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Research Article

Logarithmically Complete Monotonicity Properties Relating to the Gamma Function

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We prove that the function $f_{\alpha,\beta}(x) = \Gamma^{\beta}(x+\alpha)/x^{\alpha}\Gamma(\beta x)$ is strictly logarithmically completely monotonic on $(0,\infty)$ if $(\alpha,\beta) \in \{(\alpha,\beta) : 1/\sqrt{\alpha} \le \beta \le 1, \alpha \neq 1\} \cup \{(\alpha,\beta) : 0 < \beta \le 1, \varphi_1(\alpha,\beta) \ge 0, \varphi_2(\alpha,\beta) \ge 0\}$ and $[f_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0,\infty)$ if $(\alpha,\beta) \in \{(\alpha,\beta) : 0 < \alpha \le 1/2, 0 < \beta \le 1\} \cup \{(\alpha,\beta) : 1 \le \beta \le 1/\sqrt{\alpha} \le \sqrt{2}, \alpha \neq 1\} \cup \{(\alpha,\beta) : 1/2 \le \alpha < 1, \beta \ge 1/(1-\alpha)\}$, where $\varphi_1(\alpha,\beta) = (\alpha^2+\alpha-1)\beta^2+(2\alpha^2-3\alpha+1)\beta-\alpha$ and $\varphi_2(\alpha,\beta) = (\alpha-1)\beta^2+(2\alpha^2-5\alpha+2)\beta-1$.

1. Introduction

It is well known that the classical Euler's gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt. \tag{1.1}$$

The logarithmic derivative of $\Gamma(x)$ defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{1.2}$$

is called the psi or digamma function and $\psi^{i}(x)$ for $i \in \mathbb{N}$ are known as the polygamma or multigamma functions. These functions play central roles in the theory of special functions and have lots of extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

For extension of these functions to complex variable and for basic properties, see [1]. Over the past half century, many authors have established inequalities and monotonicity for these functions (see [2–22]).

Recall that a real-valued function $f : I \to \mathbb{R}$ is said to be completely monotonic on I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.3}$$

for all $x \in I$ and $n \geq 0$. Moreover, f is said to be strictly completely monotonic if inequality (1.3) is strict.

Recall also that a positive real-valued function $f : I \to (0, \infty)$ is said to be logarithmically completely monotonic on I if f has derivatives of all orders on I and its logarithm $\log f$ satisfies

$$(-1)^k [\log f(x)]^{(k)} \geq 0 \tag{1.4}$$

for all $x \in I$ and $k \in \mathbb{N}$. Moreover, f is said to be strictly logarithmically completely monotonic if inequality (1.4) is strict.

Recently, the completely monotonic or logarithmically completely monotonic functions have been the subject of intensive research. There has been a lot of literature about the (logarithmically) completely monotonic functions related to the gamma function, psi function, and polygamma function, for example, [17, 18, 23–37] and the references therein. In 1997, Merkle [38] proved that $F(x) = \Gamma(2x)/\Gamma^2(x)$ is strictly log-concave on $(0, \infty)$. Later, Chen [39] showed that $[F(x)]^{-1} = \Gamma^2(x)/\Gamma(2x)$ is strictly logarithmically completely monotonic on $(0, \infty)$. In [40], Li and Chen proved that $F_\beta(x) = \Gamma^\beta(x)/\Gamma(\beta x)$ is strictly logarithmically completely monotonic on $(0, \infty)$ for $\beta > 1$, and $[F_\beta(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0, \infty)$ for $0 < \beta < 1$. Qi et al. in their article [41] showed that $f_\alpha(x) = \Gamma(x + \alpha)/x^\alpha\Gamma(x)$ is strictly logarithmically complete monotonic on $(0, \infty)$ for $\alpha > 1$, and $[f_\alpha(x)]^{-1}$ is strictly logarithmically complete monotonic on $(0, \infty)$ for $0 < \alpha < 1$.

The aim of this paper is to discuss the logarithmically complete monotonicity properties of the functions

$$f_{\alpha,\beta}(x) = \frac{\Gamma^\beta(x + \alpha)}{x^\alpha \Gamma(\beta x)} \tag{1.5}$$

and $[f_{\alpha,\beta}(x)]^{-1}$ on $(0, \infty)$ where $\alpha > 0$ and $\beta > 0$. The function $f_{\alpha,\beta}(x)$ is the deformation of the functions in [40, 41] with respect to the parameters α and β. We show that the properties of logarithmically complete monotonic are also true for suitable extensions of (α, β) near by two lines $\alpha = 0$ and $\beta = 1$, which generalizes the results of [40, 41].

For $(x, y) \in (0, \infty) \times (0, \infty)$, we define two binary functions as follows:

$$\begin{aligned} \varphi_1(x, y) &= \left(x^2 + x - 1\right) y^2 + \left(2x^2 - 3x + 1\right) y - x, \\ \varphi_2(x, y) &= (x - 1) y^2 + \left(2x^2 - 5x + 2\right) y - 1. \end{aligned} \tag{1.6}$$

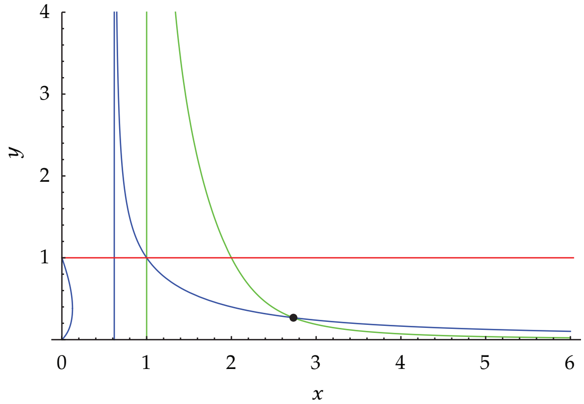

Figure 1: The blue curve is the graph of the equation $\varphi_1(x, y) = 0$ with the vertical asymptotic line $x = (\sqrt{5} - 1)/2$ and the green curve is the graph of $\varphi_2(x, y) = 0$ with the vertical asymptotic line $x = 1$.

For convenience, we need to define five subsets of $(0, \infty) \times (0, \infty)$ and refer to Figure 2,

$$\begin{aligned}
\mathbf{\Omega}_1 &= \left\{ (\alpha, \beta) : \frac{1}{\sqrt{\alpha}} \leq \beta \leq 1,\ \alpha \neq 1 \right\}, \\
\mathbf{\Omega}_2 &= \{ (\alpha, \beta) : 0 < \beta \leq 1,\ \varphi_1(\alpha, \beta) \geq 0,\ \varphi_2(\alpha, \beta) \geq 0 \}, \\
\mathbf{\Omega}_3 &= \left\{ (\alpha, \beta) : 0 < \alpha \leq \frac{1}{2},\ 0 < \beta \leq 1 \right\}, \\
\mathbf{\Omega}_4 &= \left\{ (\alpha, \beta) : 1 \leq \beta \leq \frac{1}{\sqrt{\alpha}} \leq \sqrt{2},\ \alpha \neq 1 \right\}, \\
\mathbf{\Omega}_5 &= \left\{ (\alpha, \beta) : \frac{1}{2} \leq \alpha < 1,\ \beta \geq \frac{1}{1-\alpha} \right\}.
\end{aligned} \tag{1.7}$$

We summarize the result as follows.

Theorem 1.1. *Let $\alpha > 0$, $\beta > 0$, and $f_{\alpha,\beta}(x)$ be defined as* (1.5)*; then the following statements are true:*

(1) *$f_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$ if $(\alpha, \beta) \in \mathbf{\Omega}_1 \cup \mathbf{\Omega}_2$;*

(2) *$[f_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0, \infty)$ if $(\alpha, \beta) \in \mathbf{\Omega}_3 \cup \mathbf{\Omega}_4 \cup \mathbf{\Omega}_5$.*

Note that $f_{\alpha,\beta}(x)$ is the constant 1 for $\alpha = \beta = 1$ since $\Gamma(x + 1) = x\Gamma(x)$.

2. Lemmas

In order to prove our Theorem 1.1, we need two lemmas which we present in this section.

We consider $\varphi_1(x, y)$ and $\varphi_2(x, y)$ defined as (1.6) and discuss the properties for these functions, see Figure 1 more clearly.

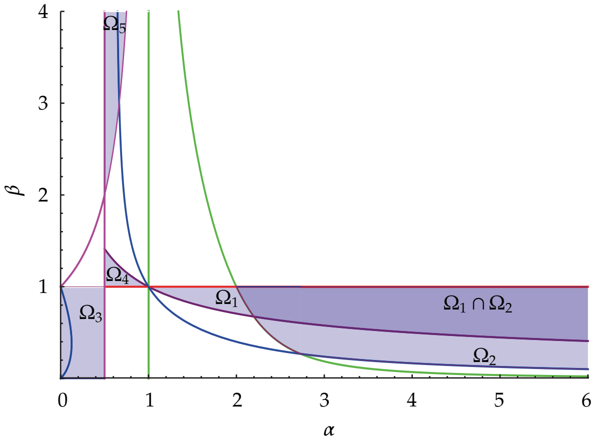

Figure 2: The shading areas are respectively denoted by the subsets $\boldsymbol{\Omega}_i$ for $i = 1, 2, \ldots, 5$. The function $f_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$ if $(\alpha, \beta) \in \boldsymbol{\Omega}_1 \cup \boldsymbol{\Omega}_2$, and $[f_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0, \infty)$ if $(\alpha, \beta) \in \boldsymbol{\Omega}_3 \cup \boldsymbol{\Omega}_4 \cup \boldsymbol{\Omega}_5$.

2.1. The Properties of Function $\varphi_1(x, y)$

The function $\varphi_1(x, y)$ can be interpreted as a quadric equation with respect to y. Let

$$\varphi_1(x, y) = a_1(x)y^2 + b_1(x)y + c_1(x), \tag{2.1}$$

where $a_1(x) = x^2 + x - 1$, $b_1(x) = 2x^2 - 3x + 1$, $c_1(x) = -x$, and its discriminant function

$$\Delta_1(x) = \sqrt{b_1^2(x) - 4a_1(x)c_1(x)} = 4x^4 - 8x^3 + 17x^2 - 10x + 1. \tag{2.2}$$

If $x = (\sqrt{5} - 1)/2$, then it is easy to see that

$$\varphi_1\left(\frac{\sqrt{5}-1}{2}, y\right) = \frac{11 - 5\sqrt{5}}{2}y - \frac{\sqrt{5}-1}{2} < 0 \tag{2.3}$$

for $y > 0$.

Let x_1, x_2 be two real roots of $\Delta_1(x)$ with $x_1 < x_2$; then we claim that $0 < x_1 < x_2 < (\sqrt{5} - 1)/2$. Indeed,

$$\Delta_1(0) = 1, \quad \lim_{x\to\infty} \Delta_1(x) = +\infty, \tag{2.4}$$

$$\Delta_1'(0) = -10, \tag{2.5}$$

$$\Delta_1'(x) = 16x^3 - 24x^2 + 34x - 10, \tag{2.6}$$

$$\Delta_1''(x) = 48x^2 - 48x + 34 > 0. \tag{2.7}$$

From (2.5)–(2.7), we know that $\Delta_1'(x)$ has only one root ξ, which is

$$\xi = \frac{1}{2} + \frac{\left(-27+\sqrt{8715}\right)^{1/3}}{26^{2/3}} - \frac{11}{2\left[6\left(-27+\sqrt{8715}\right)\right]^{1/3}} \approx 0.365\ldots. \tag{2.8}$$

Moreover, $\Delta_1'(x) < 0$ for $x \in (0,\xi)$ and $\Delta_1'(x) > 0$ for $x \in (\xi,\infty)$, which implies that $\Delta_1(x)$ is strictly decreasing on $(0,\xi)$ and strictly increasing on (ξ,∞). An easy computation shows that $\xi < (\sqrt{5}-1)/2$, $\Delta_1(\xi) < 0$, and $\Delta_1((\sqrt{5}-1)/2) > 0$. Combining with (2.4), there exist two real roots x_1, x_2 such that $0 < x_1 < x_2 < (\sqrt{5}-1)/2$. Furthermore, we conclude that $\Delta_1(x) > 0$ for $0 < x < x_1$ or $x > x_2$ and $\Delta_1(x) < 0$ for $x_1 < x < x_2$.

If $x_1 < x < x_2$, then $\varphi_1(x,y) < 0$ since $\Delta_1(x) < 0$ and $x^2 + x - 1 < 0$.

If $x_2 < x < (\sqrt{5}-1)/2$, then $a_1(x) < 0$, $b_1(x) < 0$, $c_1(x) < 0$, which implies $\varphi_1(x,y) < 0$.

If $0 < x \le x_1$ or $x > (\sqrt{5}-1)/2$, then $\Delta_1(x) \ge 0$. We can solve two roots of the equation $\varphi_1(x,y) = 0$, which are

$$\begin{aligned} \tilde{y}_1(x) &= \frac{-2x^2+3x-1-\sqrt{4x^4-8x^3+17x^2-10x+1}}{2(x^2+x-1)}, \\ y_1(x) &= \frac{-2x^2+3x-1+\sqrt{4x^4-8x^3+17x^2-10x+1}}{2(x^2+x-1)}. \end{aligned} \tag{2.9}$$

For $0 < x \le x_1$, we know that $\varphi_1(x,y) > 0$ for $y_1(x) < y < \tilde{y}_1(x)$ and $\varphi_1(x,y) < 0$ for $0 < y < y_1(x)$ or $y > \tilde{y}_1(x)$. For $x > (\sqrt{5}-1)/2$, we know that $\varphi_1(x,y) < 0$ for $0 < y < y_1(x)$ and $\varphi_1(x,y) > 0$ for $y > y_1(x)$. Moreover, we see that $y_1(x) \to +\infty$ as $x \to (\sqrt{5}-1)/2$ and $y_1(x) \to 0$ as $x \to +\infty$.

2.2. *The Properties of Function* $\varphi_2(x,y)$

The function $\varphi_2(x,y)$ can also be interpreted as a quadric equation with respect to y. Let

$$\varphi_2(x,y) = a_2(x)y^2 + b_2(x)y + c_2(x), \tag{2.10}$$

where $a_2(x) = x-1$, $b_2(x) = 2x^2-5x+2$, $c_2(x) = -1$, and its discriminant function

$$\Delta_2(x) = \sqrt{b_2^2(x) - 4a_2(x)c_2(x)} = 4x^4 - 20x^3 + 33x^2 - 16x. \tag{2.11}$$

If $x = 1$, then we have $\varphi_2(1,y) = -y-1 < 0$ for $y > 0$.

If $x < 1$, then a simple calculation leads to $\Delta_2(x) < 0$ for $0 < x < (1/6)[10 - 1/(53-6\sqrt{78})^{1/3} - (53-6\sqrt{78})^{1/3}] \approx 0.8427\ldots$. This implies that $\varphi_2(x,y) < 0$. Notice that $a_2(x) < 0$, $b_2(x) < 0$, and $c_2(x) = -1$; for $1/2 < x < 1$, then we have $\varphi_2(x,y) < 0$.

If $x > 1$, then we can solve the roots of the equation $\varphi_2(x, y) = 0$ but only one of the roots is positive, that is,

$$y_2(x) = \frac{-2x^2 + 5x - 2 + \sqrt{4x^4 - 20x^3 + 33x^2 - 16x}}{2(x-1)}. \tag{2.12}$$

Therefore, we conclude that $\varphi_2(x, y) < 0$ for $0 < y < y_2(x)$ and $\varphi_2(x, y) > 0$ for $y > y_2(x)$. Moreover, it is easy to see that $y_2(x) \to +\infty$ as $x \to 1$ and $y_2(x) \to 0$ as $x \to +\infty$.

Finally, we calculate an intersection point of $\varphi_1(x, y) = 0$ and $\varphi_2(x, y) = 0$, that is, the point

$$\left(\frac{2\sqrt{3}}{3-\sqrt{3}}, 2-\sqrt{3}\right). \tag{2.13}$$

Lemma 2.1. *The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed as*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1-e^{-t}} dt, \tag{2.14}$$

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-t}} e^{-xt} dt \tag{2.15}$$

for $x > 0$ and $n \in \mathbb{N} := \{1, 2, \ldots\}$, where $\gamma = 0.5772\ldots$ is Euler's constant.

Lemma 2.2. *Let $(\alpha, \beta) \in (0, \infty) \times (0, \infty)$ and*

$$r(t) = \left(1 - e^{-t}\right)\left(\beta e^{-\alpha\beta t} - \alpha e^{-\beta t}\right) + e^{-\beta t} - \alpha e^{-t} + \alpha - 1. \tag{2.16}$$

Then the following statements are true:

(1) *if $(\alpha, \beta) \in \mathbf{\Omega}_1 \cup \mathbf{\Omega}_2$, then $r(t) > 0$ for $t \in (0, \infty)$;*

(2) *if $(\alpha, \beta) \in \mathbf{\Omega}_3 \cup \mathbf{\Omega}_4 \cup \mathbf{\Omega}_5$, then $r(t) < 0$ for $t \in (0, \infty)$;*

(3) *if $0 < \alpha < 1/2$, $\beta > 1$ or $1/2 < \alpha < 1$, $0 < \beta < 1$, then there exist $\delta_2 \gg \delta_1 > 0$ such that $r(t) > 0$ for $t \in (0, \delta_1)$ and $r(t) < 0$ for $t \in (\delta_2, \infty)$;*

(4) *if $\alpha > 1$, $\beta > 1$, then there exist $\delta_4 \gg \delta_3 > 0$ such that $r(t) < 0$ for $t \in (0, \delta_3)$ and $r(t) > 0$ for $t \in (\delta_4, \infty)$.*

Proof. Let $r_1(t) = e^t r'(t), r_2(t) = (1/\beta)e^{(\alpha\beta-1)t} r_1'(t), r_3(t) = e^t r_2'(t)$, and $r_4(t) = e^{(\beta-\alpha\beta)t} r_3'(t)$. Then simple calculations lead to

$$\begin{aligned} r(0) &= 0, \\ r'(t) &= \left(\beta + \alpha\beta^2\right)e^{-(\alpha\beta+1)t} - (\alpha + \alpha\beta)e^{-(\beta+1)t} - \alpha\beta^2 e^{-\alpha\beta t} \\ &\quad + (\alpha\beta - \beta)e^{-\beta t} + \alpha e^{-t}, \end{aligned} \tag{2.17}$$

$$r_1(0) = r'(0) = 0, \tag{2.18}$$

$$\begin{aligned} r_1(t) &= \beta(1 + \alpha\beta)e^{-\alpha\beta t} - \alpha(1 + \beta)e^{-\beta t} - \alpha\beta^2 e^{-(\alpha\beta-1)t} \\ &\quad + \beta(\alpha - 1)e^{-(\beta-1)t} + \alpha, \end{aligned} \tag{2.19}$$

$$\begin{aligned} r_1'(t) &= -\alpha\beta^2(1 + \alpha\beta)e^{-\alpha\beta t} + \alpha\beta(1 + \beta)e^{-\beta t} \\ &\quad + \alpha\beta^2(\alpha\beta - 1)e^{-(\alpha\beta-1)t} - \beta(\alpha - 1)(\beta - 1)e^{-(\beta-1)t}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} r_2(0) &= \frac{1}{\beta} r_1'(0) = (\beta - 1)(1 - 2\alpha), \\ r_2(t) &= -\alpha\beta(1 + \alpha\beta)e^{-t} + \alpha(1 + \beta)e^{(\alpha\beta-\beta-1)t} \\ &\quad - (\alpha - 1)(\beta - 1)e^{(\alpha-1)\beta t} + \alpha\beta(\alpha\beta - 1), \end{aligned} \tag{2.21}$$

$$\begin{aligned} r_2'(t) &= \alpha\beta(1 + \alpha\beta)e^{-t} + \alpha(1 + \beta)(\alpha\beta - \beta - 1)e^{(\alpha\beta-\beta-1)t} \\ &\quad - \beta(\alpha - 1)^2(\beta - 1)e^{(\alpha-1)\beta t}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} r_3(0) &= r_2'(0) = \varphi_1(\alpha, \beta), \\ r_3(t) &= \alpha(\beta + 1)(\alpha\beta - \beta - 1)e^{(\alpha-1)\beta t} \\ &\quad - \beta(\alpha - 1)^2(\beta - 1)e^{(\alpha\beta-\beta+1)t} + \alpha\beta(1 + \alpha\beta), \end{aligned} \tag{2.23}$$

$$\begin{aligned} r_3'(t) &= \alpha\beta(\alpha - 1)(\beta + 1)(\alpha\beta - \beta - 1)e^{(\alpha-1)\beta t} \\ &\quad + \beta(\alpha - 1)^2(\beta - 1)(\beta - \alpha\beta - 1)e^{(\alpha\beta-\beta+1)t}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} r_4(0) &= r_3'(0) = \beta(\alpha - 1)\varphi_2(\alpha, \beta), \\ r_4(t) &= \beta(\alpha - 1)^2(\beta - 1)(\beta - \alpha\beta - 1)e^t \\ &\quad + \alpha\beta(\alpha - 1)(\beta + 1)(\alpha\beta - \beta - 1), \end{aligned} \tag{2.25}$$

$$r_4'(t) = \beta(\alpha - 1)^2(\beta - 1)(\beta - \alpha\beta - 1)e^t. \tag{2.26}$$

(1) If $(\alpha, \beta) \in \mathbf{\Omega}_1 \cup \mathbf{\Omega}_2$, then we divide the proof into two cases. Note that $\mathbf{\Omega}_1 \cap \mathbf{\Omega}_2 = \{(\alpha, \beta) : \max\{1/\sqrt{\alpha}, y_2(\alpha)\} \le \beta \le 1\}$, see Figure 2.

Case 1. If $(\alpha, \beta) \in \mathbf{\Omega}_1$, then $1/\sqrt{\alpha} \leq \beta \leq 1$, $\alpha \neq 1$, and it follows from (2.21) that

$$\begin{aligned} r_2(t) &= -\alpha\beta(1+\alpha\beta)e^{-t} + e^{(\alpha-1)\beta t}\left[\alpha(1+\beta)e^{-t} + (\alpha-1)(1-\beta)\right] \\ &\quad + \alpha\beta(\alpha\beta-1) \\ &> \alpha\left(1-\alpha\beta^2\right)e^{-t} + (\alpha-1)(1-\beta) + \alpha\beta(\alpha\beta-1) \\ &\geq \alpha\left(1-\alpha\beta^2\right) + (\alpha-1)(1-\beta) + \alpha\beta(\alpha\beta-1) \\ &= (\beta-1)(1-2\alpha) \\ &\geq 0. \end{aligned} \tag{2.27}$$

Therefore, $r(t) > 0$ for $t \in (0, \infty)$ follows from (2.17), (2.18) together with (2.27).

Case 2. If $(\alpha, \beta) \in \mathbf{\Omega}_2$, then $0 < \beta \leq 1$, $\varphi_1(\alpha, \beta) \geq 0$, and $\varphi_2(\alpha, \beta) \geq 0$. It follows from $\varphi_2(\alpha, \beta) \geq 0$ that $\alpha > 1$ and then (2.20) and (2.22) together with (2.24) lead to

$$r_2(0) \geq 0, \tag{2.28}$$

$$r_3(0) = \varphi_1(\alpha, \beta) \geq 0, \tag{2.29}$$

$$r_4(0) = \beta(\alpha-1)\varphi_2(\alpha, \beta) \geq 0, \tag{2.30}$$

$$r_4'(t) \geq 0. \tag{2.31}$$

This could not happen together for all qualities of (2.28)–(2.31) since the qualities of (2.29) and (2.30) hold only for $\alpha = 2\sqrt{3}/(3-\sqrt{3})$, $\beta = 2-\sqrt{3}$ while the qualities of (2.29) and (2.30) hold only for $\beta = 1$.

Therefore, $r(t) > 0$ for $t \in (0, \infty)$ follows from (2.17) and (2.18) together with (2.28)–(2.31).

(2) If $(\alpha, \beta) \in \mathbf{\Omega}_3 \cup \mathbf{\Omega}_4 \cup \mathbf{\Omega}_5$, then we divide the proof into three cases.

Case 1. If $(\alpha, \beta) \in \mathbf{\Omega}_3$, then $0 < \alpha \leq 1/2$ and $0 < \beta \leq 1 < 1/(1-\alpha)$. From (2.26), we clearly see that

$$r_4'(t) \geq 0. \tag{2.32}$$

In terms of the properties of $\varphi_2(x, y)$, we know that $\varphi_2(\alpha, \beta) < 0$ for (α, β) lying on the left-side of the green curve, see Figure 1. From (2.24), we see that

$$r_4(0) = \beta(\alpha-1)\varphi_2(\alpha, \beta) > 0. \tag{2.33}$$

Combining (2.32) with (2.33) we get that $r_3(t)$ is strictly increasing on $(0, \infty)$.

If $\varphi_1(\alpha,\beta) \geq 0$, then $0 < \beta < 1$ and $r_3(t) > 0$ follow from (2.22), which implies that $r_2(t)$ is strictly increasing in $(0,\infty)$. Thus we can obtain

$$r_2(t) < \lim_{t\to\infty} r_2(t) = \alpha\beta(\alpha\beta - 1) < 0. \tag{2.34}$$

If $\varphi_1(\alpha,\beta) < 0$, then it follows from $\lim_{t\to\infty} r_3(t) = +\infty$ or $\alpha\beta(1+\alpha\beta) > 0$ that there exists $\sigma_1 > 0$ such that $r_3(t) < 0$ for $t \in (0,\sigma_1)$ and $r_3(t) > 0$ for $t \in (\sigma_1,\infty)$. Hence, $r_2(t)$ is strictly decreasing in $(0,\sigma_1)$ and strictly increasing in (σ_1,∞). Then we can obtain

$$r_2(t) < \max\left\{ r_2(0), \lim_{t\to\infty} r_2(t) \right\} \leq 0. \tag{2.35}$$

Finally, we conclude that $r(t) < 0$ for $t \in (0,\infty)$ follows from (2.17), (2.18) together with (2.34), (2.35).

Case 2. If $(\alpha,\beta) \in \boldsymbol{\Omega}_4$, then $1/2 \leq \alpha < 1$ and $1 \leq \beta \leq 1/\sqrt{\alpha}$. It follows from (2.21) that

$$\begin{aligned}
r_2(t) &= -\alpha\beta(1+\alpha\beta)e^{-t} + e^{(\alpha-1)\beta t}\left[\alpha(1+\beta)e^{-t} + (1-\alpha)(\beta-1)\right] \\
&\quad + \alpha\beta(\alpha\beta-1) \\
&< \alpha\left(1-\alpha\beta^2\right)e^{-t} + (1-\alpha)(\beta-1) + \alpha\beta(\alpha\beta-1) \\
&\leq \alpha\left(1-\alpha\beta^2\right) + (1-\alpha)(\beta-1) + \alpha\beta(\alpha\beta-1) \\
&= (\beta-1)(1-2\alpha) \\
&\leq 0.
\end{aligned} \tag{2.36}$$

Therefore, $r(t) < 0$ for $t \in (0,\infty)$ follows from (2.17), (2.18) together with (2.36).

Case 3. If $(\alpha,\beta) \in \boldsymbol{\Omega}_5$, then $1/2 \leq \alpha < 1$ and $\beta - \alpha\beta - 1 \geq 0$. From (2.26), we know that

$$r_4'(t) \geq 0. \tag{2.37}$$

In terms of the location of $\boldsymbol{\Omega}_3$, we know that $\varphi_2(\alpha,\beta) < 0$. From (2.24), we see that

$$r_4(0) = \beta(\alpha-1)\varphi_2(\alpha,\beta) > 0. \tag{2.38}$$

It follows from (2.37) and (2.38) that $r_3(t)$ is strictly increasing on $(0,\infty)$.

If $\varphi_1(\alpha,\beta) \geq 0$, then $1/2 < \alpha < 1$ and $r_3(t) > 0$ follow that from (2.22), which implies that $r_2(t)$ is strictly increasing on $(0,\infty)$. From (2.20) and (2.21), we see that

$$r_2(0) = (\beta-1)(1-2\alpha) < 0, \qquad \lim_{t\to+\infty} r_2(t) = \alpha\beta(\alpha\beta-1) > 0. \tag{2.39}$$

Thus there exists $\sigma_2 > 0$ such that $r_2(t) < 0$ for $t \in (0, \sigma_2)$ and $r_2(t) > 0$ for $t \in (\sigma_2, \infty)$, which implies that $r_1(t)$ is strictly decreasing on $(0, \sigma_2)$ and strictly increasing on (σ_2, ∞). It follows from (2.18) and $\lim_{t\to\infty} r_1(t) = \alpha > 0$ that $\sigma_3 > \sigma_2$ such that $r_1(t) < 0$ for $t \in (0, \sigma_3)$ and $r_1(t) > 0$ for $t \in (\sigma_3, \infty)$, which implies that $r(t)$ is strictly decreasing on $(0, \sigma_3)$ and strictly increasing on (σ_3, ∞). Therefore, it follows from (2.17) and $\lim_{t\to\infty} r(t) = \alpha - 1 < 0$ that

$$r(t) < \max\left\{ r(0), \lim_{t\to\infty} r(t) \right\} = 0 \tag{2.40}$$

for $t \in (0, \infty)$.

If $\varphi_1(\alpha, \beta) < 0$, then there exists $\sigma_4 > 0$ such that $r_3(t) < 0$ for $t \in (0, \sigma_4)$ and $r_3(t) > 0$ for $t \in (\sigma_4, \infty)$ follows from $\lim_{t\to\infty} r_3(t) = \alpha\beta(1 + \alpha\beta) > 0$ or $\lim_{t\to\infty} r_3(t) = \beta[(\alpha - 1/2)^2 + \beta(2\alpha - 1) + 3/4] > 0$. This leads to $r_2(t)$ being strictly decreasing in $(0, \sigma_4)$ and strictly increasing in (σ_4, ∞). From (2.20), we clearly see that

$$r_2(0) \leq 0. \tag{2.41}$$

For special case of $\alpha\beta = 1$, that is, $\alpha = 1/2$ and $\beta = 2$, it follows from (2.41) and (2.21) that

$$r_2(t) < \max\left\{ r_2(0), \lim_{t\to\infty} r_2(t) \right\} = 0, \tag{2.42}$$

which implies that $r(t) < 0$ for $t \in (0, \infty)$ follows from (2.17) and (2.18).

For $\alpha\beta > 1$, it follows from (2.38) and $\lim_{t\to\infty} r_2(t) = \alpha\beta(\alpha\beta - 1) > 0$ that there exists $\sigma_5 > \sigma_4 > 0$ such that $r_2(t) < 0$ for $t \in (0, \sigma_5)$ and $r_2(t) > 0$ for $t \in (\sigma_5, \infty)$. Making use of the same arguments as the case of $\varphi_1(\alpha, \beta) \geq 0$, then $r(t) < 0$ for $t \in (0, \infty)$ follows from (2.17).

(3) If $0 < \alpha < 1/2$, $\beta > 1$ or $1/2 < \alpha < 1$, $0 < \beta < 1$, then we have

$$\lim_{t\to\infty} r(t) = \alpha - 1 < 0. \tag{2.43}$$

From (2.20), we know that

$$r_2(0) = \left(\beta - 1\right)(1 - 2\alpha) > 0. \tag{2.44}$$

It follows from (2.44) that there exists $\delta_1 > 0$ such that $r_2(t) > 0$ for $t \in (0, \delta_1)$, which implies that $r_1(t)$ is strictly increasing on $(0, \delta_1)$. Therefore, $r(t) > 0$ for $t \in (0, \delta_1)$ follows from (2.17) and (2.18).

From (2.43), we know that there exists $\delta_2 \gg \delta_1 > 0$ such that $r(t) < 0$ for $t \in (\delta_2, \infty)$.

(4) If $\alpha > 1$, $\beta > 1$, then we have

$$\lim_{t\to\infty} r(t) = \alpha - 1 > 0. \tag{2.45}$$

From (2.15), we know that

$$r_2(0) = (\beta - 1)(1 - 2\alpha) < 0. \tag{2.46}$$

Making use of (2.45) and (2.46) together with the same arguments as in Lemma 2.2(3), we know that there exist $\delta_4 \gg \delta_3 > 0$ such that $r_2(t) < 0$ for $t \in (0, \delta_3)$ and $r(t) > 0$ for $t \in (\delta_4, \infty)$. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. From (2.15), we have

$$\begin{aligned}
(-1)^n \left[\log f_{\alpha,\beta}(x)\right]^{(n)} &= (-1)^n \left[(-1)^n \frac{\alpha(n-1)!}{x^n} + \beta\psi^{(n-1)}(x+\alpha) - \beta^n \psi^{(n-1)}(\beta x)\right] \\
&= \alpha \int_0^\infty s^{n-1} e^{-xs} ds + \beta \int_0^\infty \frac{s^{n-1}}{1-e^{-s}} e^{-(x+\alpha)s} ds - \beta^n \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} e^{-\beta x t} dt \\
&= \alpha\beta^n \int_0^\infty t^{n-1} e^{-\beta x t} dt + \beta^{n+1} \int_0^\infty \frac{t^{n-1}}{1-e^{-\beta t}} e^{-\beta(x+\alpha)t} dt - \beta^n \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} e^{-\beta x t} dt \\
&= \beta^n \int_0^\infty \frac{t^{n-1} e^{-\beta x t}}{(1-e^{-t})(1-e^{-\beta t})} r(t) dt,
\end{aligned} \tag{3.1}$$

where

$$r(t) = (1 - e^{-t})\left(\beta e^{-\alpha\beta t} - \alpha e^{-\beta t}\right) + e^{-\beta t} - \alpha e^{-t} + \alpha - 1. \tag{3.2}$$

(1) If $(\alpha, \beta) \in \mathbf{\Omega}_1 \cup \mathbf{\Omega}_2$, then from (3.1) and (3.2) together with Lemma 2.2(1) we clearly see that

$$(-1)^n \left[\log f_{\alpha,\beta}(x)\right]^{(n)} > 0. \tag{3.3}$$

Therefore, $f_{\alpha,\beta}(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$ following from (3.3).

(2) If $(\alpha, \beta) \in \mathbf{\Omega}_3 \cup \mathbf{\Omega}_4 \cup \mathbf{\Omega}_5$, then from (3.1) we can get

$$(-1)^n \left\{\log \left[f_{\alpha,\beta}(x)\right]^{-1}\right\}^{(n)} = -\beta^n \int_0^\infty \frac{t^{n-1} e^{-\beta x t}}{(1-e^{-t})(1-e^{-\beta t})} r(t) dt, \tag{3.4}$$

where $r(t)$ is defined as (3.2).

Therefore, $[f_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0, \infty)$ following from (3.4) and Lemma 2.2 (2). □

Remark 3.1. Note that neither $f_{\alpha,\beta}(x)$ nor $[f_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0,\infty)$ for $(\alpha,\beta) \in \{(\alpha,\beta) : 0 < \alpha < 1/2, \beta > 1\} \cup \{(\alpha,\beta) : 1/2 < \alpha < 1, 0 < \beta < 1\} \cup \{(\alpha,\beta) : \alpha > 1,\ \beta > 1\}$ following from Lemma 2.2 (3) and (4), it is known that the logarithmically completely monotonicity properties of $f_{\alpha,\beta}(x)$ and $[f_{\alpha,\beta}(x)]^{-1}$ are not completely continuously depended on α and β.

Remark 3.2. Compared with Theorem 9 of [40], we can also extend $\mathbf{\Omega}_3$ onto one component of its boundaries, which is

$$\mathbf{\Omega}_3 \rightarrow \widetilde{\mathbf{\Omega}}_3 = \left\{ (\alpha,\beta) : 0 \le \alpha \le \frac{1}{2}, 0 < \beta \le 1 \right\} \setminus \{\alpha = 0, \beta = 1\}. \tag{3.5}$$

Then $[f_{\alpha,\beta}(x)]^{-1}$ is strictly logarithmically completely monotonic on $(0,\infty)$ for $(\alpha,\beta) \in \widetilde{\mathbf{\Omega}}_3$.

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