Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 759175, 13 pages
doi:10.1155/2011/759175

Research Article

A Study on Becker's Univalence Criteria

Maslina Darus and Imran Faisal

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi, Selangor D. Ehsan 43600, Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

Received 26 January 2011; Accepted 11 May 2011

Academic Editor: Allan C. Peterson

Copyright © 2011 M. Darus and I. Faisal. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study univalence properties for certain subclasses of univalent functions $\mathfrak{K}$, $\mathfrak{K}_2$, $\mathfrak{K}_{2,\mu}$, and $S(p)$, respectively. These subclasses are associated with a generalized integral operator. The extended Becker-typed univalence criteria will be studied for these subclasses.

1. Introduction and Preliminaries

Let A denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus, each $f \in A$ has a Taylor series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let A_2 be the subclass of A consisting of functions of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k. \tag{1.2}$$

Let $\mathfrak{K}$ be the univalent subclass of A which satisfies

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < 1, \quad z \in \mathbb{U}. \tag{1.3}$$

Let $\mathfrak{K}_2$ be the subclass of $\mathfrak{K}$ for which $f''(0) = 0$. Let $\mathfrak{K}_{2,\mu}$ be the subclass of $\mathfrak{K}_2$ consisting of functions of the form (1.2) which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < \mu, \quad 0 < \mu \leq 1, \ z \in \mathbb{U}. \tag{1.4}$$

Next, we define a subclass $S(p)$ of A consisting of all functions $f(z)$ that satisfy

$$\left|\left(\frac{z}{(f(z))}\right)''\right| \leq p, \quad 0 < p \leq 2, \ p \in \mathfrak{R}, \ z \in \mathbb{U}. \tag{1.5}$$

For functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) $f * g$ is defined as usual by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.6}$$

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) = z_2 F(1, a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k-1}, \quad c \neq 0, -1-, 2, \ldots, \tag{1.7}$$

where $(a)_k k$ is the famous Pochhammer symbol defined in terms of Gamma function. It is easily seen that $\varphi(2-\alpha, 2; z)$ is a convex function, since $z\varphi'(z) = \varphi(2-\alpha, 1; z) \in \mathrm{SV}^{\star}(\alpha)\cdot$

Using the fractional derivative of order α, D_z^{α} [1], Owa and Srivastava [2] introduced the operator $\Omega^{\alpha} : A \to A$ which is known as an extension of fractional derivative and fractional integral, as follows:

$$\begin{aligned} \Omega^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \quad \alpha \neq 2, 3, 4, \ldots \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k \\ &= \varphi(2, 2-\alpha; z) * f(z)\cdot \end{aligned} \tag{1.8}$$

Note that $\Omega^0 f(z) = f(z)\cdot$

For a function f in A, we define $D_{\lambda}^{n,\nu}(\alpha, \beta, \mu) f(z) : A \to A$, the linear fractional differential operator, as follows:

$$\begin{gathered} I_{\lambda}^{0,\nu}(\alpha, \beta, \mu) f(z) = f(z), \\ I_{\lambda}^{1,\nu}(\alpha, \beta, \mu) f(z) = \left(\frac{\nu - \mu + \beta - \lambda}{\nu + \beta}\right)(\Omega^{\alpha} f(z)) + \left(\frac{\mu + \lambda}{\nu + \beta}\right) z(\Omega^{\alpha} f(z)), \end{gathered}$$

$$I_{\lambda}^{2,\nu}(\alpha,\beta,\mu)f(z) = I_{\lambda}^{\alpha}\left(I_{\lambda}^{1,\nu}(\alpha,\beta,\mu)f(z)\right)$$
$$\vdots$$
$$I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) = I_{\lambda}^{\alpha}\left(I_{\lambda}^{n-1,\nu}(\alpha,\beta,\mu)f(z)\right). \tag{1.9}$$

If f is given by (1.1), then by (1.8) and (1), we see that

$$I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) = z + \sum_{k=2}^{\infty}\left(\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)^{n} a_k z^k. \tag{1.10}$$

From (1.8) and (1), $D_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z)$ can be written in terms of convolution as

$$I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z) = \underbrace{\left[\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)\cdots\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)\right]}* f(z), \tag{1.11}$$

where

$$\begin{aligned} g_{\beta,\lambda}^{\mu,\nu}(z) &= \frac{z-((\nu-\mu+\beta-\lambda)/(\nu+\beta))z^2}{(1-z)^2} \\ &= z-\left(\frac{\nu-\mu+\beta-\lambda}{\nu+\beta}\right)z^2\left(1+2z+3z^2+\cdots\right) \\ &= z+\left(1+\frac{\mu+\lambda}{\nu+\beta}\right)z^2+\left(1+2\frac{\mu+\lambda}{\nu+\beta}\right)z^3\cdots \\ &\vdots \\ g_{\beta,\lambda}^{\mu,\nu}(z) &= z+\sum_{k=2}^{\infty}\left(\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)z^k, \end{aligned} \tag{1.12}$$

$$\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z)\cdots\varphi(2,2-\alpha;z) * g_{\beta,\lambda}^{\mu,\nu}(z) = n\text{-times product}, \tag{1.13}$$

which generalizes many operators. Indeed, if we choose suitably values of α, β, μ, and ν in (1.12), we have the following.

(i) $\beta = 1$, $\mu = 0$, and $\alpha = 0$, we obtain $D_{\alpha,\lambda}^{m}f(z)$ given by Aouf et al. [3].

(ii) $\nu = 1$, $\beta = 0$, $\mu = 0$, and $\alpha = 0$, we obtain $D_{\lambda}^{m}f(z)$ given by Al-Oboudi [4].

(iii) $\nu = 1$, $\beta = 0$, $\mu = 0$, $\lambda = 1$, and $\alpha = 0$, we obtain $D^{m}f(z)$ given by Sălăgean [5].

(iv) $\nu = 1$, $\beta = 1$, $\lambda = 1$, $\mu = 0$, and $\alpha = 0$, we obtain $I^{m}f(z)$ given by Uralegaddi and Somanatha [6].

(v) $\beta = 1$, $\lambda = 1$, $\mu = 0$, and $\alpha = 0$, we obtain $I^{m}(\ell)f(z)$ given by Cho and Srivastava [7] and Cho and Kim [8].

(vi) $\nu = 1, \beta = 0, \mu = 0, \lambda = 0$, and $n = 1$, we obtain Owa and Srivastava differential operator [2].

(vii) $\nu = 1, \beta = 0$, and $\mu = 0$, we obtain $D_{\lambda}^{n,\alpha} f(z)$ given by Al-Oboudi and Al-Amoudi [9, 10].

(viii) $\beta = l$, $\mu = 0$, and $\alpha = p$, we obtain $I_p^n(\lambda, l) f(z)$ given by Catas [11].

(ix) $\beta = l$, $\mu = 0$, $\alpha = p$, and $\lambda = 1$, we obtain $I_p^n(\lambda, l) f(z)$ given by Kumar et al. and Srivastava et al., respectively [12, 13].

Next, we introduce a new family of integral operator by using generalized differential operator already defined above.

For $m \in \mathbb{N} \cup \{0\}$ and $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n, \rho \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we define a family of integral operators $\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta) : A^m \to A^m$ by

$$\Upsilon_{\gamma_i,\eta,\lambda}(n, \rho, \nu, \alpha, \beta : z) = \left\{ \rho \int_0^z t^{\rho-1} \prod_{i=1}^{m} \left(\frac{I_{\lambda}^{n,\nu}(\alpha, \beta, \mu, \eta) f_i(t)}{t} \right)^{1/\gamma_i} dt \right\}^{1/\rho}, \quad f_i \in A, \tag{1.14}$$

which generalize many integral operators. In fact, if we choose suitable values of parameters in this type of operator, we get the following interesting operators.

(i) $\nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/\alpha_i$, and $\rho = 1$, we obtain $I(f_1, \ldots, f_m)$ given by Bulut [14].

(ii) $n = 0, \nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/(\alpha - 1)$, and $\rho = n(\alpha - 1) + 1$, we obtain $F_{n,\alpha}(z)$ given by Breaz et al. [15].

(iii) $n = 0, \nu = 1, \beta = 0, \mu = 0, \alpha = 0, \gamma_i = 1/\alpha_i$, and $\rho = 1$, we obtain $F_{\alpha}(z)$ given by D. Breaz and N. Breaz [16].

For our main result, we need the following lemmas.

Lemma 1.1 (see [17, 18]). *Let c be a complex number, $|c| \leq 1$, $c \neq -1$. If $f(z) = z + a_2 z^2 + \cdots$ is a regular function in $\mathbb{U}$ and*

$$\left| c|z|^2 + \left(1 - |z|^2\right) \frac{z f''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in \mathbb{U}, \tag{1.15}$$

then the function f is regular and univalent in $\mathbb{U}$.

Lemma 1.2 (Schwarz Lemma). *Let the function $f(z)$ be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$. If $f(z)$ has one zero with multiply $\geq m$ for $z = 0$, then*

$$\left| f(z) \right| \leq \frac{M}{R^m} |z|^m, \quad \forall z \in \mathbb{U}_R, \tag{1.16}$$

and equality holds only if $f(z) = e^{i\theta}(M/R^m)|z|^m$, where θ is constant.

Lemma 1.3 (see [19]). *Let δ be a complex number with* $\operatorname{Re}\delta > 0$ *such that* $c \in \mathbb{C}$, $|c| \leq 1$, $c \neq -1$. *If $f \in A$ satisfies the condition*

$$\left| c|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zf''(z)}{\delta f'(z)} \right| \leq 1, \quad \forall z \in \mathbb{U}, \tag{1.17}$$

then the function

$$F_\delta(z) = \left\{ \delta \int_0^z t^{\delta-1} f'(t)dt \right\}^{1/\delta} \tag{1.18}$$

is analytic and univalent in $\mathbb{U}$.

Lemma 1.4 (see [20]). *If a function $f \in S(p)$, then*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq p|z|^2, \quad \forall z \in \mathbb{U}. \tag{1.19}$$

2. Univalence Properties

In this section, we will discuss the univalence properties of the new family of integral operators mentioned above.

Theorem 2.1. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots\}$ and $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{\infty} \frac{((M_i - 1)p_i + 2)M_i - 1}{|\gamma_i|(M_i 1)}, \tag{2.1}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{\infty} \frac{((M_i - 1)p_i + 2)M_i - 1}{|\gamma_i|(M_i - 1)}, \quad M_i \geq 1, \tag{2.2}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Proof. Since $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$, so by Lemma 1.4, we have

$$\left| \frac{z^2 (I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t))^2} - 1 \right| \leq p_i|z|^2, \quad \forall z \in \mathbb{U}. \tag{2.3}$$

Now, by using hypothesis, we have

$$\left| I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z) \right| \leq M_i, \tag{2.4}$$

so by Lemma 1.3, we get

$$\left|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right| \leq M_i z, \quad \because R = 1. \tag{2.5}$$

Let

$$\begin{aligned}\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu)f(z)}{z} &= 1 + \sum_{k=2}^{\infty}\left(\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left(\frac{\nu+(\mu+\lambda)(k-1)+\beta}{\nu+\beta}\right)\right)^{n} a_k z^{k-1} \neq 0 \\ &= 1 \quad \text{if } z = 0,\end{aligned} \tag{2.6}$$

so

$$\begin{aligned}\prod_{i=1}^{m}\left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)}{z}\right)^{1/\gamma_i} &= \left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(z)}{z}\right)^{1/\gamma_1}\cdots\left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(z)}{z}\right)^{1/\gamma_m} \\ &= 1.\end{aligned} \tag{2.7}$$

Let

$$F(z) = \int_0^z\left(\left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(t)}{t}\right)^{1/\gamma_1}\cdots\left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(t)}{t}\right)^{1/\gamma_m}\right)dt, \tag{2.8}$$

which implies that

$$\begin{aligned}F'(z) &= \left(\left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(t)}{t}\right)^{1/\gamma_1}\cdots\left(\frac{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(t)}{t}\right)^{1/\gamma_m}\right)', \\ \frac{zF''(z)}{F'(z)} &= \frac{1}{\gamma_1}\left(\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(z)\right)'}{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_1(z)} - 1\right) + \cdots + \frac{1}{\gamma_m}\left(\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(z)\right)'}{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_m(z)} - 1\right),\end{aligned} \tag{2.9}$$

$$\frac{zF''(z)}{F'(z)} = \sum_{i=1}^{m}\frac{1}{\gamma_i}\left(\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)} - 1\right). \tag{2.10}$$

This implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)\right)'}{I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)}\right| + 1\right), \tag{2.11}$$

or

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))^2}\right|\left|\frac{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))}{(z)}\right|+1\right). \tag{2.12}$$

Using (2.5), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))^2}\right|M_i+1\right). \tag{2.13}$$

This implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z))^2}-1\right|M_i+M_i+1\right). \tag{2.14}$$

By using (2.3), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_i|z|^2M_i+M_i+1\right), \tag{2.15}$$

which implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_iM_i+\left(M_i+M_i^2+M_i^3+\cdots\right)+1\right), \tag{2.16}$$

because $M_i, M_i^2, M_i^3, \ldots, \geq 1$ implies that

$$\begin{gathered}
\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_iM_i+\left(\frac{M_i}{M_i-1}\right)+1\right)=\sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_iM_i+\left(\frac{2M_i-1}{M_i-1}\right)\right),\\
\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{p_iM_i^2-p_iM_i+2M_i-1}{M_i-1}\right)\right) \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{(p_iM_i-p_i+2)M_i-1}{M_i-1}\right)\right),\\
\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{((M_i-1)p_i+2)M_i-1}{M_i-1}\right)\right).
\end{gathered} \tag{2.17}$$

Now, we calculate

$$\left|c|z|^{2\rho}+\left(1-|z|^{2\rho}\right)\frac{zF''(z)}{\rho F'(z)}\right| \leq |c|+\frac{1}{|\rho|}\left|\frac{zF''(z)}{F'(z)}\right| \leq |c|+\frac{1}{\Re(\rho)}\left|\frac{zF''(z)}{F'(z)}\right|. \tag{2.18}$$

This implies that

$$\left|c|z|^{2\rho}+\left(1-|z|^{2\rho}\right)\frac{zF''(z)}{\rho F'(z)}\right|<|c|+\frac{1}{\Re(\rho)}\sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{((M_i-1)p_i+2)M_i-1}{M_i-1}\right)\right)\cdot \tag{2.19}$$

By using (2.46), we conclude that

$$\left|c|z|^{2\rho}+\left(1-|z|^{2\rho}\right)\frac{zF'(z)}{\rho F(z)}\right|\leq|c|+\frac{1}{|\rho|}\left|\frac{zF''(z)}{F'(z)}\right|\leq 1\cdot \tag{2.20}$$

Hence, by Lemma 1.3, the family of integral operators $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta:z)$ is univalent. □

Corollary 2.2. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)|\leq M$, $M\geq 1$ for all $i=\{1,2,3,\ldots\}$ and $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t)\in S(p)$, $M_i=M\geq 1$, for all $i=\{1,2,3,\ldots\}$ such that*

$$\Re(\rho)\geq\sum_{i=1}^{\infty}\frac{((M-1)p+2)M-1}{|\gamma_i|(M-1)}, \tag{2.21}$$

where ρ, γ_i are complex numbers. If

$$|c|\leq 1-\frac{1}{\mathrm{Re}(\rho)}\sum_{i=1}^{\infty}\frac{((M-1)p_i+2)M-1}{|\gamma_i|(M-1)},\quad M\geq 1, \tag{2.22}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta:z)$ is univalent.

Corollary 2.3. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)|\leq M$, $M\geq 1$, for all $i=\{1,2,3,\ldots\}$ and the family $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t)\in S(p)$, $M_i=M\geq 1$, $|\gamma_i|=|\gamma|$, for all $i=\{1,2,3,\ldots\}$ such that*

$$\Re(\rho)\geq\sum_{i=1}^{\infty}\frac{((M-1)p+2)M-1}{|\gamma|(M-1)}, \tag{2.23}$$

where ρ, γ_i are complex numbers. If

$$|c|\leq 1-\frac{1}{\mathrm{Re}(\rho)}\sum_{i=1}^{\infty}\frac{((M-1)p_i+2)M-1}{|\gamma|(M-1)},\quad M\geq 1, \tag{2.24}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta:z)$ is univalent.

Using the method given in the proof of Theorem 2.1, one can prove the following results.

Theorem 2.4. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots\}$ and the family $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots\}$ and c such that*

$$\Re(\rho) \geq \sum_{i=1}^{\infty} \frac{(p_i M_i - 1)M_i + 1}{|\gamma_i| p_i M_i}, \tag{2.25}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{\infty} \frac{(p_i M_i - 1)M_i + 1}{|\gamma_i|(p_i M_i)}, \quad M_i \geq 1, \tag{2.26}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.5. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(p_i(M_i - 1) + M_i^n - 2)M_i + 1}{|\gamma_i|(M_i - 1)}, \tag{2.27}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{n} \frac{(p_i(M_i - 1) + M_i^n - 2)M_i + 1}{|\gamma_i|(M_i - 1)}, \quad M_i \geq 1, \tag{2.28}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.6. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in S(p_i)$ for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(p_i + (n(n+1)/2))M_i - 1}{|\gamma_i|}, \tag{2.29}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{n} \frac{(p_i + (n(n+1)/2))M_i - 1}{|\gamma_i|}, \quad M_i \geq 1, \tag{2.30}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.7. *Let c be a complex number, $|I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_{\lambda}^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M_i}{|\gamma_i|}, \tag{2.31}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M_i}{|\gamma_i|}, \quad M_i \geq 1, \tag{2.32}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Proof. Using the proof of Theorem 2.1, we have

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} \left(\left| \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z))^2} \right| M_i + 1 \right). \tag{2.33}$$

Since $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t) \in \Re_{2,\mu_i}$, so by using (1.4), we get

$$\left| \frac{z^2(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z))^2} - 1 \right| < \mu_i, \quad 0 < \mu \leq 1, \ z \in \mathbb{U}. \tag{2.34}$$

So from (2.33), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} \left(\left| \frac{z(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z))'}{(I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z))^2} - 1 \right| M_i + M_i + 1 \right), \tag{2.35}$$

or

$$\begin{gathered}
\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + 2M_i), \quad M_i > 1, \\
\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + 2M_i + 4M_i + \cdots + n\text{-times}), \quad M_i > 1, \\
\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + n(n+1)M_i), \quad M_i > 1.
\end{gathered} \tag{2.36}$$

Now, we evaluate the expression

$$\begin{gathered}
\left| c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right) \frac{zF''(z)}{\rho F'(z)} \right| \leq |c| + \frac{1}{|\rho|} \left| \frac{zF''(z)}{F'(z)} \right| \leq |c| + \frac{1}{\Re(\rho)} \left| \frac{zF''(z)}{F'(z)} \right|, \\
\left| c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right) \frac{zF''(z)}{\rho F'(z)} \right| \leq |c| + \frac{1}{\Re(\rho)} \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|} (\mu_i M_i + n(n+1)M_i).
\end{gathered} \tag{2.37}$$

Using (2.45) and (2.46), we conclude that

$$\left| c|z|^{2\rho} + \left(1 - |z|^{2\rho}\right) \frac{zF'(z)}{\rho F(z)} \right| \leq 1. \tag{2.38}$$

Hence by using Lemma 1.3, the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent. □

Corollary 2.8. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M$, $M \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma_i|}, \tag{2.39}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma_i|}, \quad M \geq 1, \tag{2.40}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Corollary 2.9. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M$, $M \geq 1$, $|\gamma_i| = |\gamma|$ for all $i = \{1,2,3,\ldots,n\}$ and $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma|}, \tag{2.41}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i + n(n+1))M}{|\gamma|}, \quad M \geq 1, \tag{2.42}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Using a similar method as in the proof of Theorem 2.7, one can prove the following results.

Theorem 2.10. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots,n\}$ and $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta)f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots,n\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i M_i - 1)M_i + M_i^n M_i}{|\gamma_i|(M_i - 1)}, \tag{2.43}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i M_i - 1) M_i + M_i^n M_i}{|\gamma_i|(M_i - 1)}, \quad M_i \geq 1, \tag{2.44}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Theorem 2.11. *Let c be a complex number, $|I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(z)| \leq M_i$, $M_i \geq 1$ for all $i = \{1,2,3,\ldots\}$ and $I_\lambda^{n,\nu}(\alpha,\beta,\mu,\eta) f_i(t) \in \mathfrak{K}_{2,\mu_i}$, for $i = \{1,2,3,\ldots\}$ such that*

$$\Re(\rho) \geq \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i|(M_i - 1)}, \tag{2.45}$$

where ρ, γ_i are complex numbers. If

$$|c| \leq 1 - \frac{1}{\mathrm{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i|(M_i - 1)}, \quad M_i \geq 1, \tag{2.46}$$

then the family $\Upsilon_{\gamma_i,\eta,\lambda}(n,\rho,\nu,\alpha,\beta : z)$ is univalent.

Note that some other related work involving integral operators regarding univalence criteria can also be found in [21–23].

Acknowledgment

The work presented here was partially supported by UKM-ST-06-FRGS0244-2010.

References

[1] S. Owa, "On the distortion theorems. I," *Kyungpook Mathematical Journal*, vol. 18, no. 1, pp. 53–59, 1978.
[2] S. Owa and H. M. Srivastava, "Univalent and starlike generalized hypergeometric functions," *Canadian Journal of Mathematics*, vol. 39, no. 5, pp. 1057–1077, 1987.
[3] M. K. Aouf, R. M. El-Ashwah, and S. M. El-Deeb, "Some inequalities for certain p-valent functions involving extended multiplier transformations," *Proceedings of the Pakistan Academy of Sciences*, vol. 46, no. 4, pp. 217–221, 2009.
[4] F. M. Al-Oboudi, "On univalent functions defined by a generalized Salagean operator," *International Journal of Mathematics and Mathematical Sciences*, no. 25–28, pp. 1429–1436, 2004.
[5] G. Ş. Sălăgean, "Subclasses of univalent functions," in *Complex Analysis—5th Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, vol. 1013 of *Lecture Notes in Mathematics*, pp. 362–372, Springer, Berlin, Germany, 1983.
[6] B. A. Uralegaddi and C. Somanatha, "Certain classes of univalent functions," in *Current Topics in Analytic Function Theory*, pp. 371–374, World Scientific, Singapore, 1992.
[7] N. E. Cho and H. M. Srivastava, "Argument estimates of certain analytic functions defined by a class of multiplier transformations," *Mathematical and Computer Modelling*, vol. 37, no. 1-2, pp. 39–49, 2003.
[8] N. E. Cho and T. H. Kim, "Multiplier transformations and strongly close-to-convex functions," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 3, pp. 399–410, 2003.
[9] F. M. Al-Oboudi and K. A. Al-Amoudi, "On classes of analytic functions related to conic domains," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 655–667, 2008.

[10] F. M. Al-Oboudi, "On classes of functions related to starlike functions with respect to symmetric conjugate points defined by a fractional differential operator," *Complex Analysis and Operator Theory*. In press.

[11] A. Catas, "On certain classes of p-valent functions defined by multiplier transformations," in *Proceedings of the International Symposium on Geometric Function Theory and Applications (GFTA '07)*, S. Owa and Y. Polatoglu, Eds., vol. 91, TC Istanbul Kultur University Publications, Istanbul, Turkey, August 2007.

[12] S. S. Kumar, H. C. Taneja, and V. Ravichandran, "Classes multivalent functions defined by Dziok—Srivastava linear operaor and multiplier transformations," *Kyungpook Mathematical Journal*, vol. 46, pp. 97–109, 2006.

[13] H. M. Srivastava, K. Suchithra, B. A. Stephen, and S. Sivasubramanian, "Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions of complex order," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 5, pp. 1–8, 2006.

[14] S. Bulut, "Sufficient conditions for univalence of an integral operator defined by Al-Oboudi differential operator," *Journal of Inequalities and Applications*, Article ID 957042, 5 pages, 2008.

[15] D. Breaz, N. Breaz, and H. M. Srivastava, "An extension of the univalent condition for a family of integral operators," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 41–44, 2009.

[16] D. Breaz and N. Breaz, "Two integral operators," *Universitatis Babeş-Bolyai*, vol. 47, no. 3, pp. 13–19, 2002.

[17] L. V. Ahlfors, "Sufficient conditions for quasiconformal extension," in *Discontinuous Groups and Riemann Surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pp. 23–29, Princeton University Press, Princeton, NJ, USA, 1974.

[18] J. Becker, "Löwnersche differentialgleichung und Schlichtheitskriterien," *Mathematische Annalen*, vol. 202, pp. 321–335, 1973.

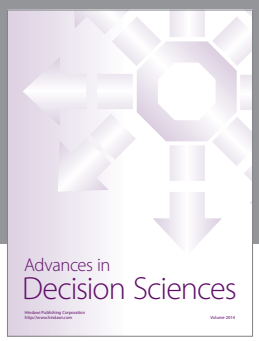
[19] V. Pescar, "A new generalization of Ahlfors's and Becker's criterion of univalence," *Malaysian Mathematical Society. Bulletin. Second Series*, vol. 19, no. 2, pp. 53–54, 1996.

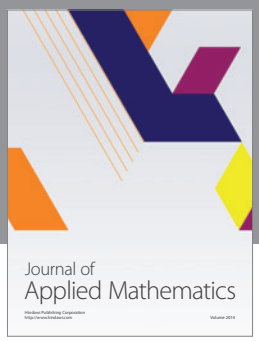
[20] V. Singh, "On a class of univalent functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 23, no. 12, pp. 855–857, 2000.

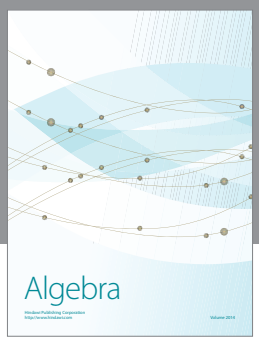
[21] N. Breaz, D. Braez, and M. Darus, "Convexity properties for some general integral operators on uniformly analytic functions classes," *Computers and Mathematics with Applications*, vol. 60, pp. 3105–3017, 2010.

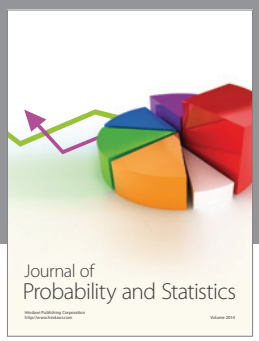
[22] A. Mohammed and M. Darus, "A new integral operator for meromorphic functions," *Acta Universitatis Apulensis*, no. 24, pp. 231–238, 2010.

[23] A. Mohammed, M. Darus, and D. Breaz, "Some properties for certain integral operators," *Acta Universitatis Apulensis*, no. 23, pp. 79–89, 2010.









Hindawi

Submit your manuscripts at
http://www.hindawi.com