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Research Article

A Sharp Double Inequality between Harmonic and Identric Means

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We find the greatest value p and the least value q in $(0, 1/2)$ such that the double inequality $H(pa + (1-p)b, pb + (1-p)a) < I(a,b) < H(qa + (1-q)b, qb + (1-q)a)$ holds for all $a, b > 0$ with $a \neq b$. Here, $H(a,b)$, and $I(a,b)$ denote the harmonic and identric means of two positive numbers a and b, respectively.

1. Introduction

The classical harmonic mean $H(a,b)$ and identric mean $I(a,b)$ of two positive numbers a and b are defined by

$$H(a,b) = \frac{2ab}{a+b}, \tag{1.1}$$

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.2}$$

respectively. Recently, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for H and I can be found in the literature [1–17].

Let $M_p(a,b) = [(a^p + b^p)/2]^{1/p}$, $L(a,b) = (a-b)/(\log a - \log b)$, $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$, and $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$ be the pth power, logarithmic,

geometric, arithmetic, and Seiffert means of two positive numbers a and b with $a \neq b$, respectively. Then it is well-known that

$$\begin{aligned} \min\{a,b\} < H(a,b) &= M_{-1}(a,b) < G(a,b) \\ &= M_0(a,b) < L(a,b) \\ &< P(a,b) < I(a,b) < A(a,b) \\ &= M_1(a,b) < \max\{a,b\} \end{aligned} \tag{1.3}$$

for all $a,b > 0$ with $a \neq b$.

Long and Chu [18] answered the question: what are the greatest value p and the least value q such that $M_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < M_q(a,b)$ for all $a,b > 0$ with $a \neq b$ and $\alpha,\beta > 0$ with $\alpha + \beta < 1$.

In [19], the authors proved that the double inequality

$$\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b) \tag{1.4}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$.

The following sharp bounds for I, $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means are presented in [20]:

$$\begin{gathered} M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \qquad M_0(a,b) < \sqrt{L(a,b)I(a,b)} < M_{1/2}(a,b), \\ M_{\log 2/(1+\log 2)}(a,b) < \frac{L(a,b)+I(a,b)}{2} < M_{1/2}(a,b) \end{gathered} \tag{1.5}$$

for all $a,b > 0$ with $a \neq b$.

Alzer and Qiu [21] proved that the inequalities

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{1.6}$$

hold for all positive real numbers a and b with $a \neq b$ if and only if $\alpha \leq 2/3$ and $\beta \geq 2/e = 0.73575$, and so forth.

For fixed $a,b > 0$ with $a \neq b$ and $x \in [0,1/2]$, let

$$f(x) = H(xa + (1-x)b, xb + (1-x)a). \tag{1.7}$$

Then it is not difficult to verify that $f(x)$ is continuous and strictly increasing in $[0,1/2]$. Note that $f(0) = H(a,b) < I(a,b)$ and $f(1/2) = A(a,b) > I(a,b)$. Therefore, it is natural to ask what are the greatest value p and the least value q in $(0,1/2)$ such that the double inequality $H(pa + (1-p)b, pb + (1-p)a) < I(a,b) < H(qa + (1-q)b, qb + (1-q)a)$ holds for all $a,b > 0$ with $a \neq b$. The main purpose of this paper is to answer these questions. Our main result is Theorem 1.1.

Theorem 1.1. *If $p, q \in (0, 1/2)$, then the double inequality*

$$\begin{aligned} &H(pa + (1-p)b, pb + (1-p)a) \\ &\quad < I(a,b) \\ &\quad < H(qa + (1-q)b, qb + (1-q)a) \end{aligned} \tag{1.8}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq (1 - \sqrt{1 - 2/e})/2$ and $q \geq (6 - \sqrt{6})/12$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (6 - \sqrt{6})/12$ and $\mu = (1 - \sqrt{1 - 2/e})/2$. Then from the monotonicity of the function $f(x) = H(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.8) we only need to prove that inequalities

$$I(a,b) < H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a), \tag{2.1}$$

$$I(a,b) > H(\mu a + (1-\mu)b, \mu b + (1-\mu)a), \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = a/b > 1$ and $r \in (0, 1/2)$, then from (1.1) and (1.2) one has

$$\begin{aligned} &\log H(ra + (1-r)b, rb + (1-r)a) - \log I(a,b) \\ &\quad = \log\left\{r(1-r)t^2 + \left[r^2 + (1-r)^2\right]t + r(1-r)\right\} \\ &\qquad - \log(t+1) - \frac{t\log t}{t-1} + 1 + \log 2. \end{aligned} \tag{2.3}$$

Let

$$\begin{aligned} g(t) &= \log\left\{r(1-r)t^2 + \left[r^2 + (1-r)^2\right]t + r(1-r)\right\} \\ &\quad - \log(t+1) - \frac{t\log t}{t-1} + 1 + \log 2. \end{aligned} \tag{2.4}$$

Then simple computations lead to

$$g(1) = 0, \tag{2.5}$$

$$\lim_{t \to +\infty} g(t) = \log[r(1-r)] + 1 + \log 2, \tag{2.6}$$

$$g'(t) = \frac{g_1(t)}{(t-1)^2}, \tag{2.7}$$

where

$$g_1(t) = \log t - \frac{(t-1)\left[\left(2r^2-2r+1\right)t^2+4r(1-r)t+2r^2-2r+1\right]}{(t+1)\left[r(1-r)t^2+(2r^2-2r+1)t+r(1-r)\right]}, \tag{2.8}$$

$$g_1(1) = 0, \tag{2.9}$$

$$\lim_{t\to+\infty} g_1(t) = +\infty, \tag{2.10}$$

$$g_1'(t) = \frac{g_2(t)}{t(t+1)^2\left[r(1-r)t^2+(2r^2-2r+1)t+r(1-r)\right]^2}, \tag{2.11}$$

where

$$\begin{aligned} g_2(t) &= r^2(1-r)^2t^6+\left(2r^4-4r^3-2r^2+4r-1\right)t^5-\left(17r^4-34r^3+25r^2-8r+1\right)t^4 \\ &\quad +4\left(7r^4-14r^3+13r^2-6r+1\right)t^3-\left(17r^4-34r^3+25r^2-8r+1\right)t^2 \\ &\quad +\left(2r^4-4r^3-2r^2+4r-1\right)t+r^2(1-r)^2, \end{aligned} \tag{2.12}$$

$$g_2(1) = 0, \tag{2.13}$$

$$\lim_{t\to+\infty} g_2(t) = +\infty, \tag{2.14}$$

$$\begin{aligned} g_2'(t) &= 6r^2(1-r)^2t^5+5\left(2r^4-4r^3-2r^2+4r-1\right)t^4-4\left(17r^4-34r^3+25r^2-8r+1\right)t^3 \\ &\quad +12\left(7r^4-14r^3+13r^2-6r+1\right)t^2-2\left(17r^4-34r^3+25r^2-8r+1\right)t \\ &\quad +2r^4-4r^3-2r^2+4r-1, \end{aligned} \tag{2.15}$$

$$g_2'(1) = 0, \tag{2.16}$$

$$\lim_{t\to+\infty} g_2'(t) = +\infty, \tag{2.17}$$

$$\begin{aligned} g_2''(t) &= 30r^2(1-r)^2t^4+20\left(2r^4-4r^3-2r^2+4r-1\right)t^3-12\left(17r^4-34r^3+25r^2-8r+1\right)t^2 \\ &\quad +24\left(7r^4-14r^3+13r^2-6r+1\right)t-2\left(17r^4-34r^3+25r^2-8r+1\right), \end{aligned} \tag{2.18}$$

$$g_2''(1) = -2\left(24r^2-24r+5\right), \tag{2.19}$$

$$\lim_{t\to+\infty} g_2''(t) = +\infty, \tag{2.20}$$

$$\begin{aligned} g_2'''(t) &= 120r^2(1-r)^2t^3+60\left(2r^4-4r^3-2r^2+4r-1\right)t^2 \\ &\quad -24\left(17r^4-34r^3+25r^2-8r+1\right)t+24\left(7r^4-14r^3+13r^2-6r+1\right), \end{aligned} \tag{2.21}$$

$$g_2'''(1) = -12\left(24r^2-24r+5\right), \tag{2.22}$$

$$\lim_{t\to\infty} g_2'''(t) = \infty, \tag{2.23}$$

$$\begin{aligned} g_2^{(4)}(t) &= 360r^2(1-r)^2t^2+120\left(2r^4-4r^3-2r^2+4r-1\right)t \\ &\quad -24\left(17r^4-34r^3+25r^2-8r+1\right), \end{aligned} \tag{2.24}$$

$$g_2^{(4)}(1) = 48\left(4r^4 - 8r^3 - 10r^2 + 14r - 3\right), \tag{2.25}$$

$$\lim_{t\to+\infty} g_2^{(4)}(t) = +\infty, \tag{2.26}$$

$$g_2^{(5)}(t) = 720r^2(1-r)^2 t + 120\left(2r^4 - 4r^3 - 2r^2 + 4r - 1\right), \tag{2.27}$$

$$g_2^{(5)}(1) = 120\left(8r^4 - 16r^3 + 4r^2 + 4r - 1\right). \tag{2.28}$$

We divide the proof into two cases.

Case 1 ($r = \lambda = (6-\sqrt{6})/12$). Then (2.19), (2.22), (2.25), and (2.28) lead to

$$g_2''(1) = 0, \tag{2.29}$$

$$g_2'''(1) = 0, \tag{2.30}$$

$$g_2^{(4)}(1) = \frac{13}{3} > 0, \tag{2.31}$$

$$g_2^{(5)}(1) = \frac{65}{3} > 0. \tag{2.32}$$

From (2.27) we clearly see that $g_2^{(5)}(t)$ is strictly increasing in $[1,+\infty)$, then inequality (2.32) leads to the conclusion that $g_2^{(5)}(t) > 0$ for $t \in [1,+\infty)$, hence $g_2^{(4)}(t)$ is strictly increasing in $[1,+\infty)$.

It follows from inequality (2.31) and the monotonicity of $g_2^{(4)}(t)$ that $g_2'''(t)$ is strictly increasing in $[1,+\infty)$. Then (2.30) implies that $g_2'''(t) > 0$ for $t \in [1,+\infty)$, so $g_2''(t)$ is strictly increasing in $[1,+\infty)$.

From (2.29) and the monotonicity of $g_2''(t)$ we clearly see that $g_2'(t)$ is strictly increasing in $[1,+\infty)$.

From (2.5), (2.7), (2.9), (2.11), (2.13), (2.16), and the monotonicity of $g_2'(t)$ we conclude that

$$g(t) > 0 \tag{2.33}$$

for $t \in (1,+\infty)$.

Therefore, inequality (2.1) follows from (2.3) and (2.4) together with inequality (2.33).

Case 2 ($r = \mu = (1-\sqrt{1-2/e})/2$). Then (2.19), (2.22), (2.25), and (2.28) lead to

$$g_2''(1) = -\frac{2}{e}(5e-12) < 0, \tag{2.34}$$

$$g_2'''(1) = -\frac{12}{e}(5e-12) < 0, \tag{2.35}$$

$$g_2^{(4)}(1) = -\frac{48}{e^2}\left(3e^2 - 7e - 1\right) < 0, \tag{2.36}$$

$$g_2^{(5)}(1) = \frac{120}{e^2}\left(2 + 2e - e^2\right) > 0. \tag{2.37}$$

From (2.27) and (2.37) we know that ${g_2}^{(4)}(t)$ is strictly increasing in $[1,+\infty)$. Then (2.26) and (2.36) lead to the conclusion that there exists $t_1 > 1$ such that ${g_2}^{(4)}(t) < 0$ for $t \in [1,t_1)$ and ${g_2}^{(4)}(t) > 0$ for $t \in (t_1,+\infty)$, hence $g_2'''(t)$ is strictly decreasing in $[1,t_1]$ and strictly increasing in $[t_1,+\infty)$.

It follows from (2.23) and (2.35) together with the piecewise monotonicity of $g_2'''(t)$ that there exists $t_2 > t_1 > 1$ such that $g_2''(t)$ is strictly decreasing in $[1,t_2]$ and strictly increasing in $[t_2,+\infty)$. Then (2.20) and (2.34) lead to the conclusion that there exists $t_3 > t_2 > 1$ such that $g_2'(t)$ is strictly decreasing in $[1,t_3]$ and strictly increasing in $[t_3,+\infty)$.

From (2.16) and (2.17) together with the piecewise monotonicity of $g_2'(t)$ we clearly see that there exists $t_4 > t_3 > 1$ such that $g_2'(t) < 0$ for $t \in (1,t_4)$ and $g_2'(t) > 0$ for $t \in (t_4,+\infty)$. Therefore, $g_2(t)$ is strictly decreasing in $[1,t_4]$ and strictly increasing in $[t_4,+\infty)$. Then (2.11)–(2.14) lead to the conclusion that there exists $t_5 > t_4 > 1$ such that $g_1(t)$ is strictly decreasing in $[1,t_5]$ and strictly increasing in $[t_5,+\infty)$.

It follows from (2.7)–(2.10) and the piecewise monotonicity of $g_1(t)$ that there exists $t_6 > t_5 > 1$ such that $g(t)$ is strictly decreasing in $[1,t_6]$ and strictly increasingin $[t_6,+\infty)$.

Note that (2.6) becomes

$$\lim_{t\to+\infty} g(t) = \log[r(1-r)] + 1 + \log 2 = 0 \tag{2.38}$$

for $r = \mu = (1-\sqrt{1-2/e})/2$.

From (2.5) and (2.38) together with the piecewise monotonicity of $g(t)$ we clearly see that

$$g(t) < 0 \tag{2.39}$$

for $t \in (1,+\infty)$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with inequality (2.39).

Next, we prove that the parameter $\lambda = (6-\sqrt{6})/12$ is the best possible parameter in $(0,1/2)$ such that inequality (2.1) holds for all $a,b > 0$ with $a \neq b$. In fact, if $r < \lambda = (6-\sqrt{6})/12$, then (2.19) leads to $g_2''(1) = -2(24r^2 - 24r + 5) < 0$. From the continuity of $g_2''(t)$ we know that there exists $\delta > 0$ such that

$$g_2''(t) < 0 \tag{2.40}$$

for $t \in (1,1+\delta)$.

It follows from (2.3)–(2.5), (2.7), (2.9), (2.11), (2.13), and (2.16) that $I(a,b) > H(ra + (1-r)b, rb + (1-r)a)$ for $a/b \in (1,1+\delta)$.

Finally, we prove that the parameter $\mu = (1-\sqrt{1-2/e})/2$ is the best possible parameter in $(0,1/2)$ such that inequality (2.2) holds for all $a,b > 0$ with $a \neq b$. In fact, if $(1-\sqrt{1-2/e})/2 = \mu < r < 1/2$, then (2.6) leads to $\lim_{t\to+\infty} g(t) > 0$. Hence, there exists $T > 1$ such that

$$g(t) > 0 \tag{2.41}$$

for $t \in (T,+\infty)$.

Therefore, $H(ra+(1-r)b, rb+(1-r)a) > I(a,b)$ for $a/b \in (T,+\infty)$, follows from (2.3) and (2.4) together with inequality (2.41). □

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