Research Article

Tripled Fixed Points of Multivalued Nonlinear Contraction Mappings in Partially Ordered Metric Spaces

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Berinde and Borcut (2011), introduced the concept of tripled fixed point for single mappings in partially ordered metric spaces. Samet and Vetro (2011) established some coupled fixed point theorems for multivalued nonlinear contraction mappings in partially ordered metric spaces. In this paper, we obtain existence of tripled fixed point of multivalued nonlinear contraction mappings in the framework of partially ordered metric spaces. Also, we give an example.

1. Introduction and Preliminaries

Let (X, d) be a metric space. Consistent with [1], we denote by $CB(X)$ the family of all nonempty closed bounded and nonempty closed subsets of X. Let $CL(X) = \{A \subset X : A \neq \emptyset$ and $A = \overline{A}\}$, where $\overline{A}$ denotes the closure of A in X. For $A, B \in CB(X)$, and $x \in X$, set $D(x, A) := \inf\{d(x, a) : a \in A\}$. We define a Hausdorff metric H on $CB(X)$ by

$$H(A, B) := \max\left\{\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)\right\}. \tag{1.1}$$

A point $x \in K$ is called a fixed point of T if $x \in Tx$.

The study of fixed points for multivalued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin [2]. Later, an interesting and rich fixed point

theory for such maps was developed. Several authors studied the problem of existence of fixed point of multivalued mappings satisfying different contractive conditions (see, e.g., [3–10]). The theory of multivalued maps has application in control theory, convex optimization, differential equations, and economics.

Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [11], further studied by Nieto and Rodríguez-López [12]. Samet and Vetro [13] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for $N = 3$ (tripled case), we have the following definition.

Definition 1.1 (see, e.g., [13]). An element $(x, y, z) \in X^3$ is called a tripled fixed point of a mapping $F : X^3 \to X$ if and only if

$$x = F(x, y, z), \qquad y = F(y, z, x), \qquad z = F(z, x, y). \tag{1.2}$$

Recently, Berinde and Borcut [14] established the existence of tripled fixed point of single-valued mappings in partially ordered metric spaces. The aim of this paper is to initiate the study of tripled fixed point of multivalued mappings in the framework of partially ordered metric spaces which in turn extend and strengthen various known results [5, 15].

2. Tripled Fixed Point Results for Multivalued Mappings

First, we introduce the following concepts.

Definition 2.1. An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to \mathrm{CL}(X)$ if

$$x \in F(x, y, z), \qquad y \in F(y, z, x), \qquad z \in F(z, x, y). \tag{2.1}$$

Definition 2.2. A mapping $f : X^3 \to \mathbb{R}$ is called lower semicontinuous if, for any sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in X and $(x, y, z) \in X^3$, one has

$$\lim_{n \to \infty} (x_n, y_n, z_n) = (x, y, z) \Longrightarrow f(x, y, z) \leq \liminf_{n \to \infty} (x_n, y_n, z_n). \tag{2.2}$$

Let (X, d) be a metric space endowed with a partial order $\preceq$ and $T : X \to X$. Define the set $\Psi \subset X^3$ by

$$\Psi = \left\{ (x, y, z) \in X^3 : T(x) \preceq T(y) \preceq T(z) \right\}. \tag{2.3}$$

Definition 2.3. A mapping $F : X^3 \to X$ is said to have a Ψ-property if

$$(x, y, z) \in \Psi \Longrightarrow F(x, y, z) \times F(y, z, x) \times F(z, y, x) \subset \Psi. \tag{2.4}$$

We give some examples to illustrate Definition 2.3.

Example 2.4. Let $X = \mathbb{R}$ be endowed with the usual order $\leq$, and $T : X \rightarrow X$. Define $F : X^3 \rightarrow$ CL(X) by

$$F(x, y, z) = \{x\} \quad \forall x, y, z \in \mathbb{R}. \tag{2.5}$$

Obviously, F has the Ψ-property.

Example 2.5. Let $X = \mathbb{R}^+$ be endowed with the usual order $\leq$, and let $T : X \rightarrow X$ be defined by $Tx = \exp(x)$. Define $F : X^3 \rightarrow$ CL(X) by

$$F(x, y, z) = \{x + z\} \quad \forall x, y, z \in \mathbb{R}^+. \tag{2.6}$$

We have $\Psi = \{(x, y, z) \in X^3, \ \exp(x) \leq \exp(y) \leq \exp(z)\}$. Moreover, F has the Ψ-property.

Now, we prove the following theorem.

Theorem 2.6. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and $\Psi \neq \emptyset$; that is, there exists $(x_0, y_0, z_0) \in \Psi$. Suppose that $F : X^3 \rightarrow$ CL(X) has a Ψ-property such that $f : X^3 \rightarrow [0, \infty)$ given by*

$$f(x, y, z) = D(x, F(x, y, z)) + D(y, F(y, z, x)) + D(z, F(z, x, y)) \quad \forall x, y, z \in X \tag{2.7}$$

is lower semicontinuous and there exists a function $\phi : [0, \infty) \rightarrow [M, 1), \ 0 < M < 1$, satisfying

$$\limsup_{r \rightarrow t^+} \phi(r) < 1 \quad \text{for each } t \in [0, \infty). \tag{2.8}$$

If for any $(x, y, z) \in \Psi$ there exist $u \in F(x, y, z), \ v \in F(y, z, x)$, and $w \in F(z, y, x)$ with

$$\sqrt{\phi(f(x, y, z))} \left[d(x, u) + d(y, v) + d(z, w)\right] \leq f(x, y, z) \tag{2.9}$$

such that

$$f(u, v, w) \leq \phi(f(x, y, z)) \left[d(x, u) + d(y, v) + d(z, w)\right], \tag{2.10}$$

then F has a tripled fixed point.

Proof. By our assumption, $\phi(f(x, y, z)) < 1$ for each $(x, y, z) \in X^3$. Hence, for any $(x, y, z) \in X^3$, there exist $u \in F(x, y, z), \ v \in F(y, z, x)$, and $w \in F(z, x, y)$ satisfying

$$\begin{aligned}
\sqrt{\phi(f(x, y, z))} d(x, u) &\leq D(x, F(x, y, z)), \\
\sqrt{\phi(f(x, y, z))} d(y, v) &\leq D(y, F(y, z, x)), \\
\sqrt{\phi(f(x, y, z))} d(z, w) &\leq D(z, F(z, x, y)).
\end{aligned} \tag{2.11}$$

Let (x_0, y_0, z_0) be an arbitrary point in Ψ. By (2.9) and (2.10), we can choose $x_1 \in F(x_0, y_0, z_0)$, $y_1 \in F(y_0, z_0, x_0)$, and $z_1 \in F(z_0, x_0, y_0)$ satisfying

$$\sqrt{\phi(f(x_0, y_0, z_0))}\big[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1)\big] \leq f(x_0, y_0, z_0) \tag{2.12}$$

such that

$$f(x_1, y_1, z_1) \leq \phi(f(x_0, y_0, z_0))\big[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1)\big]. \tag{2.13}$$

By (2.12) and (2.13), we obtain

$$
\begin{aligned}
f(x_1, y_1, z_1) &\leq \phi(f(x_0, y_0, z_0))\big[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1)\big] \\
&\leq \sqrt{\phi(f(x_0, y_0, z_0))}\left(\sqrt{\phi(f(x_0, y_0, z_0))}\big[d(x_0, x_1) + d(y_0, y_1) + d(z_0, z_1)\big]\right) \\
&\leq \sqrt{\phi(f(x_0, y_0, z_0))}\, f(x_0, y_0, z_0).
\end{aligned}
\tag{2.14}
$$

Thus,

$$f(x_1, y_1, z_1) \leq \sqrt{\phi(f(x_0, y_0, z_0))}\, f(x_0, y_0, z_0). \tag{2.15}$$

Since F has a Ψ-property and $(x_0, y_0, z_0) \in \Psi$, so we have

$$F(x_0, y_0, z_0) \times F(y_0, z_0, x_0) \times F(z_0, x_0, y_0) \subset \Psi \tag{2.16}$$

which implies that $(x_1, y_1, z_1) \in \Psi$.

Again by (2.9) and (2.10), we can choose $x_2 \in F(x_1, y_1, z_1)$, $y_2 \in F(y_1, z_1, x_1)$, and $z_2 \in F(z_1, x_1, y_1)$ satisfying

$$\sqrt{\phi(f(x_1, y_1, z_1))}\big[d(x_1, x_2) + d(y_1, y_2) + d(z_1, z_2)\big] \leq f(x_1, y_1, z_1) \tag{2.17}$$

such that

$$f(x_2, y_2, z_2) \leq \phi(f(x_1, y_1, z_1))\big[d(x_1, x_2) + d(y_1, y_2) + d(z_1, z_2)\big]. \tag{2.18}$$

Thus, we have

$$f(x_2, y_2, z_2) \leq \sqrt{\phi(f(x_1, y_1, z_1))}\, f(x_1, y_1, z_1) \tag{2.19}$$

and $(x_2, y_2, z_2) \in \Psi$.

Continuing this process, we can choose sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in X such that for each $n \in \mathbb{N}$ with $(x_n, y_n, z_n) \in \Psi$.

$x_{n+1} \in F(x_n, y_n, z_n)$, $y_{n+1} \in F(y_n, z_n, x_n)$ and $z_{n+1} \in F(z_n, x_n, y_n)$ satisfying

$$\sqrt{\phi(f(x_n, y_n, z_n))}\left[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})\right] \leq f(x_n, y_n, z_n) \tag{2.20}$$

such that

$$f(x_{n+1}, y_{n+1}, z_{n+1}) \leq \phi(f(x_n, y_n, z_n))\left[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1})\right]. \tag{2.21}$$

Hence, we obtain

$$f(x_{n+1}, y_{n+1}, z_{n+1}) \leq \sqrt{\phi(f(x_n, y_n, z_n))}\, f(x_n, y_n, z_n) \tag{2.22}$$

with $(x_{n+1}, y_{n+1}, z_{n+1}) \in \Psi$. We claim that $f(x_n, y_n, z_n) \to 0$ as $n \to \infty$. If $f(x_n, y_n, z_n) = 0$ for some $n \in \mathbb{N}$, then $D(x_n, F(x_n, y_n, z_n)) = 0$ implies that $x_n \in \overline{F(x_n, y_n, z_n)} = F(x_n, y_n, z_n)$. Analogously, $D(y_n, F(y_n, z_n, x_n)) = 0$ implies that $y_n \in F(y_n, z_n, x_n)$ and $D(z_n, F(z_n, y_n, x_n)) = 0$ implies that $z_n \in F(z_n, y_n, x_n)$. Hence, (x_n, y_n, z_n) becomes a tripled fixed point of F for such n and the result follows. Suppose that $f(x_n, y_n, z_n) > 0$ for all $n \in \mathbb{N}$.

Using (2.22) and $\phi(t) < 1$, we conclude that $\{f(x_n, y_n, z_n)\}$ is a decreasing sequence of positive real numbers. Thus, there exists a $\delta \geq 0$ such that

$$\lim_{n \to \infty} f(x_n, y_n, z_n) = \delta. \tag{2.23}$$

We will show that $\delta = 0$. Assume on contrary that $\delta > 0$. Letting $n \to \infty$ in (2.22) and by assumption (2.8), we obtain

$$\delta \leq \limsup_{f(x_n, y_n, z_n) \to \delta^+} \sqrt{\phi(f(x_n, y_n, z_n))}\, \delta < \delta, \tag{2.24}$$

a contradiction. Hence,

$$\lim_{n \to \infty} f(x_n, y_n, z_n) = 0^+. \tag{2.25}$$

Now, we prove that $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ are Cauchy sequences in (X, d). Assume that

$$\alpha = \limsup_{f(x_n, y_n, z_n) \to 0^+} \sqrt{\phi(f(x_n, y_n, z_n))}. \tag{2.26}$$

By (2.8), we conclude that $\alpha < 1$. Let k be a real number such that $\alpha < k < 1$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\sqrt{\phi(f(x_n, y_n, z_n))} < k \quad \text{for each } n \geq n_0. \tag{2.27}$$

Using (2.22), we obtain

$$f(x_{n+1}, y_{n+1}, z_{n+1}) < kf(x_n, y_n, z_n) \quad \text{for each } n \geq n_0. \tag{2.28}$$

By mathematical induction,

$$f(x_{n+1}, y_{n+1}, z_{n+1}) < k^{n+1-n_0} f(x_{n_0}, y_{n_0}, z_{n_0}) \quad \text{for each } n \geq n_0. \tag{2.29}$$

Since $\phi(t) \geq M > 0$ for all $t \geq 0$ so (2.20), and (2.29) gives that

$$\left[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \right] < \frac{k^{n-n_0}}{\sqrt{M}} f(x_{n_0}, y_{n_0}, z_{n_0}) \quad \text{for each } n \geq n_0 \tag{2.30}$$

which yields that $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ are Cauchy sequences in X. Since X is complete, there exists $(a, b, c) \in X^3$ such that

$$\lim_{n \to \infty} x_n = a, \qquad \lim_{n \to \infty} y_n = b, \qquad \lim_{n \to \infty} z_n = c. \tag{2.31}$$

Finally, we show that $(a, b, c) \in X^3$ is tripled fixed point of F. As f is lower semicontinuous, (2.25) implies that

$$0 \leq f(a, b, c) = D(a, F(a, b, c)) + D(b, F(b, c, a)) + D(c, F(c, a, b)) \leq \liminf_{n \to \infty} f(x_n, y_n, z_n) = 0. \tag{2.32}$$

Hence, $D(a, F(a, b, c)) = D(b, F(b, c, a)) = D(c, F(c, a, b)) = 0$ gives that (a, b, c) is a tripled fixed point of F. $\qquad \square$

Theorem 2.7. *Let* (X, d) *be a complete metric space endowed with a partial order* $\preceq$ *and* $\Psi \neq \emptyset$; *that is, there exists* $(x_0, y_0, z_0) \in \Psi$. *Suppose that* $F : X^3 \to \mathrm{CL}(X)$ *has a* Ψ-*property such that function* $f : X^3 \to [0, \infty)$ *defined by*

$$f(x, y, z) = D(x, F(x, y, z)) + D(y, F(y, z, x)) + D(z, F(z, x, y)) \quad \forall x, y, z \in X, \tag{2.33}$$

is lower semicontinuous and there exists a function $\phi : [0, \infty) \to [M, 1), \ 0 < M < 1$, *satisfying*

$$\limsup_{r \to t^+} \phi(r) < 1 \quad \text{for each } t \in [0, \infty). \tag{2.34}$$

If for any $(x, y, z) \in \Psi$ *there exist* $u \in F(x, y, z), \ v \in F(y, z, x)$, *and* $w \in F(z, y, x)$ *satisfying*

$$\sqrt{\phi(\Delta)} \Delta \leq D(x, f(x, y, z)) + D(y, f(y, z, x)) + D(z, f(z, x, y)) \tag{2.35}$$

such that

$$D(u, f(u, v, w)) + D(v, f(v, w, u)) + D(w, f(w, u, v)) \leq \phi(\Delta)\Delta, \qquad (2.36)$$

where $\Delta = \Delta((x, y, z), (u, v, w)) = [d(x, u) + d(y, v) + d(z, w)]$, *then F admits a tripled fixed point.*

Proof. By replacing $\phi(f(x, y, z))$ with $\phi([d(x, u) + d(y, v) + d(z, w)])$ in the proof of Theorem 2.6, we obtain sequences $\{x_n\}, \{y_n\}, \{z_n\} \subset X$ such that for each $n \in \mathbb{N}$ with

$$(x_n, y_n, z_n) \in \Psi, \qquad x_{n+1} \in F(x_n, y_n, z_n), \qquad y_{n+1} \in F(y_n, z_n, x_n), \qquad z_{n+1} \in F(z_n, x_n, y_n), \qquad (2.37)$$

such that

$$\sqrt{\phi(\Delta_n)}\Delta_n \leq D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n)), \qquad (2.38)$$

$$D(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1})) + D(y_{n+1}, F(y_{n+1}, z_{n+1}, x_{n+1})) + D(z_{n+1}, F(z_{n+1}, x_{n+1}, y_{n+1}))$$

$$\leq \sqrt{\phi(\Delta_n)}(D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n))), \qquad (2.39)$$

where

$$\Delta_n = \Delta_n((x_n, y_n, z_n), (x_{n+1}, y_{n+1}, z_{n+1})) = d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}). \qquad (2.40)$$

Again, following arguments similar to those given in the proof of Theorem 2.6, we deduce that

$$\{D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n))\} \qquad (2.41)$$

is a decreasing sequence of real numbers. Thus, there exists a $\delta > 0$ such that

$$\lim_{n \to \infty} D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + (z_n, F(z_n, x_n, y_n)) = \delta. \qquad (2.42)$$

Now, we need to prove that $\{\Delta_n\}$ admits a subsequence converging to certain η^+ for some $\eta \geq 0$. Since $\phi(t) \geq M > 0$, using (2.38), we obtain

$$\Delta_n \leq \frac{1}{\sqrt{a}}(D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n))). \qquad (2.43)$$

From (2.42) and (2.43), it is clear that the sequence

$$\{D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n))\} \qquad (2.44)$$

is bounded. Therefore, there is some $\theta \geq 0$ such that

$$\liminf_{n \to +\infty} \Delta_n = \theta. \tag{2.45}$$

From (2.37), we have $x_{n+1} \in F(x_n, y_n, z_n)$, $y_{n+1} \in F(y_n, z_n, x_n)$, and $z_{n+1} \in F(z_n, x_n, y_n)$,

$$\Delta_n \geq D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n)) \quad \text{for each } n \geq 0. \tag{2.46}$$

So comparing (2.42) to (2.45), we get that $\theta \geq \delta$. Now, we shall show that $\theta = \delta$. If $\delta = 0$, then, by (2.42) and (2.43), we get $\theta =: \liminf_{n \to +\infty} \Delta_n = 0$ and consequently $\theta = \delta = 0$. Suppose that $\delta > 0$. Assume on contrary that $\theta > \delta$. From (2.42) and (2.45), there is a positive integer n_0 such that

$$D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n)) < \delta + \frac{\theta - \delta}{4}, \tag{2.47}$$

$$\delta - \frac{\theta - \delta}{4} < \Delta_n, \tag{2.48}$$

for all $n \geq n_0$. We combine (2.38), (2.47) to (2.48) to obtain

$$\sqrt{\phi(\Delta_n)} \left(\delta - \frac{\theta - \delta}{4} \right) \leq \sqrt{\phi(\Delta_n)} \Delta_n$$

$$\leq D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n))$$

$$< \delta + \frac{\theta - \delta}{4}, \tag{2.49}$$

for all $n \geq n_0$. It follows that

$$\sqrt{\phi(\Delta_n)} \leq \frac{\theta + 3\delta}{3\theta + \delta} \quad \forall n \geq n_0. \tag{2.50}$$

By (2.39) and (2.50), we have

$$D(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1})) + D(y_{n+1}, F(y_{n+1}, z_{n+1}, x_{n+1})) + D(z_{n+1}, F(z_{n+1}, x_{n+1}, y_{n+1}))$$

$$\leq h(D(x_n, F(x_n, y_n, z_n)) + D(y_n, F(y_n, z_n, x_n)) + D(z_n, F(z_n, x_n, y_n))) \quad \forall n \geq n_0, \tag{2.51}$$

where $h = (\theta + 3\delta)/(3\theta + \delta)$. Since $\theta > \delta > 0$, therefore $h < 1$, so proceeding by induction and combining the above inequalities, it follows that

$$
\begin{aligned}
\delta &\leq D\big(x_{n_0+k_0}, F\big(x_{n_0+k_0}, y_{n_0+k_0}, z_{n_0+k_0}\big)\big) + D\big(y_{n_0+k_0}, F\big(y_{n_0+k_0}, z_{n_0+k_0}, x_{n_0+k_0}\big)\big) \\
&\quad + D\big(z_{n_0+k_0}, F\big(z_{n_0+k_0}, x_{n_0+k_0}, y_{n_0+k_0}\big)\big) \\
&\leq h^{k_0}\big[D\big(x_{n_0}, F\big(x_{n_0}, y_{n_0}, z_{n_0}\big)\big) + D\big(y_{n_0}, F\big(y_{n_0}, z_{n_0}, x_{n_0}\big)\big) + D\big(z_{n_0}, F\big(z_{n_0}, x_{n_0}, y_{n_0}\big)\big)\big] < \delta,
\end{aligned}
\tag{2.52}
$$

for a positive integer k_0. Then, we obtain a contradiction, so we must have $\theta = \delta$.

Now, we shall show that $\theta = 0$. Since

$$
\theta = \delta \leq D\big(x_n, F\big(x_n, y_n, z_n\big)\big) + D\big(y_n, F\big(y_n, z_n, x_n\big)\big) + D\big(z_n, F\big(z_n, x_n, y_n\big)\big) \leq \Delta_n, \tag{2.53}
$$

then we rewrite (2.45) as

$$
\liminf_{n \to +\infty} \Delta_n = \theta^+. \tag{2.54}
$$

Hence, there exists a subsequence $\{\Delta_{n_k}\}$ of $\{\Delta_n\}$ such that $\liminf_{k \to +\infty} \Delta_{n_k} = \theta^+$.

By (2.34), we have

$$
\limsup_{\Delta_{n_k} \to \theta^+} \sqrt{\phi(\Delta_{n_k})} < 1. \tag{2.55}
$$

From (2.39), we obtain

$$
\begin{aligned}
&D\big(x_{n_k+1}, F\big(x_{n_k+1}, y_{n_k+1}, z_{n_k+1}\big)\big) + D\big(y_{n_k+1}, F\big(y_{n_k+1}, z_{n_k+1}, x_{n_k+1}\big)\big) \\
&\quad + D\big(z_{n_k+1}, F\big(z_{n_k+1}, x_{n_k+1}, y_{n_k+1}\big)\big) \\
&\leq \sqrt{\phi(\Delta_{n_k})}\big(D\big(x_{n_k}, F\big(x_{n_k}, y_{n_k}, z_{n_k}\big)\big) + D\big(y_{n_k}, f\big(y_{n_k}, z_{n_k}, x_{n_k}\big)\big) + D\big(z_{n_k}, F\big(z_{n_k}, x_{n_k}, y_{n_k}\big)\big)\big).
\end{aligned}
\tag{2.56}
$$

Taking the limit as $k \to \infty$ and using (2.42), we have

$$
\begin{aligned}
\delta &= \limsup_{k \to +\infty}\big[D\big(x_{n_k+1}, F\big(x_{n_k+1}, y_{n_k+1}, z_{n_k+1}\big)\big) + D\big(y_{n_k+1}, F\big(y_{n_k+1}, z_{n_k+1}, x_{n_k+1}\big)\big) \\
&\qquad\qquad + D\big(z_{n_k+1}, f\big(z_{n_k+1}, x_{n_k+1}, y_{n_k+1}\big)\big)\big] \\
&\leq \limsup_{k \to +\infty}\Big[\sqrt{\phi(\Delta_{n_k})}\Big]
\end{aligned}
$$

$$\limsup_{k \to +\infty}\left(D(x_{n_k}, F(x_{n_k}, y_{n_k}, z_{n_k})) + D(y_{n_k}, F(y_{n_k}, z_{n_k}, x_{n_k})) + D(z_{n_k}, F(z_{n_k}, x_{n_k}, y_{n_k}))\right)$$

$$= \left(\limsup_{\Delta_{n_k} \to \theta^+}\sqrt{\phi(\Delta_{n_k})}\right)\delta.$$

(2.57)

Assume that $\delta > 0$, then from (2.57) we get that

$$1 \le \limsup_{\Delta_{n_k} \to \theta^+}\sqrt{\phi(\Delta_{n_k})},$$

(2.58)

a contradiction with respect to (2.55), so $\delta = 0$. Now, from (2.39), and (2.42) we have

$$\alpha = \limsup_{\Delta_n \to 0^+}\sqrt{\phi(\Delta_n)} < 1.$$

(2.59)

The rest of the proof is similar to the proof of Theorem 2.6, so it is omitted.   □

We improved and corrected the example of Samet and Vetro [13].

Example 2.8. Let $X = [0, 1]$, and let $d : X \times X \to [0, \infty)$ be the usual metric. Suppose that $T(x) = M$ for all $x \in [0, 1]$ where M is a constant in $[0, 1]$, and $F : X^3 \to CL(X)$ is defined, for all $y, z \in X$ as follow:

$$F(x, y, z) = \begin{cases} \left\{\dfrac{x^2}{4}\right\} & \text{if } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ \left\{\dfrac{15}{96}, \dfrac{1}{5}\right\} & \text{if } x = \dfrac{15}{32}. \end{cases}$$

(2.60)

Obviously, F has the Ψ-property. Set $\phi : [0, \infty) \to [0, 1)$:

$$\phi(t) = \begin{cases} \dfrac{11}{12}t & \text{if } t \in \left[0, \dfrac{2}{3}\right], \\ \dfrac{10}{16} & \text{if } t \in \left(\dfrac{2}{3}, \infty\right). \end{cases}$$

(2.61)

Consider the function

$$f(x, y, z) = \begin{cases} x + y + z - \dfrac{1}{4}(x^2 + y^2 + z^2) & \text{if } x, y, z \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ x + y - \dfrac{1}{4}(x^2 + y^2) + \dfrac{43}{160} & \text{if } x, y \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right] \text{ with } z = \dfrac{15}{32}, \\ x - \dfrac{1}{4}x^2 + \dfrac{86}{160} & \text{if } x, y \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right] \text{ with } y = z = \dfrac{15}{32}, \\ \dfrac{129}{160} & \text{if } x = y = z = \dfrac{15}{32} \end{cases}$$

(2.62)

which is lower semicontinuous. Thus, for all $x, y, z \in X$ with $x, y, z \neq 15/32$, there exist $u \in F(x, y, z) = \{x^2/4\}$, $v \in F(y, z, x) = \{y^2/4\}$, and $w \in F(z, y, x) = \{z^2/4\}$ such that

$$
\begin{aligned}
&D(u, F(u, v, w)) + D(v, F(v, w, u)) + D(w, F(w, v, u)) \\
&\quad = \frac{x^2}{4} - \frac{x^4}{64} + \frac{y^2}{4} - \frac{y^4}{64} + \frac{z^2}{4} - \frac{z^4}{64} \\
&\quad = \frac{1}{4}\left[\left(x + \frac{x^2}{4}\right)\left(x - \frac{x^2}{4}\right) + \left(y + \frac{y^2}{4}\right)\left(y - \frac{y^2}{4}\right) + \left(z + \frac{z^2}{4}\right)\left(z - \frac{z^2}{4}\right)\right] \\
&\quad \leq \frac{1}{4}\left[\left(x + \frac{x^2}{4}\right)d(x, u) + \left(y + \frac{y^2}{4}\right)d(y, v) + \left(z + \frac{z^2}{4}\right)d(z, w)\right] \\
&\quad \leq \frac{1}{4}\max\left\{\left(x + \frac{x^2}{4}\right), \left(y + \frac{y^2}{4}\right), \left(z + \frac{z^2}{4}\right)\right\}[d(x, u) + d(y, v) + d(z, w)] \\
&\quad < \frac{10}{12}\max\left\{\left(x - \frac{x^2}{4}\right), \left(y - \frac{y^2}{4}\right), \left(z - \frac{z^2}{4}\right)\right\}[d(x, u) + d(y, v) + d(z, w)] \\
&\quad \leq \phi(d(x, u) + d(y, v) + d(z, w))[d(x, u) + d(y, v) + d(z, w)].
\end{aligned}
\tag{2.63}
$$

Hence, for all $x, y, z \in X$ with $x, y, z \neq 15/32$, the conditions (2.35) and (2.36) are satisfied. Analogously, one can easily show that conditions (2.35) and (2.36) are satisfied for the cases $(x, y \in [0, 15/32) \cup (15/32, 1]$ with $z = 15/32)$ and $(x \in [0, 15/32) \cup (15/32, 1]$ and $y = z = 15/32)$. For the last case, that is, $x = y = z = 15/32$, we assume that $u = v = w = 15/96$, so it follows that

$$
[d(x, u) + d(y, v) + d(z, w)] = \frac{15}{16} > \frac{2}{3}.
\tag{2.64}
$$

As a consequence, we get that

$$
\begin{aligned}
&\sqrt{\phi(d(x, u) + d(y, v) + d(z, w))}[d(x, u) + d(y, v) + d(z, w)] \\
&\quad = \sqrt{\frac{10}{16}\frac{15}{16}} < \frac{129}{160} = D(x, F(x, y, z)) + D(y, F(y, z, x)) + D(z, F(z, x, y)), \\
&D(u, F(u, v, w)) + D(v, F(v, w, u)) + D(w, F(w, v, u)) = 3\left|\frac{15}{96} - \frac{1}{4}\left(\frac{15}{96}\right)^2\right| \\
&\quad < \frac{10}{16}\frac{15}{16} = \phi(d(x, u) + d(y, v) + d(z, w))[d(x, u) + d(y, v) + d(z, w)].
\end{aligned}
\tag{2.65}
$$

Thus, we conclude that all the conditions of Theorem 2.7 are satisfied and F admits a tripled fixed point $a = (0, 0, 0)$.

Remark 2.9. If we replace the function ϕ with the following, we get the results again:

$$\phi(t) = \begin{cases} \dfrac{7}{12}t & \text{if } t \in \left[0, \dfrac{3}{4}\right], \\[3mm] \dfrac{7}{16} & \text{if } t \in \left(\dfrac{3}{4}, \infty\right). \end{cases} \tag{2.66}$$

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