Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 924319, 13 pages
doi:10.1155/2011/924319

*Research Article*

# Refinements of the Lower Bounds of the Jensen Functional

**Iva Franjić,[1] Sadia Khalid,[2] and Josip Pečarić[2,3]**

[1] *Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*
[2] *Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore 54600, Pakistan*
[3] *Faculty of Textile Technology, University of Zagreb, Prilaz Baruna Filipovića 28A, 10000 Zagreb, Croatia*

Correspondence should be addressed to Sadia Khalid, saadiakhalid176@gmail.com

Received 1 July 2011; Accepted 4 August 2011

Academic Editor: Wing-Sum Cheung



The lower bounds of the functional defined as the difference of the right-hand and the left-hand side of the Jensen inequality are studied. Refinements of some previously known results are given by applying results from the theory of majorization. Furthermore, some interesting special cases are considered.

## 1. Introduction

The classical Jensen inequality states (see e.g., [1]).

**Theorem 1.1** (see [2]). *Let $I$ be an interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 2$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a positive $n$-tuple, that is, such that $p_i > 0$ for $i = 1, \ldots, n$, then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

*where $P_n = \sum_{i=1}^{n} p_i$. If $f$ is strictly convex, then inequality* (1.1) *is strict unless $x_1 = \cdots = x_n$.*

In this work, the functional

$$J(\mathbf{x},\mathbf{p},f)=\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right) \tag{1.2}$$

defined as the difference of the right-hand and the left-hand sides of the Jensen inequality is studied. More precisely, its lower bounds are investigated, together with various sets of assumptions under which they hold.

The lower bounds of $J(\mathbf{x},\mathbf{p},f)$ were the topic of interest in many papers. For example, the following results were proved in [3] (see also [1, page 717]). In what follows, $I$ is an interval in $\mathbb{R}$.

**Theorem 1.2.** *Let $f : I \to \mathbb{R}$ be a convex function, $\mathbf{x}\in I^n$, and let $\mathbf{p}$ be a positive $n$-tuple, then*

$$P_n\cdot J(\mathbf{x},\mathbf{p},f)\geq\max_{1\leq j\leq k\leq n}\left\{p_jf(x_j)+p_kf(x_k)-(p_j+p_k)f\left(\frac{p_jx_j+p_kx_k}{p_j+p_k}\right)\right\}\geq 0. \tag{1.3}$$

**Theorem 1.3.** *Let $f : I \to \mathbb{R}$ be a convex function and $\mathbf{x}\in I^n$. Let $\mathbf{p}$ and $\mathbf{r}$ be positive $n$-tuples such that $\mathbf{p}\geq\mathbf{r}$, that is, $p_i\geq r_i$, $i=1,\dots,n$, then*

$$P_n\cdot J(\mathbf{x},\mathbf{p},f)\geq R_n\cdot J(\mathbf{x},\mathbf{r},f)\geq 0, \tag{1.4}$$

*where $P_n=\sum_{i=1}^{n}p_i$ and $R_n=\sum_{i=1}^{n}r_i$.*

Further, in [4], the following theorem was given. An alternative proof of the same result was given in [5].

**Theorem 1.4.** *Let $f : I \to \mathbb{R}$ be a convex function, $n\geq 2$, and $\mathbf{x}\in I^n$. Let $\mathbf{p}$ and $\mathbf{q}$ be positive $n$-tuples such that $\sum_{i=1}^{n}p_i=\sum_{i=1}^{n}q_i=1$, then*

$$\max_{1\leq j\leq n}\left\{\frac{p_j}{q_j}\right\}J(\mathbf{x},\mathbf{q},f)\geq J(\mathbf{x},\mathbf{p},f)\geq\min_{1\leq j\leq n}\left\{\frac{p_j}{q_j}\right\}J(\mathbf{x},\mathbf{q},f)\geq 0. \tag{1.5}$$

For more related results, see [6–8]. The motivation for the research in this work were the following results presented in [9].

**Lemma 1.5.** *Let $f$ be a convex function on $I$, $\mathbf{p}$ a positive $n$-tuple such that $P_n=\sum_{i=1}^{n}p_i=1$ and $x_1,x_2,\dots,x_n\in I$, $n\geq 3$ such that $x_1\leq x_2\leq\cdots\leq x_n$. For fixed $x_j,x_{j+1},\dots,x_n$, where $j=2,3,\dots,n-1$, the Jensen functional $J(\mathbf{x},\mathbf{p},f)$ defined in (1.2) is minimal when $x_1=x_2=\cdots=x_{j-1}=x_j$, that is,*

$$J(\mathbf{x},\mathbf{p},f)\geq P_jf(x_j)+\sum_{i=j+1}^{n}p_if(x_i)-f\left(P_jx_j+\sum_{i=j+1}^{n}p_ix_i\right), \tag{1.6}$$

*where*

$$P_j = \sum_{i=1}^{j} p_i, \quad j = 1, \ldots, n. \tag{1.7}$$

**Lemma 1.6.** *Let $f$ be a convex function on $I$, $\mathbf{p}$ a positive $n$-tuple such that $P_n = \sum_{i=1}^{n} p_i = 1$ and $x_1, x_2, \ldots, x_n \in I$, $n \geq 3$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$. For fixed $x_1, x_2, \ldots, x_k$, where $k = 2, 3, \ldots, n-1$, the Jensen functional $J(\mathbf{x}, \mathbf{p}, f)$ defined in* (1.2) *is minimal when $x_k = x_{k+1} = \cdots = x_{n-1} = x_n$, that is,*

$$J(\mathbf{x}, \mathbf{p}, f) \geq \sum_{i=1}^{k-1} p_i f(x_i) + Q_k f(x_k) - f\left(\sum_{i=1}^{k-1} p_i x_i + Q_k x_k\right), \tag{1.8}$$

*where*

$$Q_k = \sum_{i=k}^{n} p_i, \quad k = 1, \ldots, n. \tag{1.9}$$

**Theorem 1.7.** *Let $f$ be a convex function on $I$, $\mathbf{p}$ a positive $n$-tuple such that $P_n = \sum_{i=1}^{n} p_i = 1$ and $x_1, x_2, \ldots, x_n \in I$, $n \geq 3$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$. For fixed $x_j$ and $x_k$, where $1 \leq j < k \leq n$, the Jensen functional $J(\mathbf{x}, \mathbf{p}, f)$ defined in* (1.2) *is minimal when*

$$\begin{gathered} x_1 = x_2 = \cdots = x_j, \qquad x_k = x_{k+1} = \cdots = x_n, \\ x_{j+1} = x_{j+2} = \cdots = x_{k-1} = \frac{P_j x_j + Q_k x_k}{P_j + Q_k}, \end{gathered} \tag{1.10}$$

*that is,*

$$J(\mathbf{x}, \mathbf{p}, f) \geq P_j f(x_j) + Q_k f(x_k) - (P_j + Q_k) f\left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right), \tag{1.11}$$

*where $P_j$ are as in* (1.7) *and $Q_k$ are as in* (1.9).

The key step in proving these results was the following lemma presented in the same paper.

**Lemma 1.8.** *Let $f$ be a convex function on $I$, and let $p_1$, $p_2$ be nonnegative real numbers. If $a_1, a_2, b_1, b_2 \in I$ are such that $a_1, a_2 \in [b_1, b_2]$ and*

$$p_1 a_1 + p_2 a_2 = p_1 b_1 + p_2 b_2, \tag{1.12}$$

*then*

$$p_1 f(a_1) + p_2 f(a_2) \leq p_1 f(b_1) + p_2 f(b_2). \tag{1.13}$$

Note that for a monotonic $n$-tuple $\mathbf{x}$, Theorem 1.7 is an improvement of Theorem 1.2, in a sense that (the maximum of) the right-hand side of (1.11) is greater than the middle part of (1.3), which follows directly from the Jensen inequality. The aim of this work is to give an improvement of Lemmas 1.5 and 1.6, and Theorem 1.7, in a sense that the condition of monotonicity imposed on the $n$-tuple $\mathbf{x}$ will be relaxed. Several sets of conditions under which (1.6), (1.8), and (1.11) hold shall be given. In our proofs, in addition to Lemma 1.8, the following result from the theory of majorization is needed. It was obtained in [10].

**Lemma 1.9.** *Let $f$ be a convex function on $I$, $\mathbf{p}$ a positive $n$-tuple, and $\mathbf{a}, \mathbf{b} \in I^n$ such that*

$$\sum_{i=1}^{k} p_i a_i \leq \sum_{i=1}^{k} p_i b_i \quad \text{for } k = 1, 2, \ldots, n-1, \qquad \sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i b_i. \tag{1.14}$$

*If $\mathbf{a}$ is a decreasing $n$-tuple, then one has*

$$\sum_{i=1}^{n} p_i f(a_i) \leq \sum_{i=1}^{n} p_i f(b_i), \tag{1.15}$$

*while if $\mathbf{b}$ is an increasing $n$-tuple, then we have*

$$\sum_{i=1}^{n} p_i f(b_i) \leq \sum_{i=1}^{n} p_i f(a_i). \tag{1.16}$$

*If $f$ is strictly convex and $\mathbf{a} \neq \mathbf{b}$, then* (1.15) *and* (1.16) *are strict.*

Note that for $n = 2$, inequality (1.15) holds if $a_2 \leq a_1 \leq b_1$ and if (1.12) is valid, while inequality (1.16) holds if $a_1 \leq b_1 \leq b_2$ and if (1.12) is valid.

## 2. Main Results

In what follows, $J(\mathbf{x}, \mathbf{p}, f)$ is as in (1.2), $P_j$ are as in (1.7), and $Q_k$, as in (1.9). Without any loss of generality, we assume that $P_n = 1$, since for positive $n$-tuples such that $P_n \neq 1$ results follow easily by substituting $p_i$ with $p_i / P_n$. Furthermore, for $1 \leq j < k \leq n$, we introduce the following notation:

$$\begin{aligned} J_{\min}(\mathbf{x}, \mathbf{p}, f) &= \min\{P_j, Q_k\}\left(f(x_j) + f(x_k) - 2f\left(\frac{x_j + x_k}{2}\right)\right), \\ J_{jk}(\mathbf{x}, \mathbf{p}, f) &= P_j f(x_j) + \sum_{i=j+1}^{k-1} p_i f(x_i) + Q_k f(x_k) - f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right). \end{aligned} \tag{2.1}$$

Note that $J_{1n}(\mathbf{x}, \mathbf{p}, f) = J(\mathbf{x}, \mathbf{p}, f)$.

**Theorem 2.1.** *Let $f$ be a convex function on $I$ and $\mathbf{p}$ a positive $n$-tuple such that $P_n = 1$, $n \geq 2$. Let $1 \leq j < k \leq n$ and $x_i \in I$, $i = 1, \ldots, k$. If $x_j$ is such that*

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq \frac{1}{Q_{j+1}}\left(\sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right), \tag{2.2}$$

$$\text{or} \quad \frac{1}{Q_{j+1}}\left(\sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) \leq x_j \leq \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i, \tag{2.3}$$

*then one has*

$$J_{1k}(\mathbf{x}, \mathbf{p}, f) \geq J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{2.4}$$

*Proof.* The claim is that

$$\sum_{i=1}^{j} p_i f(x_i) - f\left(\sum_{i=1}^{k-1} p_i x_i + Q_k x_k\right) \geq P_j f(x_j) - f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right). \tag{2.5}$$

As a simple consequence of the Jensen inequality (1.1), we have

$$\sum_{i=1}^{j} p_i f(x_i) \geq P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right). \tag{2.6}$$

Therefore, if we prove

$$P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right) + f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) \geq P_j f(x_j) + f\left(\sum_{i=1}^{k-1} p_i x_i + Q_k x_k\right), \tag{2.7}$$

the claim will follow. The idea is to apply Lemma 1.8 for $p_1 = P_j$, $p_2 = 1$, $a_1 = x_j$, $a_2 = \sum_{i=1}^{k-1} p_i x_i + Q_k x_k$, $b_1 = (1/P_j)\sum_{i=1}^{j} p_i x_i$, and $b_2 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$. Condition (1.12) is obviously satisfied. In addition, we need to check that

$$\begin{gathered} \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k, \\ \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq \sum_{i=1}^{k-1} p_i x_i + Q_k x_k \leq P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k. \end{gathered} \tag{2.8}$$

Easy calculation shows that both of these conditions are valid if (2.2) holds. Thus, the claim follows from Lemma 1.8. Note that we could have taken $p_1 = 1$, $p_2 = P_j$, $a_1 = \sum_{i=1}^{k-1} p_i x_i + Q_k x_k$,

$a_2 = x_j$, $b_1 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$, and $b_2 = (1/P_j)\sum_{i=1}^{j} p_i x_i$, instead. In this case, the necessary conditions would follow from (2.3). □

**Theorem 2.2.** *Let the conditions of Theorem 2.1 hold. If $x_j$ is such that*

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \le x_j \le \sum_{i=1}^{k-1} p_i x_i + Q_k x_k, \tag{2.9}$$

$$\text{or} \quad \sum_{i=1}^{k-1} p_i x_i + Q_k x_k \le x_j \le \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i, \tag{2.10}$$

*then inequality* (2.4) *holds.*

*Proof.* Proof is analogous to the proof of Theorem 2.1. Instead of Lemma 1.8, we apply Lemma 1.9 for $n = 2$ and the same choice of weights and points, or their obvious rearrangement. □

**Theorem 2.3.** *Let $f$ be a convex function on $I$ and $\mathbf{p}$ a positive $n$-tuple such that $P_n = 1$, $n \ge 2$. Let $1 \le j < k \le n$ and $x_i \in I$, $i = j, \ldots, n$. If $x_k$ is such that*

$$\frac{1}{P_{k-1}}\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i\right) \le x_k \le \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i, \tag{2.11}$$

$$\text{or} \quad \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i \le x_k \le \frac{1}{P_{k-1}}\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i\right), \tag{2.12}$$

*then one has*

$$J_{jn}(\mathbf{x}, \mathbf{p}, f) \ge J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{2.13}$$

*Proof.* Similarly as in the proof of Theorem 2.1, after first applying the Jensen inequality to the sum on the left-hand side, the claim will follow if we prove

$$\begin{aligned} Q_k f\left(\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i\right) + f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) \\ \ge Q_k(x_k) + f\left(P_j x_j + \sum_{i=j+1}^{n} p_i x_i\right). \end{aligned} \tag{2.14}$$

We can apply Lemma 1.8 for $p_1 = 1$, $p_2 = Q_k$, $a_1 = P_j x_j + \sum_{i=j+1}^{n} p_i x_i$, $a_2 = x_k$, $b_1 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$, and $b_2 = (1/Q_k)\sum_{i=k}^{n} p_i x_i$, since condition (1.12) is obviously satisfied and (2.11) ensures that the rest of the necessary conditions are fulfilled, and thus the claim is proved. After the obvious rearrangement, applying Lemma 1.8 with (2.12), the claim is recaptured. □

**Theorem 2.4.** *Let the conditions of Theorem 2.3 hold. If $x_k$ is such that*

$$P_j x_j + \sum_{i=j+1}^{n} p_i x_i \leq x_k \leq \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i, \tag{2.15}$$

$$\text{or} \quad \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i \leq x_k \leq P_j x_j + \sum_{i=j+1}^{n} p_i x_i, \tag{2.16}$$

*then inequality* (2.13) *holds.*

*Proof.* It is analogous to the proof of Theorem 2.3. Instead of Lemma 1.8, we apply Lemma 1.9 for $n = 2$ and the same choice of weights and points, or their obvious rearrangement. □

**Corollary 2.5.** *Let $f$ be a convex function on $I$ and $\mathbf{p}$ a positive $n$-tuple such that $P_n = 1$, $n \geq 2$. Let $\mathbf{x} \in I^n$ be a real $n$-tuple and $1 \leq j < k \leq n$.*

*If $x_k$ is such that*

$$\frac{1}{P_{k-1}} \sum_{i=1}^{k-1} p_i x_i \leq x_k \leq \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i, \tag{2.17}$$

$$\text{or} \quad \frac{1}{Q_k} \sum_{i=k}^{n} p_i x_i \leq x_k \leq \frac{1}{P_{k-1}} \sum_{i=1}^{k-1} p_i x_i, \tag{2.18}$$

*and $x_j$ is such that either* (2.2) *or* (2.3) *holds, then one has*

$$J(\mathbf{x}, \mathbf{p}, f) \geq J_{1k}(\mathbf{x}, \mathbf{p}, f) \geq J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{2.19}$$

*If $x_j$ is such that*

$$\frac{1}{P_j} \sum_{i=1}^{j} p_i x_i \leq x_j \leq \frac{1}{Q_{j+1}} \sum_{i=j+1}^{n} p_i x_i, \tag{2.20}$$

$$\text{or} \quad \frac{1}{Q_{j+1}} \sum_{i=j+1}^{n} p_i x_i \leq x_j \leq \frac{1}{P_j} \sum_{i=1}^{j} p_i x_i, \tag{2.21}$$

*and $x_k$ is such that either* (2.11) *or* (2.12) *holds, then one has*

$$J(\mathbf{x}, \mathbf{p}, f) \geq J_{jn}(\mathbf{x}, \mathbf{p}, f) \geq J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{2.22}$$

*Proof.* The first inequality in (2.19) follows from Theorem 2.3 for $j = 1$, and the second is a direct consequence of Theorem 2.1, while the first inequality in (2.22) follows from Theorem 2.1 for $k = n$, and the second is a consequence of Theorem 2.3. □

**Corollary 2.6.** *Let the conditions of Corollary 2.5 hold.*

*If $x_k$ is such that*

$$\sum_{i=1}^{n} p_i x_i \le x_k \le \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i, \tag{2.23}$$

$$\text{or} \quad \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i \le x_k \le \sum_{i=1}^{n} p_i x_i, \tag{2.24}$$

*and $x_j$ is such that either* (2.9) *or* (2.10) *holds, then inequality* (2.19) *holds.*

*If $x_j$ is such that*

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \le x_j \le \sum_{i=1}^{n} p_i x_i, \tag{2.25}$$

$$\text{or} \quad \sum_{i=1}^{n} p_i x_i \le x_j \le \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i, \tag{2.26}$$

*and $x_k$ is such that either* (2.15) *or* (2.16) *holds, then inequality* (2.22) *holds.*

*Proof.* The first inequality in (2.19) follows from Theorem 2.3 for $j = 1$, and the second is a direct consequence of Theorem 2.1, while the first inequality in (2.22) follows from Theorem 2.1 for $k = n$, and the second is a consequence of Theorem 2.3. □

**Theorem 2.7.** *Let $f$ be a convex function on $I$ and $\mathbf{p}$ a positive $n$-tuple such that $P_n = 1$, $n \ge 2$. Let $\mathbf{x} \in I^n$ be a real $n$-tuple, and let $1 \le j < k \le n$. If $x_j$ and $x_k$ are such that*

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \le x_j \le \sum_{i=1}^{n} p_i x_i \le x_k \le \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i, \tag{2.27}$$

*or*

$$\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i \le x_k \le \sum_{i=1}^{n} p_i x_i \le x_j \le \frac{1}{P_j}\sum_{i=1}^{j} p_i x_i, \tag{2.28}$$

*then one has*

$$J(\mathbf{x}, \mathbf{p}, f) \ge J_{jk}(\mathbf{x}, \mathbf{p}, f). \tag{2.29}$$

*Proof.* The claim is that

$$\begin{aligned} &\sum_{i=1}^{j} p_i f(x_i) + \sum_{i=k}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\quad \ge P_j f(x_j) + Q_k f(x_k) - f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right). \end{aligned} \tag{2.30}$$

After applying the Jensen inequality to the two sums on the left-hand side, we need to prove

$$\begin{aligned} &Q_k f\left(\frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i\right) + f\left(P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k\right) + P_j f\left(\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i\right) \\ &\quad \geq Q_k f(x_k) + f\left(\sum_{i=1}^{n} p_i x_i\right) + P_j f(x_j). \end{aligned} \tag{2.31}$$

Set $p_1 = Q_k$, $p_2 = 1$, $p_3 = P_j$, $a_1 = x_k, a_2 = \sum_{i=1}^{n} p_i x_i$, $a_3 = x_j, b_1 = (1/Q_k)\sum_{i=k}^{n} p_i x_i, b_2 = P_j x_j + \sum_{i=j+1}^{k-1} p_i x_i + Q_k x_k$, and $b_3 = (1/P_j)\sum_{i=1}^{j} p_i x_i$. Assumption (2.27) ensures that the necessary conditions of Lemma 1.9 for $n = 3$ are fulfilled, and so (2.31) follows from (1.15). By obvious rearrangement, utilizing (2.28), the inequality is recaptured. □

*Remark 2.8.* Note that conditions (2.9) and (2.23) combined together give a condition

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq \sum_{i=1}^{k-1} p_i x_i + Q_k x_k \leq \sum_{i=1}^{n} p_i x_i \leq x_k \leq \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i, \tag{2.32}$$

while (2.15) and (2.25) combined together give

$$\frac{1}{P_j}\sum_{i=1}^{j} p_i x_i \leq x_j \leq \sum_{i=1}^{n} p_i x_i \leq P_j x_j + \sum_{i=j+1}^{n} p_i x_i \leq x_k \leq \frac{1}{Q_k}\sum_{i=k}^{n} p_i x_i, \tag{2.33}$$

both of which are more restricting than (2.27). The same is true for combining conditions (2.10) and (2.24), or (2.16) and (2.26), and comparing the result with (2.28).

**Theorem 2.9.** *Let $f$ be a convex function on $I$ and $\mathbf{p}$ a positive $n$-tuple such that $P_n = 1$, $n \geq 2$. Let $1 \leq j < k \leq n$ and $x_i \in I$, $i = j, \ldots, k$, then one has*

$$\begin{aligned} J_{jk}(\mathbf{x}, \mathbf{p}, f) &\geq P_j f(x_j) + Q_k f(x_k) - (P_j + Q_k) f\left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right) \\ &\geq J_{\min}(\mathbf{x}, \mathbf{p}, f) \geq 0. \end{aligned} \tag{2.34}$$

*Proof.* The first inequality is an immediate consequence of the Jensen inequality. The other two follow immediately from (1.4). □

*Remark 2.10.* Inequalities (2.19), (2.22), and (2.34) recapture results from Lemmas 1.5 and 1.6, and Theorem 1.7 as special cases, since an increasing $n$-tuple $\mathbf{x}$ fulfils conditions (2.2) and (2.17), that is, (2.11) and (2.20). A decreasing $n$-tuple $\mathbf{x}$, on the other hand, fulfills conditions (2.3) and (2.18), that is, (2.12) and (2.21). The proofs of Theorem 2.9 and Corollary 2.5, that is, Theorems 2.1 and 2.3, are in fact analogous to the proofs of Theorem 1.7, Lemmas 1.5 and 1.6 from [9].

## 3. Some Special Cases

In this section, we consider some special cases of the presented results. The same special cases were considered in [9], but here we obtain them under more relaxed conditions on the $n$-tuple $\mathbf{x}$. More precisely, Corollaries 2.5 and 2.6, or Theorem 2.7, after applying Theorem 2.9, yield

$$J(\mathbf{x},\mathbf{p},f) \geq P_j f(x_j) + Q_k f(x_k) - (P_j + Q_k) f\left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right). \tag{3.1}$$

**Corollary 3.1.** *Let the conditions of Corollaries 2.5 and 2.6, or Theorem 2.7 hold, then*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq P_j a_j + Q_k a_k - (P_j + Q_k) a_j^{P_j/(P_j+Q_k)} a_k^{Q_k/(P_j+Q_k)}. \tag{3.2}$$

*Proof.* This follows from (3.1) for $f(x) = e^x$, using notation $a_i = e^{x_i}$. □

**Corollary 3.2.** *Let the conditions of Corollaries 2.5 and 2.6, or Theorem 2.7 hold, and let in addition $x_i > 0$, $i = 1, \ldots, n$, then*

$$\frac{\sum_{i=1}^{n} p_i x_i}{\prod_{i=1}^{n} x_i^{p_i}} \geq \frac{1}{x_j^{P_j} x_k^{Q_k}} \left(\frac{P_j x_j + Q_k x_k}{P_j + Q_k}\right)^{P_j+Q_k}. \tag{3.3}$$

*Proof.* Follows from (3.1) for $f(x) = -\ln x$. □

**Corollary 3.3.** *Let the conditions of Corollaries 2.5 and 2.6, or Theorem 2.7 hold, and let in addition $x_i > 0$, $i = 1, \ldots, n$, then*

$$\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{\sum_{i=1}^{n} p_i x_i} \geq \frac{P_j Q_k (x_k - x_j)^2}{x_j x_k (P_j x_j + Q_k x_k)}. \tag{3.4}$$

*Proof.* This follows from (3.1) for $f(x) = 1/x$. □

In [9], additional bounds of $J(\mathbf{x},\mathbf{p},f)$, lower than those obtained in the previous corollaries, were derived for the case $f(x) = e^x$ and $f(x) = 1/x$. Now, note that from Theorem 2.9, under conditions of Corollaries 2.5 and 2.6, or Theorem 2.7, we have

$$J(\mathbf{x},\mathbf{p},f) \geq J_{\min}(\mathbf{x},\mathbf{p},f) \geq 0. \tag{3.5}$$

Next, we compare estimates obtained from (3.5) with those obtained in [9].

*Case 1.* For $\mathrm{f(x) = e^x}$, using notation $\mathrm{a_i = e^{x_i}}$, inequality (3.5) takes the form

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq \min\{P_j, Q_k\} \left(\sqrt{a_k} - \sqrt{a_j}\right)^2. \tag{3.6}$$

In [9], under the assumption that $\mathbf{a}$ is an increasing $n$-tuple, the following inequality was obtained

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq C\left(\sqrt{a_k} - \sqrt{a_j}\right)^2, \tag{3.7}$$

where

$$C = \begin{cases} \dfrac{2P_j Q_k}{P_j + Q_k}, & P_j \leq Q_k, \\ Q_k, & P_j \geq Q_k. \end{cases} \tag{3.8}$$

Note that when $P_j \geq Q_k$, (3.6) recaptures this result. However, when $P_j \leq Q_k$, the constant $C$ is better, since $2P_j Q_k/(P_j + Q_k) \geq P_j$.

*Case 2.* For $f(x) = 1/x$ and $x_i > 0$, $i = 1, \ldots, n$, inequality (3.5) takes the form

$$\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{\sum_{i=1}^{n} p_i x_i} \geq \min\{P_j, Q_k\} \frac{(x_k - x_j)^2}{x_j x_k (x_j + x_k)}. \tag{3.9}$$

In [9], under the assumption that $\mathbf{x}$ is an increasing $n$-tuple such that $x_1 > 0$, the following inequality was obtained:

$$\sum_{i=1}^{n} \frac{p_i}{x_i} - \frac{1}{\sum_{i=1}^{n} p_i x_i} \geq C \frac{\left(\sqrt{x_k} - \sqrt{x_j}\right)^2}{x_j x_k}, \tag{3.10}$$

where

$$C = \begin{cases} P_j, & P_j \leq 3Q_k, \\ \dfrac{4P_j Q_k}{P_j + Q_k}, & P_j \geq 3Q_k. \end{cases} \tag{3.11}$$

In order to compare these two estimates, first assume that $P_j \leq Q_k$. Since

$$P_j \frac{(x_k - x_j)^2}{x_j x_k (x_j + x_k)} \geq P_j \frac{\left(\sqrt{x_k} - \sqrt{x_j}\right)^2}{x_j x_k} \Longleftrightarrow \left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \geq x_j + x_k, \tag{3.12}$$

it follows that the estimate in (3.9) is better than the one in (3.10).

Next, assume that $Q_k \leq P_j \leq 2Q_k$. First, observe that

$$Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \geq P_j(x_k + x_j) \Longleftrightarrow P_j \leq Q_k \frac{\left(\sqrt{x_k} + \sqrt{x_j}\right)^2}{x_k + x_j}. \tag{3.13}$$

Simple calculation reveals that

$$1 \leq \frac{\left(\sqrt{x_k} + \sqrt{x_j}\right)^2}{x_k + x_j} \leq 2, \tag{3.14}$$

and so we conclude that the estimate in (3.9) is better than the one in (3.10) when $Q_k \leq P_j \leq Q_k((\sqrt{x_k} + \sqrt{x_j})^2/(x_k+x_j))$, while when $Q_k((\sqrt{x_k} + \sqrt{x_j})^2/(x_k+x_j)) \leq P_j \leq 2Q_k$, the estimate in (3.10) is better than the one in (3.9).

Further, assume that $2Q_k \leq P_j \leq 3Q_k$. In this case, the estimate in (3.10) is better than the one in (3.9), that is,

$$P_j(x_k + x_j) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2. \tag{3.15}$$

Namely,

$$\begin{gathered} P_j(x_k + x_j) \geq 2Q_k(x_k + x_j), \\ 2Q_k(x_k + x_j) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \Longleftrightarrow \left(\sqrt{x_k} - \sqrt{x_j}\right)^2 \geq 0. \end{gathered} \tag{3.16}$$

Finally, if $3Q_k \leq P_j$, the estimate in (3.10) is again better than the one in (3.9), that is,

$$\frac{4P_jQ_k}{P_j + Q_k}(x_j + x_k) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2. \tag{3.17}$$

This is equivalent to

$$P_j\left(3x_j + 3x_k - 2\sqrt{x_jx_k}\right) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2. \tag{3.18}$$

In this case, we have

$$P_j\left(3x_j + 3x_k - 2\sqrt{x_jx_k}\right) \geq Q_k\left(3x_j + 3x_k - 2\sqrt{x_jx_k}\right), \tag{3.19}$$

and since

$$Q_k\left(3x_j + 3x_k - 2\sqrt{x_jx_k}\right) \geq Q_k\left(\sqrt{x_k} + \sqrt{x_j}\right)^2 \Longleftrightarrow \left(\sqrt{x_k} - \sqrt{x_j}\right)^2 \geq 0, \tag{3.20}$$

the claim follows.

## Acknowledgments

This research work was partially funded by the Higher Education Commission, Pakistan. The research of the authors was supported by the Croatian Ministry of Science, Education and

Sports, under the Research Grants nos. 058-1170889-1050 (first author) and 117-1170889-0888 (third author).

## References

[1] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[2] J. L. W. V. Jensen, "Sur les fonctions convexes et les inégalités entre les valeurs moyennes," *Acta Mathematica*, vol. 30, no. 1, pp. 175–193, 1906.

[3] S. S. Dragomir, J. Pečarić, and L. E. Persson, "Properties of some functionals related to Jensen's inequality," *Acta Mathematica Hungarica*, vol. 70, no. 1-2, pp. 129–143, 1996.

[4] S. S. Dragomir, "Bounds for the normalised Jensen functional," *Bulletin of the Australian Mathematical Society*, vol. 74, no. 3, pp. 471–478, 2006.

[5] J. Barić, M. Matić, and J. E. Pečarić, "On the bounds for the normalized Jensen functional and Jensen-Steffensen inequality," *Mathematical Inequalities & Applications*, vol. 12, no. 2, pp. 413–432, 2009.

[6] S. Abramovich, S. Ivelić, and J. E. Pečarić, "Improvement of Jensen-Steffensen's inequality for superquadratic functions," *Banach Journal of Mathematical Analysis*, vol. 4, no. 1, pp. 159–169, 2010.

[7] S. Ivelić, A. Matković, and J. E. Pečarić, "On a Jensen-Mercer operator inequality," *Banach Journal of Mathematical Analysis*, vol. 5, no. 1, pp. 19–28, 2011.

[8] M. Khosravi, J. S. Aujla, S. S. Dragomir, and M. S. Moslehian, "Refinements of Choi-Davis-Jensen's inequality," *Bulletin of Mathematical Analysis and Applications*, vol. 3, no. 2, pp. 127–133, 2011.

[9] V. Cirtoaje, "The best lower bound depended on two fixed variables for Jensen's inequality with ordered variables," *Journal of Inequalities and Applications*, vol. 2010, Article ID 128258, 12 pages, 2010.

[10] N. Latif, J. Pečarić, and I. Perić, "On majorization of vectors,Favard and Berwald inequalities," In press.