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Research Article

Oscillation of Second-Order Neutral Functional Differential Equations with Mixed Nonlinearities

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We study the following second-order neutral functional differential equation with mixed nonlinearities $(r(t)|(u(t)+p(t)u(t-\sigma))'|^{\alpha-1}(u(t)+p(t)u(t-\sigma))')' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) + q_1(t)|u(\tau_1(t))|^{\beta-1}u(\tau_1(t)) + q_2(t)|u(\tau_2(t))|^{\gamma-1}u(\tau_2(t)) = 0$, where $\gamma > \alpha > \beta > 0$, $\int_{t_0}^{\infty}(1/r^{1/\alpha}(t))\mathrm{d}t < \infty$. Oscillation results for the equation are established which improve the results obtained by Sun and Meng (2006), Xu and Meng (2006), Sun and Meng (2009), and Han et al. (2010).

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order neutral functional differential equation with mixed nonlinearities

$$\begin{aligned}\left(r(t)\left|(u(t)+p(t)u(t-\sigma))'\right|^{\alpha-1}(u(t)+p(t)u(t-\sigma))'\right)' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) \\ +q_1(t)|u(\tau_1(t))|^{\beta-1}u(\tau_1(t)) + q_2(t)|u(\tau_2(t))|^{\gamma-1}u(\tau_2(t)) = 0, \quad t \geq t_0,\end{aligned} \tag{1.1}$$

where $\gamma > \alpha > \beta > 0$ are constants, $r \in C^1([t_0,\infty),(0,\infty))$, $p \in C([t_0,\infty),[0,1))$, $q_i \in C([t_0,\infty),\mathbb{R})$, $i = 0,1,2$, are nonnegative, $\sigma \geq 0$ is a constant. Here, we assume that there exists $\tau \in C^1([t_0,\infty),\mathbb{R})$ such that $\tau(t) \leq \tau_i(t)$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, and $\tau'(t) > 0$ for $t \geq t_0$.

One of our motivations for studying (1.1) is the application of this type of equations in real word life problems. For instance, neutral delay equations appear in modeling of networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar; see the Euler equation in some variational problems, in the theory of automatic control and in neuromechanical systems in which inertia plays an important role. We refer the reader to Hale [1] and Driver [2], and references cited therein.

Recently, there has been much research activity concerning the oscillation of second-order differential equations [3–8] and neutral delay differential equations [9–20]. For the particular case when $p(t) = 0$, (1.1) reduces to the following equation:

$$\begin{aligned}\left(r(t)|u(t)|^{\alpha-1}u(t)\right)' &+ q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) \\ &+ q_1(t)|u(\tau_1(t))|^{\beta-1}u(\tau_1(t)) + q_2(t)|u(\tau_2(t))|^{\gamma-1}u(\tau_2(t)) = 0, \quad t \geq t_0.\end{aligned} \tag{1.2}$$

Sun and Meng [6] established some oscillation criteria for (1.2), under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} \mathrm{d}t < \infty, \tag{1.3}$$

they only obtained the sufficient condition [6, Theorem 5], which guarantees that every solution u of (1.2) oscillates or tends to zero.

Sun and Meng [7] considered the oscillation of second-order nonlinear delay differential equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) = 0, \quad t \geq t_0 \tag{1.4}$$

and obtained some results for oscillation of (1.4), for example, under the case (1.3), they obtained some results which guarantee that every solution u of (1.4) oscillates or tends to zero, see [7, Theorem 2.2].

Xu and Meng [10] discussed the oscillation of the second-order neutral delay differential equation

$$\left(r(t)\left|(u(t) + p(t)u(t-\tau))'\right|^{\alpha-1}(u(t) + p(t)u(t-\tau))'\right)' + q(t)f(u(\sigma(t))) = 0, \quad t \geq t_0 \tag{1.5}$$

and established the sufficient condition [10, Theorem 2.3], which guarantees that every solution u of (1.5) oscillates or tends to zero.

Han et al. [11] examined the oscillation of second-order neutral delay differential equation

$$\left(r(t)\psi(u(t))\left|(u(t) + p(t)u(t-\tau))'\right|^{\alpha-1}(u(t) + p(t)u(t-\tau))'\right)' + q(t)f(u(\sigma(t))) = 0, \quad t \geq t_0 \tag{1.6}$$

and established some sufficient conditions for oscillation of (1.6) under the conditions (1.3) and

$$\sigma(t) \le t - \tau. \tag{1.7}$$

The condition (1.7) can be restrictive condition, since the results cannot be applied on the equation

$$\left(e^{2t}\left(u(t) + \frac{1}{2}u(t-2)\right)'\right)' + \lambda\left(e^{2t} + \frac{1}{2}e^{2t+2}\right)u(t-1) = 0, \quad t \ge t_0. \tag{1.8}$$

The aim of this paper is to derive some sufficient conditions for the oscillation of solutions of (1.1). The paper is organized as follows. In Section 2, we establish some oscillation criteria for (1.1) under the assumption (1.3). In Section 3, we will give three examples to illustrate the main results. In Section 4, we give some conclusions for this paper.

2. Main Results

In this section, we give some new oscillation criteria for (1.1).

Below, for the sake of convenience, we denote

$$z(t) := u(t) + p(t)u(t-\sigma), \qquad R(t) := \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s,$$

$$\xi(t) := r^{1/\alpha}(\tau(t)) \int_{t_1}^{t} \left(\frac{1}{r(\tau(s))}\right)^{1/\alpha} \tau'(s) \mathrm{d}s,$$

$$Q_0(t) := (1 - p(\tau_0(t)))^{\alpha} q_0(t), \qquad Q_1(t) := (1 - p(\tau_1(t)))^{\beta} q_1(t),$$

$$Q_2(t) := (1 - p(\tau_2(t)))^{\gamma} q_2(t),$$

$$\zeta_0(t) := q_0(t)\left(\frac{1}{1 + p(\rho(t))}\right)^{\alpha}, \qquad \zeta_1(t) := q_1(t)\left(\frac{1}{1 + p(\rho(t))}\right)^{\beta},$$

$$\zeta_2(t) := q_2(t)\left(\frac{1}{1 + p(\rho(t))}\right)^{\gamma},$$

$$h_0(t) := q_0(t)\left(\frac{1}{1 + p(t)}\right)^{\alpha}, \qquad h_1(t) := q_1(t)\left(\frac{1}{1 + p(t)}\right)^{\beta},$$

$$h_2(t) := q_2(t)\left(\frac{1}{1 + p(t)}\right)^{\gamma},$$

$$\delta(t) := \int_{\rho(t)}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s, \qquad \pi(t) := \int_{t}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s, \qquad k_1 := \frac{\gamma-\beta}{\gamma-\alpha}, \qquad k_2 := \frac{\gamma-\beta}{\alpha-\beta},$$
$$\varphi(t) := q_0(t)\left(\frac{\delta(t)}{1+p(\rho(t))}\right)^{\alpha} + q_1(t)\left(\frac{\delta(t)}{1+p(\rho(t))}\right)^{\beta} + q_2(t)\left(\frac{\delta(t)}{1+p(\rho(t))}\right)^{\gamma}. \tag{2.1}$$

Theorem 2.1. *Assume that* (1.3) *holds,* $p'(t) \geq 0$*, and there exists* $\rho \in C^1([t_0,\infty),\mathbb{R})$*, such that* $\rho(t) \geq t$*,* $\rho'(t) > 0$*,* $\tau_i(t) \leq \rho(t) - \sigma, i = 0,1,2$*. If for all sufficiently large* t_1*,*

$$\int^{\infty} \left\{ R^{\alpha}(\tau(t))\left[Q_0(t) + [k_1 Q_1(t)]^{1/k_1}[k_2 Q_2(t)]^{1/k_2}\right] - \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)} \right\} \mathrm{d}t = \infty, \tag{2.2}$$

$$\int^{\infty} \left\{ \left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha}(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\rho'(t)}{\delta(t) r^{1/\alpha}(\rho(t))} \right\} \mathrm{d}t = \infty, \tag{2.3}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that u is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(t) > 0$ for all large t. The case of $u(t) < 0$ can be considered by the same method. From (1.1) and (1.3), we can easily obtain that there exists a $t_1 \geq t_0$ such that

$$z(t) > 0, \quad z'(t) > 0, \qquad \left[r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right]' \leq 0, \tag{2.4}$$

or

$$z(t) > 0, \quad z'(t) < 0, \qquad \left[r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right]' \leq 0. \tag{2.5}$$

If (2.4) holds, we have

$$r(t)\left(z'(t)\right)^{\alpha} \leq r(\tau(t))\left(z'(\tau(t))\right)^{\alpha}, \qquad t \geq t_1. \tag{2.6}$$

From the definition of z, we obtain

$$u(t) = z(t) - p(t)u(t-\sigma) \geq z(t) - p(t)z(t-\sigma) \geq \left(1-p(t)\right)z(t). \tag{2.7}$$

Define

$$\omega(t) = R^{\alpha}(\tau(t))\frac{r(t)(z'(t))^{\alpha}}{(z(\tau(t)))^{\alpha}}, \qquad t \geq t_1. \tag{2.8}$$

Then, $\omega(t) > 0$ for $t \geq t_1$. Noting that $z'(t) > 0$, we get $z(\tau_i(t)) \geq z(\tau(t))$ for $i = 0, 1, 2$. Thus, from (1.1), (2.7), and (2.8), it follows that

$$\begin{aligned}\omega'(t) \leq{}& \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))}{r^{1/\alpha}(\tau(t))}\frac{r(t)(z'(t))^{\alpha}}{(z(\tau(t)))^{\alpha}} - R^{\alpha}(\tau(t))(1-p(\tau_0(t)))^{\alpha}q_0(t) \\ &- R^{\alpha}(\tau(t))\left[(1-p(\tau_1(t)))^{\beta}q_1(t)z^{\beta-\alpha}(\tau(t)) + (1-p(\tau_2(t)))^{\gamma}q_2(t)z^{\gamma-\alpha}(\tau(t))\right] \\ &- \alpha R^{\alpha}(\tau(t))\frac{r(t)(z'(t))^{\alpha}}{(z(\tau(t)))^{\alpha+1}}z'(\tau(t))\tau'(t).\end{aligned} \tag{2.9}$$

By (2.4), (2.9), and $\tau'(t) > 0$, we get

$$\begin{aligned}\omega'(t) \leq{}& \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))}{r^{1/\alpha}(\tau(t))}\frac{r(t)(z'(t))^{\alpha}}{(z(\tau(t)))^{\alpha}} - R^{\alpha}(\tau(t))(1-p(\tau_0(t)))^{\alpha}q_0(t) \\ &- R^{\alpha}(\tau(t))\left[(1-p(\tau_1(t)))^{\beta}q_1(t)z^{\beta-\alpha}(\tau(t)) + (1-p(\tau_2(t)))^{\gamma}q_2(t)z^{\gamma-\alpha}(\tau(t))\right].\end{aligned} \tag{2.10}$$

In view of (2.4), (2.6), and (2.10), we have

$$\begin{aligned}\omega'(t) \leq{}& \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))}{r^{1/\alpha}(\tau(t))}\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(\tau(t)))^{\alpha}} - R^{\alpha}(\tau(t))(1-p(\tau_0(t)))^{\alpha}q_0(t) \\ &- R^{\alpha}(\tau(t))\left[(1-p(\tau_1(t)))^{\beta}q_1(t)z^{\beta-\alpha}(\tau(t)) + (1-p(\tau_2(t)))^{\gamma}q_2(t)z^{\gamma-\alpha}(\tau(t))\right].\end{aligned} \tag{2.11}$$

By (2.4), we obtain

$$\begin{aligned}z(\tau(t)) &= z(\tau(t_1)) + \int_{t_1}^{t} z'(\tau(s))\tau'(s)\mathrm{d}s \\ &= z(\tau(t_1)) + \int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\left[r(\tau(s))(z'(\tau(s)))^{\alpha}\right]^{1/\alpha}\tau'(s)\mathrm{d}s \\ &\geq r^{1/\alpha}(\tau(t))z'(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)\mathrm{d}s,\end{aligned} \tag{2.12}$$

that is,

$$z(\tau(t)) \geq \xi(t)z'(\tau(t)). \tag{2.13}$$

Set

$$a := \left[k_1Q_1(t)z^{\beta-\alpha}(\tau(t))\right]^{1/k_1}, \quad b := \left[k_2Q_2(t)z^{\gamma-\alpha}(\tau(t))\right]^{1/k_2}, \quad p := k_1, \quad q := k_2. \tag{2.14}$$

Using Young's inequality

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad a, b \in \mathbb{R},\ p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{2.15}$$

we have

$$Q_1(t) z^{\beta-\alpha}(\tau(t)) + Q_2(t) z^{\gamma-\alpha}(\tau(t)) \ge [k_1 Q_1(t)]^{1/k_1} [k_2 Q_2(t)]^{1/k_2}. \tag{2.16}$$

Hence, by (2.11), (2.13), and (2.16), we obtain

$$\omega'(t) \le \frac{\alpha \tau'(t) R^{\alpha-1}(\tau(t)) r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)} - R^{\alpha}(\tau(t)) \Big[Q_0(t) + [k_1 Q_1(t)]^{1/k_1} [k_2 Q_2(t)]^{1/k_2} \Big]. \tag{2.17}$$

Integrating (2.17) from t_1 to t, we get

$$0 < \omega(t) \le \omega(t_1), \tag{2.18}$$

$$-\int_{t_1}^{t} \left\{ R^{\alpha}(\tau(s)) \Big[Q_0(s) + [k_1 Q_1(s)]^{1/k_1} [k_2 Q_2(s)]^{1/k_2} \Big] - \frac{\alpha \tau'(s) R^{\alpha-1}(\tau(s)) r^{1-1/\alpha}(\tau(s))}{\xi^{\alpha}(s)} \right\} ds. \tag{2.19}$$

Letting $t \to \infty$ in (2.19), we get a contradiction to (2.2). If (2.5) holds, we define the function υ by

$$\upsilon(t) = \frac{r(t)(-z'(t))^{\alpha-1} z'(t)}{z^{\alpha}(\rho(t))}, \quad t \ge t_1. \tag{2.20}$$

Then, $\upsilon(t) < 0$ for $t \ge t_1$. It follows from $[r(t)|z'(t)|^{\alpha-1} z'(t)]' \le 0$ that $r(t)|z'(t)|^{\alpha-1} z'(t)$ is nonincreasing. Thus, we have

$$r^{1/\alpha}(s) z'(s) \le r^{1/\alpha}(t) z'(t), \quad s \ge t. \tag{2.21}$$

Dividing (2.21) by $r^{1/\alpha}(s)$ and integrating it from $\rho(t)$ to l, we obtain

$$z(l) \le z(\rho(t)) + r^{1/\alpha}(t) z'(t) \int_{\rho(t)}^{l} \frac{ds}{r^{1/\alpha}(s)}, \quad l \ge \rho(t). \tag{2.22}$$

Letting $l \to \infty$ in the above inequality, we obtain

$$0 \le z(\rho(t)) + r^{1/\alpha}(t) z'(t) \delta(t), \quad t \ge t_1, \tag{2.23}$$

that is,

$$r^{1/\alpha}(t)\delta(t)\frac{z'(t)}{z(\rho(t))} \geq -1, \quad t \geq t_1. \tag{2.24}$$

Hence, by (2.20), we have

$$-1 \leq v(t)\delta^{\alpha}(t) \leq 0, \quad t \geq t_1. \tag{2.25}$$

Differentiating (2.20), we get

$$v'(t) = \frac{\left(r(t)(-z'(t))^{\alpha-1}z'(t)\right)' z^{\alpha}(\rho(t)) - \alpha r(t)(-z'(t))^{\alpha-1}z'(t)z^{\alpha-1}(\rho(t))z'(\rho(t))\rho'(t)}{z^{2\alpha}(\rho(t))}, \tag{2.26}$$

by the above equality and (1.1), we obtain

$$\begin{aligned} v'(t) = &-q_0(t)\frac{u^{\alpha}(\tau_0(t))}{z^{\alpha}(\rho(t))} - q_1(t)\frac{u^{\beta}(\tau_1(t))}{z^{\alpha}(\rho(t))} - q_2(t)\frac{u^{\gamma}(\tau_2(t))}{z^{\alpha}(\rho(t))} \\ &-\frac{\alpha r(t)(-z'(t))^{\alpha-1}z'(t)z^{\alpha-1}(\rho(t))z'(\rho(t))\rho'(t)}{z^{2\alpha}(\rho(t))}. \end{aligned} \tag{2.27}$$

Noticing that $p'(t) \geq 0$, from [10, Theorem 2.3], we see that $u'(t) \leq 0$ for $t \geq t_1$, so by $\tau_i(t) \leq \rho(t) - \sigma$, $i = 0, 1, 2$, we have

$$\begin{aligned} \frac{u^{\alpha}(\tau_0(t))}{z^{\alpha}(\rho(t))} &= \left(\frac{u(\tau_0(t))}{u(\rho(t)) + p(\rho(t))u(\rho(t)-\sigma)}\right)^{\alpha} \\ &= \left(\frac{1}{(u(\rho(t))/u(\tau_0(t))) + p(\rho(t))(u(\rho(t)-\sigma)/u(\tau_0(t)))}\right)^{\alpha} \\ &\geq \left(\frac{1}{1+p(\rho(t))}\right)^{\alpha}, \\ \frac{u^{\beta}(\tau_1(t))}{z^{\alpha}(\rho(t))} &= \left(\frac{u(\tau_1(t))}{u(\rho(t)) + p(\rho(t))u(\rho(t)-\sigma)}\right)^{\beta} z^{\beta-\alpha}(\rho(t)) \\ &= \left(\frac{1}{(u(\rho(t))/u(\tau_1(t))) + p(\rho(t))(u(\rho(t)-\sigma)/u(\tau_1(t)))}\right)^{\beta} z^{\beta-\alpha}(\rho(t)) \\ &\geq \left(\frac{1}{1+p(\rho(t))}\right)^{\beta} z^{\beta-\alpha}(\rho(t)), \end{aligned}$$

$$\begin{aligned}\left(u^{\gamma}(\tau_2(t))/z^{\alpha}(\rho(t))\right) &= \left(\frac{u(\tau_2(t))}{u(\rho(t))+p(\rho(t))u(\rho(t)-\sigma)}\right)^{\gamma} z^{\gamma-\alpha}(\rho(t))\\ &= \left(\frac{1}{(u(\rho(t))/u(\tau_2(t)))+p(\rho(t))(u(\rho(t)-\sigma)/u(\tau_2(t)))}\right)^{\gamma} z^{\gamma-\alpha}(\rho(t))\\ &\geq \left(\frac{1}{1+p(\rho(t))}\right)^{\gamma} z^{\gamma-\alpha}(\rho(t)).\end{aligned} \tag{2.28}$$

On the other hand, from $(r(t)(-z'(t))^{\alpha-1}z'(t))' \leq 0$, $\rho(t) \geq t$, we obtain

$$z'(\rho(t)) \leq \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(\rho(t))} z'(t). \tag{2.29}$$

Thus, by (2.20) and (2.27), we get

$$v'(t) \leq -\left[\zeta_0(t)+\zeta_1(t)z^{\beta-\alpha}(\rho(t))+\zeta_2(t)z^{\gamma-\alpha}(\rho(t))\right] - \frac{\alpha\rho'(t)}{r^{1/\alpha}(\rho(t))}(-v(t))^{(\alpha+1)/\alpha}. \tag{2.30}$$

Set

$$a := \left[k_1\zeta_1(t)z^{\beta-\alpha}(\rho(t))\right]^{1/k_1}, \quad b := \left[k_2\zeta_2(t)z^{\gamma-\alpha}(\rho(t))\right]^{1/k_2},\ p := k_1,\ q := k_2. \tag{2.31}$$

Using Young's inequality (2.15), we obtain

$$\zeta_1(t)z^{\beta-\alpha}(\rho(t))+\zeta_2(t)z^{\gamma-\alpha}(\rho(t)) \geq [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}. \tag{2.32}$$

Hence, from (2.30), we have

$$v'(t) \leq -\left[\zeta_0(t)+[k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right] - \frac{\alpha\rho'(t)}{r^{1/\alpha}(\rho(t))}(-v(t))^{(\alpha+1)/\alpha}, \tag{2.33}$$

that is,

$$v'(t) + \left[\zeta_0(t)+[k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right] + \frac{\alpha\rho'(t)}{r^{1/\alpha}(\rho(t))}(-v(t))^{(\alpha+1)/\alpha} \leq 0, \quad t \geq t_1. \tag{2.34}$$

Multiplying (2.34) by $\delta^{\alpha}(t)$ and integrating it from t_1 to t implies that

$$\begin{aligned}
&\delta^{\alpha}(t)v(t)-\delta^{\alpha}(t_1)v(t_1)+\alpha\int_{t_1}^{t} r^{-1/\alpha}(\rho(s))\rho'(s)\delta^{\alpha-1}(s)v(s)\mathrm{d}s\\
&\quad+\int_{t_1}^{t}\left[\zeta_0(s)+[k_1\zeta_1(s)]^{1/k_1}[k_2\zeta_2(s)]^{1/k_2}\right]\delta^{\alpha}(s)\mathrm{d}s\\
&\quad+\alpha\int_{t_1}^{t}\frac{\delta^{\alpha}(s)\rho'(s)}{r^{1/\alpha}(\rho(s))}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s\le 0.
\end{aligned}\tag{2.35}$$

Set $p:=(\alpha+1)/\alpha$, $q:=\alpha+1$, and

$$a:=(\alpha+1)^{\alpha/(\alpha+1)}\delta^{\alpha^2/(\alpha+1)}(t)v(t),\qquad b:=\frac{\alpha}{(\alpha+1)^{\alpha/(\alpha+1)}}\delta^{-1/(\alpha+1)}(t).\tag{2.36}$$

Using Young's inequality (2.15), we get

$$-\alpha\delta^{\alpha-1}(t)v(t)\le\alpha\delta^{\alpha}(t)(-v(t))^{(\alpha+1)/\alpha}+\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(t)}.\tag{2.37}$$

Thus,

$$-\frac{\alpha\rho'(t)\delta^{\alpha-1}(t)v(t)}{r^{1/\alpha}(\rho(t))}\le\alpha\rho'(t)\frac{\delta^{\alpha}(t)(-v(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\rho(t))}+\rho'(t)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(t)r^{1/\alpha}(\rho(t))}.\tag{2.38}$$

Therefore, (2.35) yields

$$\begin{aligned}
&\delta^{\alpha}(t)v(t)\le\delta^{\alpha}(t_1)v(t_1),\\
&-\int_{t_1}^{t}\left\{\left[\zeta_0(s)+[k_1\zeta_1(s)]^{1/k_1}[k_2\zeta_2(s)]^{1/k_2}\right]\delta^{\alpha}(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\rho'(s)}{\delta(s)r^{1/\alpha}(\rho(s))}\right\}\mathrm{d}s.
\end{aligned}\tag{2.39}$$

Letting $t\to\infty$ in the above inequality, by (2.3), we get a contradiction with (2.25). This completes the proof of Theorem 2.1. □

From Theorem 2.1, when $\rho(t)=t$, we have the following result.

Corollary 2.2. *Assume that* (1.3) *holds,* $p'(t)\ge 0$*, and* $\tau_i(t)\le t-\sigma$, $i=0,1,2$. *If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int^{\infty}\left\{\left[h_0(t)+[k_1h_1(t)]^{1/k_1}[k_2h_2(t)]^{1/k_2}\right]\pi^{\alpha}(t)\;-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(t)r^{1/\alpha}(t)}\right\}\mathrm{d}t=\infty,\tag{2.40}$$

then (1.1) *is oscillatory.*

Theorem 2.3. *Assume that* (1.3) *holds,* $p'(t) \geq 0$*, and there exists* $\rho \in C^1([t_0, \infty), \mathbb{R})$*, such that* $\rho(t) \geq t$*,* $\rho'(t) > 0$*,* $\tau_i(t) \leq \rho(t) - \sigma$*,* $i = 0, 1, 2$*. If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int^{\infty} \left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha+1}(t)\mathrm{d}t = \infty, \tag{2.41}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that u is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(t) > 0$ for all large t. The case of $u(t) < 0$ can be considered by the same method. From (1.1) and (1.3), we can easily obtain that there exists a $t_1 \geq t_0$ such that (2.4) or (2.5) holds.

If (2.4) holds, proceeding as in the proof of Theorem 2.1, we obtain a contradiction with (2.2).

If (2.5) holds, we proceed as in the proof of Theorem 2.1, then we get (2.25) and (2.34). Multiplying (2.34) by $\delta^{\alpha+1}(t)$ and integrating it from t_1 to t implies that

$$\begin{aligned}
&\delta^{\alpha+1}(t)v(t) - \delta^{\alpha+1}(t_1)v(t_1) + (\alpha+1)\int_{t_1}^{t} r^{-1/\alpha}(\rho(s))\rho'(s)\delta^{\alpha}(s)v(s)\mathrm{d}s \\
&\quad + \int_{t_1}^{t}\left[\zeta_0(s) + [k_1\zeta_1(s)]^{1/k_1}[k_2\zeta_2(s)]^{1/k_2}\right]\delta^{\alpha+1}(s)\mathrm{d}s \\
&\quad + \alpha\int_{t_1}^{t}\frac{\delta^{\alpha+1}(s)\rho'(s)}{r^{1/\alpha}(\rho(s))}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s \leq 0.
\end{aligned} \tag{2.42}$$

In view of (2.25), we have $-v(t)\delta^{\alpha+1}(t) \leq \delta(t) < \infty$, $t \to \infty$. From (1.3), we get

$$\begin{aligned}
&\int_{t_1}^{t} -r^{-1/\alpha}(\rho(s))\rho'(s)\delta^{\alpha}(s)v(s)\mathrm{d}s \leq \int_{t_1}^{t} r^{-1/\alpha}(\rho(s))\rho'(s)\mathrm{d}s = \int_{\rho(t_1)}^{\rho(t)} r^{-1/\alpha}(u)\mathrm{d}u < \infty, \quad t \longrightarrow \infty, \\
&\int_{t_1}^{t}\frac{\delta^{\alpha+1}(s)\rho'(s)}{r^{1/\alpha}(\rho(s))}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s \leq \int_{\rho(t_1)}^{\rho(t)} r^{-1/\alpha}(u)\mathrm{d}u < \infty, \quad t \longrightarrow \infty.
\end{aligned} \tag{2.43}$$

Letting $t \to \infty$ in (2.42) and using the last inequalities, we obtain

$$\int^{\infty} \left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha+1}(t)\mathrm{d}t < \infty, \tag{2.44}$$

which contradicts (2.41). This completes the proof of Theorem 2.3. ☐

From Theorem 2.3, when $\rho(t) = t$, we have the following result.

Corollary 2.4. *Assume that* (1.3) *holds,* $p'(t) \geq 0$, $\tau_i(t) \leq t - \sigma$, $i = 0, 1, 2$. *If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int^{\infty} \left[h_0(t) + [k_1 h_1(t)]^{1/k_1} [k_2 h_2(t)]^{1/k_2} \right] \pi^{\alpha+1}(t) \mathrm{d}t = \infty, \tag{2.45}$$

then (1.1) *is oscillatory.*

Theorem 2.5. *Assume that* (1.3) *holds,* $p'(t) \geq 0$, *and there exists* $\rho \in C^1([t_0, \infty), \mathbb{R})$, *such that* $\rho(t) \geq t$, $\rho'(t) > 0$, $\tau_i(t) \leq \rho(t) - \sigma$, $i = 0, 1, 2$. *If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int_{t_1}^{\infty} r^{-1/\alpha}(v) \left[\int_{t_1}^{v} \varphi(u) \mathrm{d}u \right]^{1/\alpha} \mathrm{d}v = \infty, \tag{2.46}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that u is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(t) > 0$ for all large t. The case of $u(t) < 0$ can be considered by the same method. From (1.1) and (1.3), we can easily obtain that there exists a $t_1 \geq t_0$ such that (2.4) or (2.5) holds.

If (2.4) holds, proceeding as in the proof of Theorem 2.1, we obtain a contradiction with (2.2).

If (2.5) holds, we proceed as in the proof of Theorem 2.1, and we get (2.21). Dividing (2.21) by $r^{1/\alpha}(s)$ and integrating it from $\rho(t)$ to l, letting $l \to \infty$, yields

$$z(\rho(t)) \geq -r^{1/\alpha}(t) z'(t) \int_{\rho(t)}^{\infty} r^{-1/\alpha}(s) \mathrm{d}s = -r^{1/\alpha}(t) z'(t) \delta(t) \geq -r^{1/\alpha}(t_1) z'(t_1) \delta(t) := a\delta(t). \tag{2.47}$$

By (1.1), we have

$$\left(r(t)\left(-z'(t)\right)^{\alpha}\right)' = q_0(t) u^{\alpha}(\tau_0(t)) + q_1(t) u^{\beta}(\tau_1(t)) + q_2(t) u^{\gamma}(\tau_2(t)). \tag{2.48}$$

Noticing that $p'(t) \geq 0$, from [10, Theorem 2.3], we see that $u'(t) \leq 0$ for $t \geq t_1$, so by $\tau_i(t) \leq \rho(t) - \sigma$, $i = 0, 1, 2$, we get

$$\begin{aligned} \frac{u(\tau_i(t))}{z(\rho(t))} &= \frac{u(\tau_i(t))}{u(\rho(t)) + p(\rho(t)) u(\rho(t) - \sigma)} \\ &= \frac{1}{(u(\rho(t))/u(\tau_i(t))) + p(\rho(t))(u(\rho(t) - \sigma)/u(\tau_i(t)))} \geq \frac{1}{1 + p(\rho(t))}. \end{aligned} \tag{2.49}$$

Hence, we obtain

$$\left(r(t)\left(-z'(t)\right)^{\alpha}\right)' \geq b\varphi(t), \tag{2.50}$$

where $b = \min\{a^{\alpha}, a^{\beta}, a^{\gamma}\}$. Integrating the above inequality from t_1 to t, we have

$$r(t)\left(-z'(t)\right)^{\alpha} \geq r(t_1)\left(-z'(t_1)\right)^{\alpha} + b\int_{t_1}^{t} \varphi(u)\mathrm{d}u \geq b\int_{t_1}^{t} \varphi(u)\mathrm{d}u. \tag{2.51}$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t_1) - z(t) \geq b^{1/\alpha}\int_{t_1}^{t} r^{-1/\alpha}(v)\left[\int_{t_1}^{v} \varphi(u)\mathrm{d}u\right]^{1/\alpha}\mathrm{d}v, \tag{2.52}$$

which contradicts (2.46). This completes the proof of Theorem 2.5. □

3. Examples

In this section, three examples are worked out to illustrate the main results.

Example 3.1. Consider the second-order neutral delay differential equation (1.8), where $\lambda > 0$ is a constant.

Let $r(t) = \mathrm{e}^{2t}$, $p(t) = 1/2$, $\sigma = 2$, $q_0(t) = \lambda(2\mathrm{e}^{2t} + \mathrm{e}^{2t+2})/2$, $\alpha = 1$, $\tau_0(t) = t - 1$, $q_1(t) = q_2(t) = 0$, and $\tau(t) = \tau_0(t)$, then

$$R(t) = \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)}\mathrm{d}s = \frac{\left(\mathrm{e}^{-2t_0} - \mathrm{e}^{-2t}\right)}{2},$$

$$\xi(t) = r^{1/\alpha}(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)\mathrm{d}s = \frac{\left(\mathrm{e}^{2(t-t_1)} - 1\right)}{2}, \tag{3.1}$$

$$Q_0(t) = \frac{q_0(t)}{2} = \frac{\lambda\left(2\mathrm{e}^{2t} + \mathrm{e}^{2t+2}\right)}{4}, \qquad \zeta_0(t) = \frac{2q_0(t)}{3} = \frac{\lambda\left(2\mathrm{e}^{2t} + \mathrm{e}^{2t+2}\right)}{3}.$$

Setting $\rho(t) = t + 1$, we have $\tau_0(t) = t - 1 \leq \rho(t) - \sigma$, $\delta(t) = \mathrm{e}^{-2t-2}/2$. Therefore, for all sufficiently large t_1,

$$\begin{aligned}
&\int^{\infty}\left\{R^{\alpha}(\tau(t))\left[Q_0(t) + [k_1Q_1(t)]^{1/k_1}[k_2Q_2(t)]^{1/k_2}\right] - \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)}\right\}\mathrm{d}t = \infty,\\
&\int^{\infty}\left\{\left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha}(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\rho'(t)}{\delta(t)r^{1/\alpha}(\rho(t))}\right\}\mathrm{d}t\\
&\quad = \int^{\infty}\frac{\lambda\left(2\mathrm{e}^{-2} + 1\right) - 3}{6}\mathrm{d}t = \infty
\end{aligned} \tag{3.2}$$

if $\lambda > 3/(2\mathrm{e}^{-2} + 1)$. Hence, by Theorem 2.1, (1.8) is oscillatory when $\lambda > 3/(2\mathrm{e}^{-2} + 1)$.

Note that [11, Theorem 2.1] and [11, Theorem 2.2] cannot be applied in (1.8), since $\tau_0(t) > t-2$. On the other hand, applying [11, Theorem 3.2] to that (1.8), we obtain that (1.8) is oscillatory if $\lambda > 3/(e^{-2}+2e^{-4})$. So our results improve the results in [11].

Example 3.2. Consider the second-order neutral delay differential equation

$$\left(e^t\left(u(t)+\frac{1}{2}u\left(t-\frac{\pi}{4}\right)\right)'\right)' + 12\sqrt{65}e^t u\left(t-\frac{1}{8}\arcsin\frac{\sqrt{65}}{65}\right) = 0, \quad t \ge t_0. \tag{3.3}$$

Let $r(t) = e^t$, $p(t) = 1/2$, $\sigma = \pi/4$, $q_0(t) = 12\sqrt{65}e^t$, $q_1(t) = q_2(t) = 0$, $\alpha = 1$, $\tau_0(t) = t-(\arcsin\sqrt{65}/65)/8$, $\rho(t) = t+\pi/4$, and $\tau(t) = t-\pi/4$, then

$$\begin{gathered} R(t) = \int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}ds = e^{-t_0}-e^{-t}, \qquad \xi(t) = r^{1/\alpha}(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)ds = e^{t-t_1}-1, \\ Q_0(t) = \frac{q_0(t)}{2} = 6\sqrt{65}e^t, \qquad \zeta_0(t) = \frac{2q_0(t)}{3} = 8\sqrt{65}e^t, \qquad \delta(t) = e^{-t-\pi/4}. \end{gathered} \tag{3.4}$$

Therefore, for all sufficiently large t_1,

$$\begin{aligned} &\int^{\infty}\left\{R^{\alpha}(\tau(t))\left[Q_0(t)+[k_1Q_1(t)]^{1/k_1}[k_2Q_2(t)]^{1/k_2}\right] - \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)}\right\}dt = \infty, \\ &\int^{\infty}\left\{\left[\zeta_0(t)+[k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha}(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\rho'(t)}{\delta(t)r^{1/\alpha}(\rho(t))}\right\}dt \\ &\quad = \int^{\infty}\left(8\sqrt{65}e^{-\pi/4}-\frac{1}{4}\right)dt = \infty. \end{aligned} \tag{3.5}$$

Hence, by Theorem 2.1, (3.3) oscillates. For example, $u(t) = \sin 8t$ is a solution of (3.3).

Example 3.3. Consider the second-order neutral differential equation

$$\left(e^t z'(t)\right)' + e^{2\lambda_* t}u(\lambda_0 t) + q_1(t)u^{1/3}(\lambda_1 t) + q_2(t)u^{5/3}(\lambda_2 t) = 0, \quad t \ge t_0, \tag{3.6}$$

where $z(t) = u(t)+u(t-1)/2$, $\lambda_i > 0$ for $i = 0,1,2$, are constants, $q_1(t) > 0$, $q_2(t) > 0$ for $t \ge t_0$.

Let $r(t) = e^t$, $\sigma = 1$, $q_0(t) = e^{2\lambda_* t}$, $\lambda_* = \max\{\lambda_0, \lambda_1, \lambda_2\}$, $\tau_i(t) = \lambda_i t$, $\tau(t) = \lambda t$, $0 < \lambda < \min\{\lambda_0, \lambda_1, \lambda_2, 1\}$, $\rho(t) = \lambda_* t + 1$, $\alpha = 1$, $\beta = 1/3$, and $\gamma = 5/3$, then $k_1 = k_2 = 2$,

$$\begin{gathered} R(t) = \int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}ds = e^{-t_0}-e^{-t}, \\ \xi(t) = r^{1/\alpha}(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)ds = e^{\lambda(t-t_1)}-1, \qquad \delta(t) = e^{-\lambda_* t-1}. \end{gathered} \tag{3.7}$$

It is easy to see that (2.2) and (2.41) hold for all sufficiently large t_1. Hence, by Theorem 2.3, (3.6) is oscillatory.

4. Conclusions

In this paper, we consider the oscillatory behavior of second-order neutral functional differential equation (1.1). Our results can be applied to the case when $\tau_i(t) > t$, $i = 0, 1, 2$; these results improve the results given in [6, 7, 10, 11].

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