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Research Article

Invariant Sets of Impulsive Differential Equations with Particularities in ω-Limit Set

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Sufficient conditions for the existence and asymptotic stability of the invariant sets of an impulsive system of differential equations defined in the direct product of a torus and an Euclidean space are obtained.

1. Introduction

The evolution of variety of processes in physics, chemistry, biology, and so forth, frequently undergoes short-term perturbations. It is sometimes convenient to neglect the duration of the perturbations and to consider these perturbations to be "instantaneous." This leads to the necessity of studying the differential equations with discontinuous trajectories, the so-called impulsive differential equations. The fundamentals of the mathematical theory of impulsive differential equations are stated in [1–4]. The theory is developing intensively due to its applied value in simulations of the real world phenomena.

At the same time, this paper is closely related to the oscillation theory. In the middle of the 20th century, a sharp turn towards the investigations of the oscillating processes that were characterized as "almost exact" iterations within "almost the same" periods of time took place. Quasiperiodic oscillations were brought to the primary focus of investigations of the oscillation theory [5].

Quasiperiodic oscillations are a sufficiently complicated and sensitive object for investigating. The practical value of indicating such oscillations is unessential. Due to the instability of frequency basis, quasiperiodic oscillation collapses easily and may be transformed into periodic oscillation via small shift of the right-hand side of the system. This fact has led to search for more rough object than the quasiperiodic solution. Thus the minimal set that is covered by the trajectories of the quasiperiodic motions becomes the main object of investigations. As it is known, such set is a torus. The first profound assertions regarding the

invariant toroidal manifolds were obtained by Bogoliubov et al. [6, 7]. Further results in this area were widely extended by many authors.

Consider the system of differential equations

$$\frac{dz}{dt} = F(z), \tag{1.1}$$

where the function $F(z)$ is defined in some subset D of the $(m+n)$-dimensional Euclidean space E^{m+n}, continuous and satisfies a Lipschitz condition. Let M be an invariant toroidal manifold of the system. While investigating the trajectories that begin in the neighborhood of the manifold M, it is convenient to make the change of variables from Euclidean coordinates $(z_1, \ldots, z_{m+n})$ to so-called local coordinates $\varphi = (\varphi_1, \ldots, \varphi_m)$, $x = (x_1, \ldots, x_n)$, where φ is a point on the surface of an m-dimensional torus $\mathcal{T}^m$ and x is a point in an n-dimensional Euclidean space E^n. The change of variables is performed in a way such that the equation, which defines the invariant manifold M, transforms into $x = 0$, $\varphi \in \mathcal{T}^m$ in the new coordinates. In essence, the manifold $x = 0$, $\varphi \in \mathcal{T}^m$ is the m-dimensional torus in the space $\mathcal{T}^m \times E^n$. The character of stability of the invariant torus M is closely linked with stability of the set $x = 0$, $\varphi \in \mathcal{T}^m$: from stability, asymptotic stability, and instability of the manifold M, there follow the stability, asymptotic stability, and instability of the torus $x = 0$, $\varphi \in \mathcal{T}^m$ correspondingly and vice versa. This is what determines the relevance and value of the investigation of conditions for the existence and stability of invariant sets of the systems of differential equations defined in $\mathcal{T}^m \times E^n$. Theory of the existence and perturbation, properties of smoothness, and stability of invariant sets of systems defined in $\mathcal{T}^m \times E^n$ are considered in [8].

2. Preliminaries

The main object of investigation of this paper is the system of differential equations, defined in the direct product of an m-dimensional torus $\mathcal{T}^m$ and an n-dimensional Euclidean space E^n that undergo impulsive perturbations at the moments when the phase point φ meets a given set in the phase space. Consider the system

$$\begin{gathered} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = A(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} = B(\varphi)x + g(\varphi), \end{gathered} \tag{2.1}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)^T \in \mathcal{T}^m$, $x = (x_1, \ldots, x_n)^T \in E^n$, $a(\varphi)$ is a continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector function that satisfies a Lipschitz condition

$$\|a(\varphi'') - a(\varphi')\| \le L\|\varphi'' - \varphi'\| \tag{2.2}$$

for every $\varphi', \varphi'' \in \mathcal{T}^m$. $A(\varphi), B(\varphi)$ are continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ square matrices; $f(\varphi)$, $g(\varphi)$ are continuous (piecewise continuous with first kind discontinuities in the set Γ) 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector functions.

Some aspects regarding existence and stability of invariant sets of systems similar to (2.1) were considered by different authors in [9–12].

We regard the point $\varphi = (\varphi_1, \ldots, \varphi_m)^T$ as a point of the m-dimensional torus $\mathcal{T}^m$ so that the domain of the functions $A(\varphi), B(\varphi), f(\varphi), g(\varphi)$, and $a(\varphi)$ is the torus $\mathcal{T}^m$. We assume that the set Γ is a subset of the torus $\mathcal{T}^m$, which is a manifold of dimension $m-1$ defined by the equation $\Phi(\varphi) = 0$ for some continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ function.

The system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi) \tag{2.3}$$

defines a dynamical system on the torus $\mathcal{T}^m$. Denote by $\varphi_t(\varphi)$ the solution of (2.3) that satisfies the initial condition $\varphi_0(\varphi) = \varphi$. The Lipschitz condition (2.2) guarantees the existence and uniqueness of such solution. Moreover, the solutions $\varphi_t(\varphi)$ satisfies a group property [8]

$$\varphi_t(\varphi_\tau(\varphi)) = \varphi_{t+\tau}(\varphi) \tag{2.4}$$

for all $t, \tau \in \mathbb{R}$ and $\varphi \in \mathcal{T}^m$.

Denote by $t_i(\varphi)$, $i \in \mathbb{Z}$ the solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$ that are the moments of impulsive action in system (2.1). Let the function $\Phi(\varphi)$ be such that the solutions $t = t_i(\varphi)$ exist, since otherwise, system (2.1) would not be an impulsive system. Assume that

$$\begin{gathered} \lim_{i \to \pm\infty} t_i(\varphi) = \pm\infty, \\ \lim_{T \to \pm\infty} \frac{i(t, t+T)}{T} = p < \infty \end{gathered} \tag{2.5}$$

uniformly with respect to $t \in \mathbb{R}$, where $i(a, b)$ is the number of the points $t_i(\varphi)$ in the interval (a, b). Hence, the moments of impulsive perturbations $t_i(\varphi)$ satisfy the equality [10, 11]

$$t_i(\varphi_{-t}(\varphi)) - t_i(\varphi) = t. \tag{2.6}$$

Together with system (2.1), we consider the linear system

$$\begin{gathered} \frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} = B(\varphi_{t_i(\varphi)}(\varphi))x + g(\varphi_{t_i(\varphi)}(\varphi)) \end{gathered} \tag{2.7}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. We obtain system (2.7) by substituting $\varphi_t(\varphi)$ for φ in the second and third equations of system (2.1). By invariant set of system (2.1), we understand a set that is defined by a function $u(\varphi)$, which has a period 2π with respect to each of the components φ_v, $v = 1, \ldots, m$, such that the function $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution of system (2.7) for every $\varphi \in \mathcal{T}^m$.

We call a point φ^* an ω-limit point of the trajectory $\varphi_t(\varphi)$ if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that

$$\begin{gathered}\lim_{n\to+\infty} t_n = +\infty,\\ \lim_{n\to+\infty} \varphi_{t_n}(\varphi) = \varphi^*.\end{gathered} \tag{2.8}$$

The set of all ω-limit points for a given trajectory $\varphi_t(\varphi)$ is called ω-limit set of the trajectory $\varphi_t(\varphi)$ and denoted by Ω_φ.

Referring to system (2.7), the matrices $A(\varphi_t(\varphi))$ and $B(\varphi_t(\varphi))$, that influence the behavior of the solution $x(t,\varphi)$ of the system (2.7), depend not only on the functions $A(\varphi)$ and $B(\varphi)$ but also on the behavior of the trajectories $\varphi_t(\varphi)$. Moreover, in [9], sufficient conditions for the existence and stability of invariant sets of a system similar to (2.1) were obtained in terms of a Lyapunov function $V(\varphi,x)$ that satisfies some conditions in the domain $Z = \{\varphi\in\Omega,\ x\in\overline{J}_h\}$, where $\overline{J}_h = \{x\in E^n,\ \|x\|\le h,\ h>0\}$,

$$\Omega = \bigcup_{\varphi\in\mathcal{T}^m} \Omega_\varphi. \tag{2.9}$$

Since the Lyapunov function has to satisfy some conditions not on the whole surface of the torus $\mathcal{T}^m$ but only in the ω-limit set Ω, it is interesting to consider system (2.1) with specific properties in the domain Ω.

3. Main Result

Consider system (2.1) assuming that the matrices $A(\varphi)$ and $B(\varphi)$ are constant in the domain Ω:

$$\begin{gathered}A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A},\\ B(\varphi)\big|_{\varphi\in\Omega} = \widetilde{B}.\end{gathered} \tag{3.1}$$

Therefore, for every $\varphi\in\mathcal{T}^m$

$$\begin{gathered}\lim_{t\to+\infty} A(\varphi_t(\varphi)) = \widetilde{A},\\ \lim_{t\to+\infty} B(\varphi_t(\varphi)) = \widetilde{B}.\end{gathered} \tag{3.2}$$

We will obtain sufficient conditions for the existence and asymptotic stability of an invariant set of the system (2.1) in terms of the eigenvalues of the matrices $\widetilde{A}$ and $\widetilde{B}$. Denote by

$$\begin{gathered}\gamma = \max_{j=1,\dots,n} \operatorname{Re}\lambda_j\left(\widetilde{A}\right),\\ \alpha^2 = \max_{j=1,\dots,n} \lambda_j\left(\left(E+\widetilde{B}\right)^T\left(E+\widetilde{B}\right)\right).\end{gathered} \tag{3.3}$$

Similar systems without impulsive perturbations have been considered in [13].

Theorem 3.1. *Let the moments of impulsive perturbations $\{t_i(\varphi)\}$ be such that uniformly with respect to $t \in \mathbb{R}$ there exists a finite limit*

$$\lim_{\tilde{T}\to\infty} \frac{i\left(t, t+\tilde{T}\right)}{\tilde{T}} = p. \tag{3.4}$$

If the following inequality holds

$$\gamma + p \ln \alpha < 0, \tag{3.5}$$

then system (2.1) *has an asymptotically stable invariant set.*

Proof. Consider a homogeneous system of differential equations

$$\begin{gathered} \frac{dx}{dt} = A(\varphi_t(\varphi))x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} = B(\varphi_{t_i(\varphi)}(\varphi))x \end{gathered} \tag{3.6}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. By $\Omega_\tau^t(\varphi)$, we denote the fundamental matrix of system (3.6), which turns into an identity matrix at the point $t = \tau$, that is, $\Omega_\tau^\tau(\varphi) \equiv E$. It can be readily verified [4] that $\Omega_\tau^t(\varphi)$ satisfies the equalities

$$\begin{gathered} \frac{\partial}{\partial t}\Omega_\tau^t(\varphi) = A(\varphi_t(\varphi))\Omega_\tau^t(\varphi), \\ \Omega_\tau^t(\varphi) = \Omega_\tau^t(\varphi + 2\pi e_k), \\ \Omega_{t+\tau}^t(\varphi_{-t}(\varphi)) = \Omega_\tau^0(\varphi) \end{gathered} \tag{3.7}$$

for all $t, \tau \in \mathbb{R}$ and $\varphi \in \mathcal{T}^m$. Rewrite system (3.6) in the form

$$\begin{gathered} \frac{dx}{dt} = \tilde{A}x + \left(A(\varphi_t(\varphi)) - \tilde{A}\right)x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} = \tilde{B}x + \left(B(\varphi_{t_i(\varphi)}(\varphi)) - \tilde{B}\right)x. \end{gathered} \tag{3.8}$$

The fundamental matrix $\Omega_\tau^t(\varphi)$ of the system (3.6) may be represented in the following way [4]:

$$\begin{aligned} \Omega_\tau^t(\varphi) = X_\tau^t(\varphi) &+ \int_\tau^t X_s^t(\varphi)\left(A(\varphi_s(\varphi)) - \tilde{A}\right)\Omega_s^t(\varphi)\,ds \\ &+ \sum_{\tau \le t_i(\varphi) < t} X_{t_i(\varphi)}^t(\varphi)\left(B(\varphi_{t_i(\varphi)}(\varphi)) - \tilde{B}\right)\Omega_\tau^{t_i(\varphi)}(\varphi), \end{aligned} \tag{3.9}$$

where $X_\tau^t(\varphi)$ is the fundamental matrix of the homogeneous impulsive system with constant coefficients

$$\begin{aligned} \frac{dx}{dt} &= \tilde{A}x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= \tilde{B}x \end{aligned} \tag{3.10}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. Taking into account that the matrix $X_\tau^t(\varphi)$ satisfies the estimate [14]

$$\left\| X_\tau^t(\varphi) \right\| \leq K e^{-\mu(t-\tau)}, \quad t \geq \tau \tag{3.11}$$

for every $\varphi \in \mathcal{T}^m$ and some $K \geq 1$, where $\gamma + p \ln \alpha < -\mu < 0$, we obtain

$$\begin{aligned} \left\| \Omega_\tau^t(\varphi) \right\| \leq{}& K e^{-\mu(t-\tau)} + \int_\tau^t K e^{-\mu(t-s)} \left\| A(\varphi_s(\varphi)) - \tilde{A} \right\| \left\| \Omega_\tau^s(\varphi) \right\| ds \\ &+ \sum_{\tau \leq t_i(\varphi) < t} K e^{-\mu(t-t_i(\varphi))} \left\| B(\varphi_{t_i(\varphi)}(\varphi)) - \tilde{B} \right\| \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\|. \end{aligned} \tag{3.12}$$

It follows from (3.2) that for arbitrary small ε_A and ε_B, there exists a moment T such that

$$\begin{aligned} \left\| A(\varphi_t(\varphi)) - \tilde{A} \right\| &\leq \varepsilon_A, \\ \left\| B(\varphi_t(\varphi)) - \tilde{B} \right\| &\leq \varepsilon_B \end{aligned} \tag{3.13}$$

for all $t \geq T$. Hence, multiplying (3.12) by $e^{\mu(t-\tau)}$, utilizing (3.13), and weakening the inequality, we obtain

$$\begin{aligned} e^{\mu(t-\tau)} \left\| \Omega_\tau^t(\varphi) \right\| \leq{}& K + \int_\tau^T K e^{\mu(s-\tau)} \left\| A(\varphi_s(\varphi)) - \tilde{A} \right\| \left\| \Omega_\tau^s(\varphi) \right\| ds \\ &+ \int_\tau^t K \varepsilon_A e^{\mu(s-\tau)} \left\| \Omega_\tau^s(\varphi) \right\| ds \\ &+ \sum_{\tau \leq t_i(\varphi) < T} K e^{\mu(t_i(\varphi)-\tau)} \left\| B(\varphi_{t_i(\varphi)}(\varphi)) - \tilde{B} \right\| \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\| \\ &+ \sum_{\tau \leq t_i(\varphi) < t} K \varepsilon_B e^{\mu(t_i(\varphi)-\tau)} \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\|. \end{aligned} \tag{3.14}$$

Using the Gronwall-Bellman inequality for piecewise continuous functions [4], we obtain the estimate for the fundamental matrix $\Omega_\tau^t(\varphi)$ of the system (3.6)

$$\left\| \Omega_\tau^t(\varphi) \right\| \leq K_1 e^{-(\mu - K\varepsilon_A - p \ln(1+K\varepsilon_B))(t-\tau)}, \tag{3.15}$$

where

$$K_1 = K + \int_\tau^T K e^{\mu(s-\tau)} \left\| A(\varphi_s(\varphi)) - \widetilde{A} \right\| \left\| \Omega_\tau^s(\varphi) \right\| ds + \sum_{\tau \le t_i(\varphi) < T} K e^{\mu(t_i(\varphi)-\tau)} \left\| B(\varphi_{t_i(\varphi)}(\varphi)) - \widetilde{B} \right\| \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\|. \tag{3.16}$$

Choosing ε_A and ε_B so that $\mu > K\varepsilon_A + p\ln(1 + K\varepsilon_B)$, the following estimate holds

$$\left\| \Omega_\tau^t(\varphi) \right\| \le K_1 e^{-\gamma_1(t-\tau)} \tag{3.17}$$

for all $t \ge \tau$ and some $K_1 \ge 1$, $\gamma_1 > 0$.

Estimate (3.17) is a sufficient condition for the existence and asymptotic stability of an invariant set of system (2.1). Indeed, it is easy to verify that invariant set $x = u(\varphi)$ of the system (2.1) may be represented as

$$u(\varphi) = \int_{-\infty}^0 \Omega_\tau^0(\varphi) f(\varphi_\tau(\varphi)) d\tau + \sum_{t_i(\varphi)<0} \Omega_{t_i(\varphi)}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)). \tag{3.18}$$

The integral and the sum from (3.18) converge since inequality (3.17) holds and limit (3.4) exists. Utilizing the properties (3.7) of the matrix $\Omega_\tau^t(\varphi)$ (2.4), and (2.6), one can show that the function $u(\varphi_t(\varphi))$ satisfies the equation

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)) \tag{3.19}$$

for $t \ne t_i(\varphi)$ and has discontinuities $B(\varphi_{t_i(\varphi)}(\varphi))u(\varphi_{t_i(\varphi)}(\varphi)) + g(\varphi_{t_i(\varphi)}(\varphi))$ at the points $t = t_i(\varphi)$. It means that the function $x(t,\varphi) = u(\varphi_t(\varphi))$ is a solution of the system (2.7). Hence, $u(\varphi)$ defines the invariant set of system (2.1).

Let us prove the asymptotic stability of the invariant set. Let $x = x(t,\varphi)$ be an arbitrary solutions of the system (2.7), and $x^* = u(\varphi_t(\varphi))$ is the solution that belongs to the invariant set. The difference of these solutions admits the representation

$$x(t,\varphi) - u(\varphi_t(\varphi)) = \Omega_0^t(\varphi)(x(0,\varphi) - u(\varphi)). \tag{3.20}$$

Taking into account estimate (3.17), the following limit exists

$$\lim_{t\to\infty} \left\| x(t,\varphi) - u(\varphi_t(\varphi)) \right\| = 0. \tag{3.21}$$

It proves the asymptotic stability of the invariant set $x = u(\varphi)$. □

4. Perturbation Theory

Let us show that small perturbations of the right-hand side of the system (2.1) do not ruin the invariant set. Let $A_1(\varphi)$ and $B_1(\varphi)$ be continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ square matrices. Consider the perturbed system

$$\begin{gathered} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = (A(\varphi) + A_1(\varphi))x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} = (B(\varphi) + B_1(\varphi))x + g(\varphi). \end{gathered} \tag{4.1}$$

Theorem 4.1. *Let the moments of impulsive perturbations $\{t_i(\varphi)\}$ be such that uniformly with respect to $t \in \mathbb{R}$, there exists a finite limit*

$$\lim_{\tilde{T}\to\infty} \frac{i\left(t, t+\tilde{T}\right)}{\tilde{T}} = p \tag{4.2}$$

and the following inequality holds

$$\gamma + p \ln \alpha < 0. \tag{4.3}$$

Then there exist sufficiently small constants $a_1 > 0$ and $b_1 > 0$ such that for any continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ functions $A_1(\varphi)$ and $B_1(\varphi)$ such that

$$\begin{gathered} \max_{\varphi\in\mathcal{T}^m} \|A_1(\varphi)\| \le a_1, \\ \max_{\varphi\in\mathcal{T}^m} \|B_1(\varphi)\| \le b_1, \end{gathered} \tag{4.4}$$

system (4.1) *has an asymptotically stable invariant set.*

Proof. The constants a_1 and b_1 exist since the matrices $A_1(\varphi)$ and $B_1(\varphi)$ are continuous functions defined in the torus $\mathcal{T}^m$, which is a compact manifold.

Consider the impulsive system that corresponds to system(4.1)

$$\begin{gathered} \frac{dx}{dt} = A(\varphi_t(\varphi))x + A_1(\varphi_t(\varphi))x, \quad t \ne t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} = B(\varphi_{t_i(\varphi)}(\varphi))x + B_1(\varphi_{t_i(\varphi)}(\varphi))x \end{gathered} \tag{4.5}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. The fundamental matrix $\Psi_\tau^t(\varphi)$ of the system (4.5) may be represented in the following way

$$\begin{aligned}\Psi_\tau^t(\varphi) = \Omega_\tau^t(\varphi) &+ \int_\tau^t \Omega_s^t(\varphi) A_1(\varphi_s(\varphi)) \Psi_s^t(\varphi) ds \\ &+ \sum_{\tau \le t_i(\varphi) < t} \Omega_{t_i(\varphi)}^t(\varphi) B_1(\varphi_{t_i(\varphi)}(\varphi)) \Psi_\tau^{t_i(\varphi)}(\varphi),\end{aligned} \tag{4.6}$$

where $\Omega_\tau^t(\varphi)$ is the fundamental matrix of the system (3.6). Then taking estimate (3.17) into account,

$$\begin{aligned}e^{\gamma_1(t-\tau)} \left\| \Psi_\tau^t(\varphi) \right\| \le K_1 &+ \int_\tau^t K_1 a_1 e^{\gamma_1(s-\tau)} \left\| \Psi_\tau^s(\varphi) \right\| ds \\ &+ \sum_{\tau \le t_i(\varphi) < t} K_1 b_1 e^{\gamma_1(t_i(\varphi)-\tau)} \left\| \Psi_\tau^{t_i(\varphi)}(\varphi) \right\|.\end{aligned} \tag{4.7}$$

Using the Gronwall-Bellman inequality for piecewise continuous functions, we obtain the estimate for the fundamental matrix $\Psi_\tau^t(\varphi)$ of the system (4.5)

$$\left\| \Psi_\tau^t(\varphi) \right\| \le K_1 e^{-(\gamma_1 - K_1 a_1 - p \ln(1+K_1 b_1))(t-\tau)}. \tag{4.8}$$

Let the constants a_1 and b_1 be such that $\gamma_1 > K_1 a_1 + p \ln(1 + K_1 b_1)$. Hence, the matrix $\Psi_\tau^t(\varphi)$ satisfies the estimate

$$\left\| \Psi_\tau^t(\varphi) \right\| \le K_2 e^{-\gamma_2(t-\tau)} \tag{4.9}$$

for all $t \ge \tau$ and some $K_2 \ge 1$, $\gamma_2 > 0$. As in Theorem 3.1, from estimate (4.9), we conclude that the system (4.1) has an asymptotically stable invariant set $x = u(\varphi)$, which admits the representation

$$u(\varphi) = \int_{-\infty}^0 \Psi_\tau^0(\varphi) f(\varphi_\tau(\varphi)) d\tau + \sum_{t_i(\varphi) < 0} \Psi_{t_i(\varphi)}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)). \tag{4.10}$$

□

Consider the nonlinear system of differential equations with impulsive perturbations of the form

$$\begin{aligned}\frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= F(\varphi, x), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi \in \Gamma} &= I(\varphi, x),\end{aligned} \tag{4.11}$$

where $\varphi \in \mathcal{T}^m$, $x \in \overline{J}_h$, $a(\varphi)$ is a continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector function and satisfies Lipschitz conditions (2.2); $F(\varphi, x)$ and $I(\varphi, x)$ are continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ functions that have continuous partial derivatives with respect to x up to the second order inclusively. Taking these assumptions into account, system (4.11) may be rewritten in the following form:

$$\begin{gathered} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = A_0(\varphi)x + A_1(\varphi, x)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} = B_0(\varphi)x + B_1(\varphi, x)x + g(\varphi), \end{gathered} \tag{4.12}$$

where

$$A(\varphi, x) = \int_0^1 \frac{\partial F(\varphi, \tau x)}{\partial(\tau x)} d\tau, \qquad B(\varphi, x) = \int_0^1 \frac{\partial I(\varphi, \tau x)}{\partial(\tau x)} d\tau, \tag{4.13}$$

$A_0(\varphi) = A(\varphi, 0)$, $A_1(\varphi, x) = A(\varphi, x) - A(\varphi, 0)$, $B_0(\varphi) = B(\varphi, 0)$, $B_1(\varphi, x) = B(\varphi, x) - B(\varphi, 0)$, $f(\varphi) = F(\varphi, 0)$, and $g(\varphi) = I(\varphi, 0)$. We assume that the matrices $A_0(\varphi)$ and $B_0(\varphi)$ are constant in the domain Ω:

$$\begin{gathered} A_0(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A}, \\ B_0(\varphi)\big|_{\varphi\in\Omega} = \widetilde{B} \end{gathered} \tag{4.14}$$

and the inequality $\gamma + p \ln \alpha < 0$ holds.

We will construct the invariant set of system (4.12) using an iteration method proposed in [8]. As initial invariant set M_0, we consider the set $x = 0$, as M_k—the invariant set of the system

$$\begin{gathered} \frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = A_0(\varphi)x + A_1(\varphi, u_{k-1}(\varphi))x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} = B_0(\varphi)x + B_1(\varphi, u_{k-1}(\varphi))x + g(\varphi), \end{gathered} \tag{4.15}$$

where $x = u_{k-1}(\varphi)$ is the invariant set on $(k-1)$-step.

Using Theorem 4.1, the invariant set $x = u_k(\varphi)$, $k = 1, 2, \ldots$ may be represented as

$$u_k(\varphi) = \int_{-\infty}^0 \Psi_\tau^0(\varphi, k) f(\varphi_\tau(\varphi)) d\tau + \sum_{t_i(\varphi)<0} \Psi_{t_i(\varphi)+0}^0(\varphi, k) g(\varphi_{t_i(\varphi)}(\varphi)), \tag{4.16}$$

where $\Psi_\tau^t(\varphi, k)$ is the fundamental matrix of the homogeneous system

$$\begin{aligned} \frac{dx}{dt} &= \left(A_0(\varphi_t(\varphi)) + A_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right)x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= \left(B_0(\varphi_t(\varphi)) + B_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right)x \end{aligned} \tag{4.17}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter and satisfies the estimate

$$\left\|\Psi_\tau^t(\varphi, k)\right\| \leq K_2 e^{-\gamma_2(t-\tau)} \tag{4.18}$$

for all $t \geq \tau$ and some $K_2 \geq 1$, $\gamma_2 > 0$ only if

$$\begin{aligned} &\max_{\varphi \in \mathcal{T}^m} \left\|A_1(\varphi, u_{k-1}(\varphi))\right\| \leq a_1, \\ &\max_{\varphi \in \mathcal{T}^m} \left\|B_1(\varphi, u_{k-1}(\varphi))\right\| \leq b_1. \end{aligned} \tag{4.19}$$

Let us prove that the invariant sets $x = u_k(\varphi)$ belong to the domain $\overline{J}_h$. Denote by

$$\begin{aligned} &\max_{\varphi \in \mathcal{T}^m} \left\|f(\varphi)\right\| \leq M_f, \\ &\max_{\varphi \in \mathcal{T}^m} \left\|g(\varphi)\right\| \leq M_g. \end{aligned} \tag{4.20}$$

Since the torus $\mathcal{T}^m$ is a compact manifold, such constants M_f and M_g exist. Analogously to [4], using the representation (4.16) and estimate (4.18), we obtain that

$$\left\|u_k(\varphi)\right\| \leq \frac{K_2}{\gamma_2} M_f + \frac{K_2}{1 - e^{-\gamma_2 \theta_1}} M_g, \tag{4.21}$$

where θ_1 is a minimum gap between moments of impulsive actions. Condition (3.4) guarantees that such constant θ_1 exists. Assume that the constants K_2 and γ_2 are such that $\|u(\varphi)\| \leq h$.

Let us obtain the conditions for the convergence of the sequence $\{u_k(\varphi)\}$. For this purpose, we estimate the difference $w_{k+1}(\varphi) = u_{k+1}(\varphi) - u_k(\varphi)$ and take into account that the functions $u_k(\varphi_t(\varphi))$ satisfy the relations

$$\begin{aligned} \frac{d}{dt} u_k(\varphi_t(\varphi)) &= \left(A_0(\varphi_t(\varphi)) + A_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right) \\ &\quad \times u_k(\varphi_t(\varphi)) + f(\varphi_t(\varphi)), \quad t \neq t_i(\varphi), \\ \Delta u_k(\varphi_t(\varphi))|_{t=t_i(\varphi)} &= \left(B_0(\varphi_t(\varphi)) + B_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right) \\ &\quad \times u_k(\varphi_t(\varphi)) + g(\varphi_t(\varphi)) \end{aligned} \tag{4.22}$$

for all $\varphi \in \mathcal{T}^m$, $k = 1, 2, \ldots$. Hence, the difference $w_{k+1}(\varphi) = u_{k+1}(\varphi) - u_k(\varphi)$ is the invariant set of the linear impulsive system

$$\begin{gathered}\frac{d\varphi}{dt} = a(\varphi),\\ \frac{dx}{dt} = (A_0(\varphi) + A_1(\varphi, u_k(\varphi)))x + (A_1(\varphi, u_k(\varphi)) - A_1(\varphi, u_{k-1}(\varphi)))u_k(\varphi), \quad \varphi \notin \Gamma,\\ \Delta x|_{\varphi\in\Gamma} = (B_0(\varphi) + B_1(\varphi, u_k(\varphi)))x + (B_1(\varphi, u_k(\varphi)) - B_1(\varphi, u_{k-1}(\varphi)))u_k(\varphi).\end{gathered} \tag{4.23}$$

Then, taking (4.21) into account,

$$\begin{aligned}\max_{\varphi\in\mathcal{T}^m}\|u_{k+1}(\varphi) - u_k(\varphi)\| \le{}& \frac{K_2}{\gamma_2}\|A_1(\varphi, u_k(\varphi)) - A_1(\varphi, u_{k-1}(\varphi))\|\|u_k(\varphi)\|\\ &+ \frac{K_2}{1 - e^{-\gamma_2\theta}}\|B_1(\varphi, u_k(\varphi)) - B_1(\varphi, u_{k-1}(\varphi))\|\|u_k(\varphi)\|.\end{aligned} \tag{4.24}$$

Let the functions $A_1(\varphi, x)$ and $B_1(\varphi, x)$ satisfy the Lipschitz condition with constants L_A and L_B correspondingly. Then

$$\begin{aligned}\max_{\varphi\in\mathcal{T}^m}\|u_{k+1}(\varphi) - u_k(\varphi)\| &\le \frac{K_2}{\gamma_2}L_A h\|u_k(\varphi) - u_{k-1}(\varphi)\| + \frac{K_2}{1 - e^{-\gamma_2\theta}}L_B h\|u_k(\varphi) - u_{k-1}(\varphi)\|\\ &= \left(\frac{K_2 h}{\gamma_2}L_A + \frac{K_2 h}{1 - e^{-\gamma_2\theta}}L_B\right)\|u_k(\varphi) - u_{k-1}(\varphi)\|.\end{aligned} \tag{4.25}$$

Assuming that the constants L_A and L_B are so small that

$$\frac{K_2 h}{\gamma_2}L_A + \frac{K_2 h}{1 - e^{-\gamma_2\theta}}L_B < 1, \tag{4.26}$$

we conclude that the sequence $\{u_k(\varphi)\}$ converges uniformly with respect to $\varphi \in \mathcal{T}^m$ and

$$\lim_{k\to\infty} u_k(\varphi) = u(\varphi). \tag{4.27}$$

Thus, the invariant set $x = u(\varphi)$ admits the representation

$$u(\varphi) = \int_{-\infty}^{0} \Psi_\tau^0(\varphi) f(\varphi_\tau(\varphi))d\tau + \sum_{t_i(\varphi)<0} \Psi_{t_i(\varphi)}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)), \tag{4.28}$$

where $\Psi_\tau^t(\varphi)$ is the fundamental matrix of the homogeneous system

$$\begin{gathered}\frac{dx}{dt} = \left(A(\varphi_t(\varphi)) + A_1(\varphi_t(\varphi), u(\varphi_t(\varphi)))\right)x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} = \left(B(\varphi_t(\varphi)) + B(\varphi_t(\varphi), u(\varphi_t(\varphi)))\right)x\end{gathered} \tag{4.29}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter and satisfies the estimation

$$\left\|\Psi_\tau^t(\varphi)\right\| \leq K_2 e^{-\gamma_2(t-\tau)} \tag{4.30}$$

for all $t \geq \tau$ and some $K_2 \geq 1$, $\gamma_2 > 0$. The following assertion has been proved.

Theorem 4.2. *Let the matrices $A_0(\varphi)$ and $B_0(\varphi)$ be constant in the domain Ω:*

$$\begin{gathered}A_0(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A}, \\ B_0(\varphi)\big|_{\varphi\in\Omega} = \widetilde{B},\end{gathered} \tag{4.31}$$

uniformly with respect to $t \in \mathbb{R}$, there exists a finite limit

$$\lim_{\widetilde{T}\to\infty} \frac{i\left(t, t+\widetilde{T}\right)}{\widetilde{T}} = p \tag{4.32}$$

and the following inequality holds

$$\gamma + p \ln \alpha < 0, \tag{4.33}$$

where

$$\begin{gathered}\gamma = \max_{j=1,\dots,n} \operatorname{Re} \lambda_j\left(\widetilde{A}\right), \\ \alpha^2 = \max_{j=1,\dots,n} \lambda_j\left(\left(E+\widetilde{B}\right)^T\left(E+\widetilde{B}\right)\right).\end{gathered} \tag{4.34}$$

Then there exist sufficiently small constants a_1 and b_1 and sufficiently small Lipschitz constants L_A and L_B such that for any continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ matrices $F(\varphi, x)$ and $I(\varphi, x)$, which have continuous partial derivatives with respect to x up to the second order inclusively, such that

$$\begin{gathered}\max_{\varphi\in\mathcal{T}^m, x\in\overline{J}_h} \left\|A_1(\varphi, x)\right\| \leq a_1, \\ \max_{\varphi\in\mathcal{T}^m, x\in\overline{J}_h} \left\|B_1(\varphi, x)\right\| \leq b_1\end{gathered} \tag{4.35}$$

and for any $x', x'' \in \overline{J}_h$

$$\begin{aligned} \|A_1(\varphi, x') - A_1(\varphi, x'')\| &\le L_A \|x' - x''\|, \\ \|B_1(\varphi, x') - B_1(\varphi, x'')\| &\le L_B \|x' - x''\|, \end{aligned} \tag{4.36}$$

system (4.11) *has an asymptotically stable invariant set.*

5. Conclusion

In summary, we have obtained sufficient conditions for the existence and asymptotic stability of invariant sets of a linear impulsive system of differential equations defined in $\mathcal{T}^m \times E^n$ that has specific properties in the ω-limit set Ω of the trajectories $\varphi_t(\varphi)$. We have proved that it is sufficient to impose some restrictions on system (2.1) only in the domain Ω to guarantee the existence and asymptotic stability of the invariant set.

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