(e) \to f(k,r)(e)$ in E for each $e \in E$ whenever $(k_\alpha, r_\alpha) \to (k,r)$ in $K_{\Sigma,\Gamma} \otimes \{0,1\}$ as the following shows.

Theorem 2.1. *$C^b(K_\Sigma, Z(E)_s) \times C^{b_*}(K_\Sigma, Z(E)_s)$ and $C^b(K_{\Sigma,\Gamma} \otimes \{0,1\}, Z(E)_s)$ are isometrically Ries isomorphic spaces.*

Proof. Define the map

$$\pi : C^b(K_\Sigma, Z(E)_s) \times C^{b_*}(K_\Sigma, Z(E)_s) \longrightarrow C^b(K_{\Sigma,\Gamma} \otimes \{0,1\}, Z(E)_s), \tag{2.3}$$

by

$$\pi(f,d)(k,r)(e) = f(k)(e) + rd(k)(e), \tag{2.4}$$

for each $(k,r) \in K \times \{0,1\}$ and $e \in E$.

Let $(k_\alpha, r_\alpha) \to (k,r)$ in $K_{\Sigma,\Gamma} \otimes \{0,1\}$. Then, $k_\alpha \to k$ in K_Σ so that $f(k_\alpha)(e) \to f(k)(e)$ and $r_\alpha d(k_\alpha)(e) \to rd(k)(e)$ in E for each $e \in E$. Hence, $f(k_\alpha)(e) + r_\alpha d(k_\alpha)(e) \to f(k)(e) + rd(k)(e)$ in E for each $e \in E$ so that the map π is well defined. It follows immediately that π is an isometry, as $\pi(f,d)$ agrees with $f + d$ on $K \times \{1\}$ and with f on $K \times \{0\}$. It is obvious that $\pi(f,d) \geq 0 \Leftrightarrow (f,d) \geq 0$.

It remains to show that π is onto. Let $h \in C^b(K_{\Sigma,\Gamma} \otimes \{0,1\}, Z(E)_s)$ be given. Define

$$f(k)(e) = h(k,0)(e), \quad d(k)(e) = h(k,1)(e) - h(k,0)(e), \tag{2.5}$$

for each $k \in K$ and $e \in E$. The norm boundedness of f and d follows directly from the norm boundedness of h. If $k_\alpha \to k$ in K_Σ, then $(k_\alpha, 0) \to (k,0)$ in $K_{\Sigma,\Gamma} \otimes \{0,1\}$ so that

$$f(k_\alpha)(e) = h(k_\alpha, 0)(e) \longrightarrow h(k,0)(e) = f(k)(e), \tag{2.6}$$

in E for each $e \in E$, hence $f \in C^b(K_\Sigma, Z(E)_s)$.

To show that $d \in C^{b_*}(K_\Sigma, Z(E)_s)$, let $(k_\alpha, r_\alpha) \to (k,r) \in K_{\Sigma,\Gamma} \otimes \{0,1\}$. We now examine the possibilities.

Suppose first that $r = 1$. Then, (r_α) is eventually 1. As $(k_\alpha, 0) \to (k,0)$ in $K_{\Sigma,\Gamma} \otimes \{0,1\}$, we have $r_\alpha d(k_\alpha)(e) \to rd(k)(e)$ in E for each $e \in E$ in this possibility.

Suppose now that $(k_\alpha, r_\alpha) \to (k,0)$ and assume that $r_\alpha d(k_\alpha)(e)$ does not converge to zero in E. Then, there is a subnet (r_{α_β}) of (r_α) such that $r_{\alpha_\beta} = 1$ and $\varepsilon < \|d(k_{\alpha_\beta})(e)\|$ for each β and for some $\varepsilon > 0$. On the other hand, since $(k_{\alpha_\beta}, 1) \to (k,0)$ and $(k_{\alpha_\beta}, 0) \to (k,0)$ in $K_{\Sigma,\Gamma} \otimes \{0,1\}$, we have $h(k_{\alpha_\beta}, 1)(e) \to h(k,0)(e)$ and $h(k_{\alpha_\beta}, 0)(e) \to h(k,0)(e)$ so that $d(k_{\alpha_\beta})(e) = h(k_{\alpha_\beta}, 1)(e) - h(k_{\alpha_\beta}, 0)(e) \to 0$. This contradiction shows that $d \in C^{b_*}(K_\Sigma, Z(E)_s)$. It is clear that $\pi(f,d) = h$, and this completes the proof. □

Since $Z(C(K_\Sigma, E))$ and $Z(C(K_{\Sigma,\Gamma} \otimes \{0,1\}, E))$ can be identified with $C^b(K_\Sigma, Z(E)_s)$ and $C^b(K_{\Sigma,\Gamma} \otimes \{0,1\}, Z(E)_s)$, respectively, we immediately have the following from the previous theorem.

Corollary 2.2. *$Z(C(K_{\Sigma,\Gamma} \otimes \{0,1\}, E)$ and $Z(C(K_\Sigma, E)) \times C^{b_*}(K_\Sigma, Z(E)_s)$ are isometrically Riesz isomorphic spaces.*

Let K_Γ be a discrete topology, and let E be a Banach lattice. The set of all bounded functions $f : K \to Z(E)$ such that the set $\{k : \varepsilon < \|f(k)(e)\| \text{ for all } e \in E\}$ is finite will be denoted by $c_0(K_\Gamma, Z(E)_s)$.

Lemma 2.3. *Let K_Σ be a compact Hausdorff space, and let Γ be a discrete topology on K. Then, $C^{b_*}(K_\Sigma, Z(E)_s) = c_0(K_\Gamma, Z(E)_s)$.*

Proof. Let $f \in c_0(K_\Gamma, Z(E)_s)$. Suppose that $f \notin C^{b_*}(K_\Sigma, Z(E)_s)$. Then, there exists a net $(k_\alpha, 1)$ in $A(K)$ such that $(k_\alpha, 1) \to (k, 0) \in A(K)$ and $\varepsilon < \|f(k_{\alpha_\beta})(e)\|$ for some subnet (k_{α_β}) of (k_α), $\varepsilon > 0$, and for each $e \in E$. So, (k_{α_β}) has finite range which is a contradiction. Conversely, assume that $f \in C^{b_*}(K_\Sigma, Z(E)_s)$ but $f \notin c_0(K_\Gamma, Z(E)_s)$. Then, there exist some $e \in E$ and a sequence (k_n) such that $\varepsilon < \|f(k_n)(e)\|$ for each n and $k_n \neq k_m$ whenever $n \neq m$. Then, there exists a subnet (k_{n_α}) of k_n such that $(k_{n_\alpha}, 1) \to (k, 0)$ so that $f(k_{n_\alpha})(e) \to 0$ which is impossible and this completes the proof. □

By Theorem 2.1 and the previous lemma, we have the following.

Theorem 2.4. *Let K_Σ be a compact Hausdorff space, and let Γ be a discrete topology on K. Then, $C^b(A(K), Z(E)_s)$ and $C^b(K_\Sigma, Z(E)_s) \times c_0(K_\Gamma, Z(E)_s)$ are isometrically Riesz isomorphic spaces.*

As the centre of $CD_0(K_\Sigma, E)$ can be identified with $C^b(A(K), Z(E)_s)$, we immediately have Theorem 3.1 of [8] as follows.

Corollary 2.5. *Let K_Σ be a compact Hausdorff space without isolated points, and let Γ be a discrete topology on K. Then, the centre of $CD_0(K_\Sigma, E)$ and $Z(C(K_\Sigma, E)) \times c_0(K_\Gamma, Z(E)_s)$ are isometrically Riesz isomorphic spaces.*

Note that in the corollary above, if all the operators $T \in Z(E)$ are norm attaining; that is, there exists some $e \in E$ with $\|e\| = 1$ such that $\|T\| = \|T(e)\|$, then $c_0(K_\Gamma, Z(E)_s)$ can be replaced by $c_0(K_\Gamma, Z(E))$.

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