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Research Article

Compatible and Incompatible Nonuniqueness Conditions for the Classical Cauchy Problem

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In the first part of this paper sufficient conditions for nonuniqueness of the classical Cauchy problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$ are given. As the essential tool serves a method which estimates the "distance" between two solutions with an appropriate Lyapunov function and permits to show that under certain conditions the "distance" between two different solutions vanishes at the initial point. In the second part attention is paid to conditions that are obtained by a formal inversion of uniqueness theorems of Kamke-type but cannot guarantee nonuniqueness because they are incompatible.

1. Introduction

Consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{1.1}$$

where $t_0 \in \mathbb{R}$, $t \in J := [t_0, t_0 + a]$ with $a > 0$, $x, x_0 \in \mathbb{R}^n$ and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$.

In the first part (Section 2) we give sufficient conditions for nonuniqueness of the classical n-dimensional Cauchy problem (1.1). As the essential tool serves a method which estimates the "distance" between two solutions with an appropriate Lyapunov function and permits to show that under certain conditions the "distance" between two different solutions vanishes at the initial point. In the second part (Section 3) we analyze for the one-dimensional case a set of conditions that takes its origin in an inversion of the uniqueness theorem by Kamke (see, e.g., [1, page 56]) but cannot guarantee nonuniqueness since it contains an inner contradiction. Several attempts were made to get nonuniqueness criteria by using conditions that are (in a certain sense) reverse uniqueness conditions of Kamke type. But this inversion process has to be handled very carefully. It can yield incompatible conditions. This is illustrated by a general set of conditions (in Theorems 3.2, 3.5 and 3.6) that would ensure nonuniqueness, but unfortunately they are inconsistent.

In this paper we study Cauchy problems where f is continuous at the initial point. Related results can be found in [1–5]. In literature there are several investigations for the discontinuous case [1, 6–13] with different qualitative behaviour.

2. Main Result

In the following let $\mathbb{R}_+ := [0, \infty)$, $b > 0$, $\rho > 0$ and

$$S^n_\rho(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\| < \rho\}, \tag{2.1}$$

where $\|\cdot\|$ means the Euclidean norm.

Definition 2.1. We say that the initial value problem (1.1) has at least two different solutions on the interval J if there exist solutions $\varphi(t)$, $\psi(t)$ defined on J and $\varphi \not\equiv \psi$.

The following notions are used in our paper (see, e.g., [14, pages 136 and 137]).

Definition 2.2. A function $\varphi : [0, \rho) \to \mathbb{R}_+$ is said to belong to the class $\mathcal{K}_\rho$ if it is continuous, strictly increasing on $[0, \rho)$ and $\varphi(0) = 0$.

Definition 2.3. A function $V : J \times S^n_\rho(0) \to \mathbb{R}_+$ with $V(t, 0) \equiv 0$ is said to be positive definite if there exists a function $\varphi \in \mathcal{K}_\rho$ such that the relation

$$V(t, x) \geq \varphi(\|x\|) \tag{2.2}$$

is satisfied for $(t, x) \in J \times S^n_\rho(0)$.

For the convenience of the reader we recall the definition of a uniformly Lipschitzian function with respect to a given variable.

Definition 2.4. A function $V(t, \cdot) : S^n_\rho(0) \to \mathbb{R}_+$ is said to be Lipschitzian uniformly with respect to $t \in J$ if for arbitrarily given $x^* \in S^n_\rho(0)$ there exists a constant $k = k(x^*)$ such that

$$\left\|V\left(t, x_1^*\right) - V\left(t, x_2^*\right)\right\| \leq k\left\|x_1^* - x_2^*\right\| \tag{2.3}$$

holds for every $t \in J$ and for every x_1^*, x_2^* within a small neighbourhood of x^* in $S^n_\rho(0)$.

In [1, 15, 16] generalized derivatives of a Lipschitzian function along solutions of an associated differential system are analyzed. A slight modification of Theorem 4.3 [15, Appendix I] is the following lemma.

Lemma 2.5. *Let $V : J \times S^n_\rho(0) \to \mathbb{R}_+$ be continuous and let $V(t, \cdot) : S^n_\rho(0) \to \mathbb{R}_+$ be Lipschitzian uniformly with respect to $t \in J$. Let $x_1, x_2 : J \to S^n_\rho(0)$ be any two solutions of*

$$\dot{x} = f(t, x), \tag{2.4}$$

where $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Then for the upper right Dini derivative the equality

$$\begin{aligned} &D^+ V(t, x_2(t) - x_1(t)) \\ &\quad := \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x_2(t + h) - x_1(t + h)) - V(t, x_2(t) - x_1(t))] \\ &\quad = \limsup_{h \to 0^+} \frac{1}{h} \left[V\left(t + h, x_2(t) - x_1(t) + h\left(f(t, x_2(t)) - f(t, x_1(t))\right)\right) - V(t, x_2(t) - x_1(t))\right] \end{aligned} \tag{2.5}$$

holds.

In the proof of Theorem 2.8 we require the following lemmas which are slight adaptations of Theorem 1.4.1 [14, page 15] and Theorem 1.3.1 [1, page 10] for the left side of the initial point.

Lemma 2.6. *Let E be an open (t, u)-set in $\mathbb{R}^2$, let $g : E \to \mathbb{R}$ be a continuous function, and let u be the unique solution of*

$$\dot{u} = g(t, u), \quad u(t_2) = u_2, \tag{2.6}$$

to the left with $t_2 > t_0$, $(t_2, u_2) \in E$. Further, we assume that the scalar continuous function $m : (t_0, t_2] \to \mathbb{R}$ with $(t, m(t)) \in E$ satisfies $m(t_2) \le u(t_2)$ and

$$D^+ m(t) \ge g(t, m(t)), \quad t_0 < t \le t_2. \tag{2.7}$$

Then

$$m(t) \le u(t) \tag{2.8}$$

holds as far as the solution u exists left of t_2 in $(t_0, t_2]$.

Lemma 2.7. *Let $S := \{(t, x) : t_0 - a \le t \le t_0,\ |x - x_0| \le b\}$ and $f : S \to \mathbb{R}$ be continuous and nondecreasing in x for each fixed t in $[t_0 - a, t_0]$. Then, the initial value problem* (1.1) *has at most one solution in $[t_0 - a, t_0]$.*

Theorem 2.8 (main result). *Suppose that*

(i) *$f : J \times S^n_b(x_0) \to \mathbb{R}^n$ is a continuous function such that*

$$M := \sup\left\{ \| f(t, x) \| : t \in J,\ x \in S^n_b(x_0) \right\} < \frac{b}{a}. \tag{2.9}$$

Let x_1 *be a solution of problem* (1.1) *on* J. *Let, moreover, there exist numbers* $t_1 \in (t_0, t_0+a]$, $r \in (0, 2b)$ *and continuous functions* $g : (t_0, t_1] \times \mathbb{R}_+ \to \mathbb{R}$, $V : [t_0, t_1] \times S_r^n(0) \to \mathbb{R}_+$ *such that*

(ii) g *is nondecreasing in the second variable, and the problem*

$$\dot{u} = g(t, u), \qquad \lim_{t \to t_0+} u(t) = 0 \tag{2.10}$$

has a positive solution u^* *on* $(t_0, t_1]$;

(iii) V *is positive definite and* $V(t, \cdot) : S_r^n(0) \to \mathbb{R}_+$ *is Lipschitzian uniformly with respect to* $t \in J$;

(iv) *for* $t_0 < t \le t_1$, $\|y - x_1(t)\| < r$, *the inequality*

$$\dot{V}(t, y - x_1(t)) \ge g(t, V(t, y - x_1(t))) \tag{2.11}$$

holds where

$$\begin{aligned} &\dot{V}(t, y - x_1(t)) \\ &\quad := \limsup_{h \to 0^+} \frac{1}{h}\big[V(t+h, y - x_1(t) + h[f(t,y) - f(t, x_1(t))]) - V(t, y - x_1(t))\big]. \end{aligned} \tag{2.12}$$

Then the set of different solutions of problem (1.1) *on interval* J *has the cardinality of the continuum.*

Remark 2.9. If condition (i) is fulfilled then, as it is well known, problem (1.1) is globally solvable and every global solution admits the estimate

$$\|x(t) - x_0\| \le M(t - t_0), \quad t \in J. \tag{2.13}$$

Moreover, for any local solution x_* of problem (1.1), defined on some interval $[t_0, t_1] \subset J$, there exists a global solution x of that problem such that $x(t) = x_*(t)$ for $t \in [t_0, t_1]$.

Remark 2.10. For the case $M = 0$ the initial value problem is unique and the assumptions of Theorem 2.8 cannot be satisfied. Therefore, without loss of generality, we assume $M > 0$ in the proof below.

Proof. At first we show that (1.1) has at least two different solutions on $[t_0, t_1^*]$, where $t_1^* \le t_1$, $t_1^* \le t_0 + \min\{a, b/(3M)\}$ is sufficiently close to t_0. We construct a further solution of (1.1) by finding a point (t_2, x_2) not lying on the solution $x_1(t)$ and starting from this point backwards to the initial point (t_0, x_0).

First we show that there exist values t_2 and x_2, $t_0 < t_2 \le t_1^*$, $\|x_2 - x_0\| \le 2b/3$ such that

$$u^*(t_2) = V(t_2, x_2 - x_1(t_2)) \tag{2.14}$$

holds for the nontrivial solution $u^*(t)$ of $\dot{u} = g(t,u)$. From Lemma 2.7 it follows that $u^*(t)$ is determined uniquely to the left by the initial data $(t_2, u^*(t_2))$. We consider the ε-tubes

$$S(\varepsilon) := \left\{(t,x) : t_0 \le t \le t_1^*,\ \|x - x_1(t)\| = \varepsilon\right\} \tag{2.15}$$

for $\varepsilon > 0$ around the solution $x_1(t)$. There exists $\varepsilon_1 > 0$ such that $S(\varepsilon)$ with $0 < \varepsilon \le \varepsilon_1 < r$ is contained in the set

$$\left\{(t,x) : t_0 \le t \le t_1^*,\ \|x - x_0\| \le \frac{2b}{3}\right\}. \tag{2.16}$$

For $0 \le \delta \le \varepsilon_1$, $t \in [t_0, t_1^*]$ we define

$$\begin{gathered}\Psi(\delta,t) := \max_{\|x-x_1(t)\|=\delta} V(t, x - x_1(t)),\\ \Psi(\delta) := \max_{t\in[t_0,t_1^*]} \Psi(\delta,t) \equiv \max_{(t,x)\in S(\delta)} V(t, x - x_1(t)).\end{gathered} \tag{2.17}$$

The function $\Psi(\delta,t)$ is continuous in t for $t_0 \le t \le t_1^*$. Since $\lim_{\delta\to 0}\Psi(\delta) = 0$, there exists a δ_2, $0 < \delta_2 \le \min\{\varepsilon_1, b/3\}$, such that $\Psi(\delta_2) \le u^*(t_1^*)$. It is clear that inequalities

$$\Psi\left(\delta_2, t_1^*\right) \le \Psi(\delta_2) \le u^*\left(t_1^*\right) \tag{2.18}$$

and (due to positive definiteness of V)

$$\Psi(\delta_2, t_0) > 0 = \lim_{t\to t_0+} u^*(t) \tag{2.19}$$

hold. We define a function

$$\omega(t) := \Psi(\delta_2, t) - u^*(t), \tag{2.20}$$

continuous on $[t_0, t_1^*]$. Taking into account inequalities $\omega(t_0) > 0$ and $\omega(t_1^*) \le 0$ we conclude that there exists t_2, $t_0 < t_2 \le t_1^*$, with

$$\Psi(\delta_2, t_2) = u^*(t_2). \tag{2.21}$$

The value $\Psi(\delta_2, t_2)$ is taken by $V(t_2, x - x_1(t_2))$ at a point $x = x_2$ such that $\|x_2 - x_1(t_2)\| = \delta_2$ and clearly (in view of the construction) $x_2 \ne x_1(t_2)$. The above statement is proved and (2.14) is valid for (t_2, x_2) determined above.

Now consider the initial value problem

$$\dot{x} = f(t,x), \qquad x(t_2) = x_2. \tag{2.22}$$

Obviously $t_2 - t_0 \le b/(3M)$ since

$$0 < t_2 - t_0 \le t_1^* - t_0 \le \min\left\{a, \frac{b}{3M}\right\} \le \frac{b}{3M} \tag{2.23}$$

and $\|x_2 - x_0\| \le 2b/3$ because

$$\begin{aligned}
\|x_2 - x_0\| &= \|x_2 - x_1(t_2) + x_1(t_2) - x_0\| \\
&\le \|x_2 - x_1(t_2)\| + \|x_1(t_2) - x_0\| = \delta_2 + \left\| \int_{t_0}^{t_2} f(s, x_1(s)) ds \right\| \\
&\le \delta_2 + M(t_2 - t_0) \le \delta_2 + M\frac{b}{3M} = \delta_2 + \frac{b}{3} \le \frac{2b}{3}.
\end{aligned} \tag{2.24}$$

Peano's theorem implies that there exists a solution $x_2(t)$ of problem (2.22) on $t_0 \le t \le t_2$. We will show that $x_2(t_0) = x_0$. Set

$$m(t) := V(t, x_2(t) - x_1(t)). \tag{2.25}$$

Note that $m(t_2) = u^*(t_2)$. Lemma 2.5 and condition (iv) imply

$$\begin{aligned}
D^+ m(t) &:= \limsup_{h \to 0^+} \frac{m(t+h) - m(t)}{h} \\
&= D^+ V(t, x_2(t) - x_1(t)) \\
&= \dot{V}(t, x_2(t) - x_1(t)) \ge g(t, V(t, x_2(t) - x_1(t))) = g(t, m(t))
\end{aligned} \tag{2.26}$$

for $t_0 < t \le t_2$.

Applying Lemma 2.6 we get $m(t) \le u^*(t)$ for $t_0 < t \le t_2$. As $m(t) \ge 0$ for $t_0 < t \le t_2$ and m is continuous at t_0, we find $m(t_0) = 0$. Therefore we have $x_2(t_0) = x_1(t_0) = x_0$ and, as noted above, $x_2(t_2) = x_2 \ne x_1(t_2)$. Thus problem (1.1) has two different solutions.

According to the well-known Kneser theorem [17, Theorem 4.1, page 15] the set of solutions of problem (1.1) either consists of one element or has the cardinality of the continuum. Consequently, if problem (1.1) has two different solutions on interval $[t_0, t_1^*]$ and condition (i) is satisfied, then the set of different solutions of problem (1.1) on interval J has the cardinality of the continuum. The proof is completed. □

Remark 2.11. Note that in the scalar case with $V(t, x) := |x|$ condition (2.11) has the form

$$\big(f(t, y) - f(t, x_1(t))\big) \cdot \operatorname{sign}\big(y - x_1(t)\big) \ge g\big(t, |y - x_1(t)|\big). \tag{2.27}$$

Example 2.12. Consider for $a = 0.1$, $b = 1$, $t_0 = 0$ and $x_0 = 0$ the scalar differential equation

$$\dot{x} = f(t,x) := \begin{cases} 2x^{1/3} - \dfrac{1}{2} \cdot t^{1/2} \cdot \sin \dfrac{|x|}{t} & \text{if } t \neq 0, \\ 2x^{1/3} & \text{if } t = 0, \end{cases} \tag{2.28}$$

with the initial condition $x(0) = 0$. Let us show that the set of different solutions of this problem on interval J has the cardinality of $\mathbb{R}$. Obviously we can set $x_1(t) \equiv 0$. Put

$$g(t,u) := 2u^{1/3} - \frac{1}{2} \cdot t^{1/2}, \qquad u^*(t) := t^{3/2}, \qquad V(t,x) := |x|. \tag{2.29}$$

Conditions (i), (ii), and (iii) are satisfied. Let us verify that the last condition (iv) is valid, too. We get

$$\begin{aligned} \dot{V}(t, y - x_1(t)) = \dot{V}(t,y) &= (\operatorname{sign} y) \cdot \left[2y^{1/3} - \frac{1}{2} \cdot t^{1/2} \cdot \sin \frac{|y|}{t} \right] \\ &\geq 2|y|^{1/3} - \frac{1}{2} \cdot t^{1/2} = 2V(t,y)^{1/3} - \frac{1}{2} \cdot t^{1/2} = g(t, V(t,y)) \\ &= g(t, V(t, y - x_1(t))). \end{aligned} \tag{2.30}$$

Thus, all conditions of Theorem 2.8 hold and, consequently, the set of different solutions on J of given problem has the cardinality of $\mathbb{R}$.

3. Incompatible Conditions

In this section we show that the formulation of condition (iv) in Theorem 2.8 without knowledge of a solution of the Cauchy problem can lead to an incompatible set of conditions. In the proof of Theorem 3.2 for the one-dimensional case we use the following result given by Nekvinda [18, page 1].

Lemma 3.1. *Let $D \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a continuous function in D. Let equation*

$$\dot{x} = f(t,x) \tag{3.1}$$

has the property of left uniqueness. For any $t_0 \in \mathbb{R}$ let A be the set of all $x_0 \in \mathbb{R}$ such that $(t_0, x_0) \in D$ and, for some $\varepsilon > 0$, the initial-value problem (1.1) *has more than one solution in the interval $[t_0, t_0 + \varepsilon)$. Then A is at most countable.*

Theorem 3.2. *The set of conditions (i)–(iv):*

(i) *$f : R_0 \to \mathbb{R}$ with $R_0 := \{(t,x) \in J \times \mathbb{R},\ |x - x_0| \leq b\}$ is continuous;*

(ii) $g : (t_0, t_0 + a] \times (0, \infty) \to \mathbb{R}_+$ *is continuous, nondecreasing in the second variable, and has the following property: there exists a continuous function* $u^*(t)$ *on* J*, which satisfies the differential equation*

$$\dot{u}(t) = g(t, u) \tag{3.2}$$

for $t_0 < t \le t_0 + a$ *with* $u^*(t_0) = 0$ *and does not vanish for* $t \neq t_0$*;*

(iii) $V : J \times S^1_{2b}(0) \to \mathbb{R}_+$ *is continuous, positive definite, and Lipschitzian uniformly with respect to* $t \in J$*;*

(iv) *for* $t_0 < t \le t_0 + a$*,* $|x - x_0| \le b$*,* $|y - x_0| \le b$*,* $x \neq y$*,*

$$\dot{V}(t, x - y) \ge g(t, V(t, x - y)), \tag{3.3}$$

where we define

$$\dot{V}(t, x - y) := \limsup_{h \to 0^+} \frac{1}{h} \left[V(t + h, x - y + h[f(t, x) - f(t, y)]) - V(t, x - y) \right] \tag{3.4}$$

contains a contradiction.

Proof. Any initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x^* \tag{3.5}$$

with $|x^* - x_0| \le b$ has at least two different solutions due to Theorem 2.8. Thus we have an uncountable set of nonuniqueness points. We show that solutions passing through different initial points are left unique. Suppose that it does not hold. Let $x_1(t)$ be a solution starting from (t_0, x_1), and let $x_2(t)$ be a solution starting from (t_0, x_2) with $x_2 \neq x_1$. If we assume that these solutions cross at a point $t_1 > t_0$ and if we set

$$m(t) := V(t, x_1(t) - x_2(t)) \tag{3.6}$$

then $m(t_0) > 0$, $m(t_1) = 0$. Therefore there exists a point $t \in (t_0, t_1)$ such that (we apply Lemma 2.5)

$$D^+ m(t) = D^+ V(t, x_1(t) - x_2(t)) = \dot{V}(t, x_1(t) - x_2(t)) < 0, \tag{3.7}$$

in contradiction to (3.3). Thus we obtain left uniqueness. From Lemma 3.1 we conclude in contrast to the above conclusion that the set of nonuniqueness points (t_0, x^*) can be at most countable. □

In [1, Theorem 1.24.1, page 99] the following nonuniqueness result (see [14, Theorem 2.2.7, page 55], too) is given which uses an inverse Kamke's condition (condition (3.9) below).

Theorem 3.3. *Let $g(t,u)$ be continuous on $0 < t \le a$, $0 \le u \le 2b$, $g(t,0) \equiv 0$, and $g(t,u) > 0$ for $u > 0$. Suppose that, for each t_1, $0 < t_1 < a$, $u(t) \not\equiv 0$ is a differentiable function on $0 < t < t_1$, and continuous on $0 \le t < t_1$ for which $\dot{u}_+(0)$ exists,*

$$\begin{aligned} \dot{u} &= g(t,u), \qquad 0 < t < t_1, \\ u(0) &= \dot{u}_+(0) = 0. \end{aligned} \tag{3.8}$$

Let $f \in C[R_0, \mathbb{R}]$, where $R_0 : 0 \le t \le a,\ |x| \le b$, and, for $(t,x), (t,y) \in R_0,\ t \ne 0$,

$$\left| f(t,x) - f(t,y) \right| \ge g(t, |x-y|). \tag{3.9}$$

Then, the scalar problem $\dot{x} = f(t,x)$, $x(0) = 0$ has at least two solutions on $0 \le t \le a$.

Remark 3.4. In the proof of Theorem 3.3 at first $f(t,0) = 0$ is assumed. Putting $y = 0$ in (3.9) leads to the inequality

$$\left| f(t,x) \right| \ge g(t, |x|). \tag{3.10}$$

As $f(t,x)$ is continuous and $g(t,u) > 0$ for $u > 0$ it follows that $f(t,x)$ must have constant sign for each of the half planes $x > 0$ and $x < 0$. For the upper half plane this implies that

$$\begin{aligned} f(t,x) &\ge g(t,x), \\ f(t,x) &\le -g(t,x). \end{aligned} \tag{3.11}$$

For the first inequality nonuniqueness is shown in [1]. But a similar argumentation cannot be used for the second inequality as the following example in [5] shows. We consider the initial value problem $\dot{x} = f(t,x)$, $x(0) = 0$, with

$$f(t,x) = \begin{cases} -\sqrt{x} & \text{if } x \ge 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases} \tag{3.12}$$

and $g(t,u) := \sqrt{u}$. Thus inequality $|f(t,x)| = \sqrt{|x|} \ge g(t,|x|)$ holds. In the upper half-plane we have $f(t,x) \le -g(t,x)$. The function $u(t) = t^2/4$ is a nontrivial solution of the comparison equation. Therefore all assumptions are fulfilled, but the initial value problem has at most one solution because of Theorem 1.3.1 [1, page 10].

The next theorem analyzes in the scalar case (for $(t_0, x_0) = (0,0)$) that even fulfilling a rather general condition (see condition (3.14) in the following theorem) cannot ensure nonuniqueness since the set of all conditions contains an inner contradiction. The proof was motivated by the paper [5].

Theorem 3.5. *There exists no system of three functions f, g, and V satisfying the following suppositions:*

(i) $f : R_0 \to \mathbb{R}$ *with* $R_0 := \{(t,x) \in \mathbb{R} \times \mathbb{R},\ 0 \le t \le a, 0 \le x \le b\}$ *is a continuous function;*

(ii) *the continuous function* $g : (0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, $g(t,0) := 0$ *if* $t \in (0,a]$, *has the following property: there exists a continuously differentiable function* $u^*(t)$ *on* $0 \le t \le a$, *satisfying the differential equation*

$$\dot{u} = g(t,u) \tag{3.13}$$

for $0 < t \le a$ *such that* $u^*(0) = 0$ *and* $u^*(t) > 0$ *for* $t \neq 0$*;*

(iii) *the continuous function* $V : [0,a] \times S_b^1(0) \to \mathbb{R}_+$ *is positive definite, and for all* $0 < t \le a$, $0 < x < b$ *continuously differentiable;*

(iv) *for* $0 < t \le a$, $0 < y < x \le b$,

$$\dot{V}(t, x-y) \ge g(t, V(t, x-y)) \ge 0, \tag{3.14}$$

where we define

$$\dot{V}(t,x-y) := V_1'(t,x-y) + V_2'(t,x-y) \cdot [f(t,x) - f(t,y)] \tag{3.15}$$

and subscript indices denote the derivative with respect to the first and second argument, respectively;

(v) *there exist a positive constant* ϑ *and a function* $\xi : (0,b] \to (0,\infty)$ *such that for* $0 < t \le a$ *and* $0 < x \le b$

$$0 \le V'_1(t,x) \le \vartheta \cdot \xi(x), \quad 0 < V_2'(t,x) \le \vartheta \cdot \frac{\xi(x)}{x}, \qquad V(t,x) \ge \xi(x); \tag{3.16}$$

(vi) *for* $t \in [0,a]$ *and* x, y *with* $0 < y < x \le b$ *the inequality*

$$f(t,x) - f(t,y) \ge 0 \tag{3.17}$$

holds.

Proof. Let us show that the above properties are not compatible. For fixed numbers x, y with $0 < y < x \le b$ consider the auxiliary function

$$F(t) := \frac{f(t,x) - f(t,y)}{x-y} + 1, \quad t \in [0,a]\ . \tag{3.18}$$

Clearly, F is continuous and assumes a (positive) maximum. Set

$$K = \max_{[0,a]} F(t) \geq 1. \tag{3.19}$$

If the function g fulfills the inequality

$$g(t,u) \leq \Lambda \cdot u \tag{3.20}$$

with a positive constant Λ in a domain $0 < t \leq A \leq a$, $0 \leq u \leq B$, $B > 0$, then the initial value problem

$$\dot{u} = g(t,u), \qquad u(0) = 0 \tag{3.21}$$

has the unique trivial solution $u = 0$. Really, since $u^*(t) > 0$ for $t \in (0,a]$, by integrating inequality

$$\frac{\dot{u}^*(t)}{u^*(t)} \leq \Lambda \tag{3.22}$$

with limits t, $A^* \in (0,A)$ we get

$$u^*(A^*) \leq u^*(t)\exp[\Lambda(A^* - t)] \tag{3.23}$$

and for $t \rightarrow 0^+$

$$u^*(A^*) \leq 0 \tag{3.24}$$

which contradicts positivity of u^*. Therefore problem (3.21) has only the trivial solution. Hence, there exist a sequence $\{(t_n,u_n)\}$ with $t_n \in (0,a]$, $u_n > 0$, $\lim_{n\to\infty}(t_n,u_n) = (0,0)$ and a sequence $\{\lambda_n\}$, $\lambda_n > 0$, with $\lim_{n\to\infty}\lambda_n = \infty$ such that the inequality

$$g(t_n,u_n) > \lambda_n u_n \tag{3.25}$$

holds for every n. Consider now the relation

$$V(t,x) = 0. \tag{3.26}$$

Due to the properties of V we conclude that for all sufficiently small positive numbers t_n, u_n (i.e., for all sufficiently large n) there exists a (sufficiently small and positive) number $\widetilde{u}_n$ such that the equation

$$V(t_n,x) = u_n \tag{3.27}$$

has the solution $x = \tilde{u}_n$. Thus a sequence $\{\tilde{u}_n\}$ with $\tilde{u}_n > 0$ and $\lim_{n\to\infty}\tilde{u}_n = 0$ corresponds to the sequence $\{(t_n, u_n)\}$. For every n define a number j_n as

$$j_n = \left\lceil \frac{x-y}{\tilde{u}_n} - 1 \right\rceil, \tag{3.28}$$

where $\lceil\cdot\rceil$ is the ceiling function. Without loss of generality we can suppose that

$$\frac{x-y}{\tilde{u}_n} > 4. \tag{3.29}$$

Obviously,

$$\frac{x-y}{\tilde{u}_n} - 1 \le j_n < \frac{x-y}{\tilde{u}_n}. \tag{3.30}$$

Moreover, without loss of generality we can suppose that for every sufficiently large n the inequality

$$\lambda_n > 2\vartheta K \tag{3.31}$$

holds. Set

$$\begin{aligned} x_0 &:= y, \\ x_1 &:= y + \tilde{u}_n, \\ x_2 &:= y + 2\tilde{u}_n, \\ &\vdots \\ x_{j_n} &:= y + j_n \cdot \tilde{u}_n, \\ x_{j_n+1} &:= x. \end{aligned} \tag{3.32}$$

Consider for all sufficiently large n the expression

$$\mathcal{E}_n := j_n V_1'(t_n, \tilde{u}_n) + V_2'(t_n,\ \tilde{u}_n) \cdot \left[f(t_n, x) - f(t_n, y) \right]. \tag{3.33}$$

Then

$$\begin{aligned}
\mathcal{E}_n &= j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot \sum_{i=1}^{j_n+1}\left[f(t_n,x_i)-f(t_n,x_{i-1})\right] \\
&= j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot \sum_{i=1}^{j_n}\left[f(t_n,x_i)-f(t_n,x_{i-1})\right] + \left[f(t_n,x)-f\left(t_n,x_{j_n}\right)\right] \\
&\geq [\text{due to (vi)}] \geq j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot \sum_{i=1}^{j_n}\left[f(t_n,x_i)-f(t_n,x_{i-1})\right] \\
&= [\text{due to (iv) and (v)}] = j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot j_n\left[\frac{-V_1'(t_n,\tilde{u}_n)+\dot{V}(t_n,\tilde{u}_n)}{V_2'(t_n,\tilde{u}_n)}\right] \\
&\geq [\text{due to (iv)}] \\
&\geq j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot j_n\left[\frac{-V_1'(t_n,\tilde{u}_n)+g(t_n,V(t_n,\tilde{u}_n))}{V'_2(t_n,\tilde{u}_n)}\right] \\
&= j_n\cdot g(t_n,V(t_n,\tilde{u}_n)) = [\text{due to (3.27)}] = j_n\cdot g(t_n,u_n) \geq [\text{due to (3.25)}] \\
&\geq j_n\lambda_n u_n \geq [\text{due to (3.31)}] \geq j_n u_n\cdot 2\vartheta K \geq [\text{due to (3.30)}] \\
&\geq \left(\frac{x-y}{\tilde{u}_n}-1\right) u_n\cdot 2\vartheta K \\
&= \left(\frac{x-y}{\tilde{u}_n}-1\right) V(t_n,\tilde{u}_n)\cdot 2\vartheta K \\
&\geq [\text{due to (3.16) }] \geq \left(\frac{x-y}{\tilde{u}_n}-1\right)\xi(\tilde{u}_n)\cdot 2\vartheta K \\
&= (x-y-\tilde{u}_n)\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n}\cdot 2\vartheta K \\
&\geq [\text{due to (3.29)}] \geq \frac{3}{4}\cdot(x-y)\cdot 2\vartheta K\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n} \\
&= \frac{3}{2}\cdot(x-y)\cdot\vartheta K\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n} > 0.
\end{aligned} \tag{3.34}$$

Estimating the expression $\mathcal{E}_n$ from above we get (see (3.32))

$$\begin{aligned}
\mathcal{E}_n &\leq \frac{x-y}{\tilde{u}_n}V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot\left[f(t_n,x)-f(t_n,y)\right] \\
&\leq [\text{due to (v)}] \\
&\leq \frac{x-y}{\tilde{u}_n}\cdot\vartheta\cdot\xi(\tilde{u}_n) + \vartheta\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n}\cdot(K-1)(x-y) = \vartheta\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n}\cdot K(x-y).
\end{aligned} \tag{3.35}$$

These two above estimations yield

$$0 < \frac{3}{2} \cdot (x - y) \cdot \vartheta K \cdot \frac{\xi(\widetilde{u}_n)}{\widetilde{u}_n} \leq \mathcal{E}_n \leq (x - y) \cdot \vartheta K \cdot \frac{\xi(\widetilde{u}_n)}{\widetilde{u}_n}, \tag{3.36}$$

in contrast to $(3/2) \nleq 1$. Since the initially taken points x and y, $0 < y < x$, can be chosen arbitrarily close to zero, the theorem is proved. □

The following result is a consequence of Theorem 3.5 if $V(t,x) := |x|$, $\xi(x) := x$ and $\vartheta = 1$. Condition (3.38) below was discussed previously in [5].

Theorem 3.6. *There exists no system of two functions f and g satisfying the following suppositions:*

(i) *$f : R_0 \to \mathbb{R}$ with $R_0 := \{(t,x) \in \mathbb{R} \times \mathbb{R},\ 0 \leq t \leq a, 0 \leq x \leq b\}$ is a continuous function;*

(ii) *the continuous function $g : (0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, $g(t,0) := 0$ if $t \in (0,a]$, has the following property: there exists a continuously differentiable function $u^*(t)$ on $0 \leq t \leq a$, satisfying the differential equation*

$$\dot{u}(t) = g(t,u) \tag{3.37}$$

for $0 < t \leq a$ such that $u^*(0) = 0$ and $u^*(t) > 0$ for $t \neq 0$;

(iii) *for $0 < t \leq a$, $0 < y < x \leq b$*

$$f(t,x) - f(t,y) \geq g(t,x-y) \geq 0; \tag{3.38}$$

(iv) *for $0 < y < x \leq b$ the inequality $f(0,x) - f(0,y) \geq 0$ holds.*

Remark 3.7. Let us note that in the singular case, that is, when we permit that the function $f(t,x)$ is not continuous at $t = 0$, the given sets of conditions in Theorems 3.5 and 3.6 can be compatible. This can be seen from the proof where the continuity of f is substantial. Such singular case was considered in [13].

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