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Research Article

H_∞ Estimation for a Class of Lipschitz Nonlinear Discrete-Time Systems with Time Delay

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The issue of H_∞ estimation for a class of Lipschitz nonlinear discrete-time systems with time delay and disturbance input is addressed. First, through integrating the H_∞ filtering performance index with the Lipschitz conditions of the nonlinearity, the design of robust estimator is formulated as a positive minimum problem of indefinite quadratic form. Then, by introducing the Krein space model and applying innovation analysis approach, the minimum of the indefinite quadratic form is obtained in terms of innovation sequence. Finally, through guaranteeing the positivity of the minimum, a sufficient condition for the existence of the H_∞ estimator is proposed and the estimator is derived in terms of Riccati-like difference equations. The proposed algorithm is proved to be effective by a numerical example.

1. Introduction

In control field, nonlinear estimation is considered to be an important task which is also of great challenge, and it has been a very active area of research for decades [1–7]. Many kinds of methods on estimator design have been proposed for different types of nonlinear dynamical systems. Generally speaking, there are three approaches widely adopted for nonlinear estimation. In the first one, by using an extended (nonexact) linearization of the nonlinear systems, the estimator is designed by employing classical linear observer techniques [1]. The second approach, based on a nonlinear state coordinate transformation which renders the dynamics driven by nonlinear output injection and the output linear on the new coordinates, uses the quasilinear approaches to design the nonlinear estimator [2–4]. In the last one, methods are developed to design nonlinear estimators for systems which consist of an observable linear part and a locally or globally Lipschitz nonlinear part [5–7]. In this paper, the problem of H_∞ estimator design is investigated for a class of Lipschitz nonlinear discrete-time systems with time delay and disturbance input.

In practice, most nonlinearities can be regarded as Lipschitz, at least locally when they are studied in a given neighborhood [6]. For example, trigonometric nonlinearities occurring in many robotic problems, non-linear softening spring models frequently used in mechanical systems, nonlinearities which are square or cubic in nature, and so forth. Thus, in recent years, increasing attention has been paid to estimator design for Lipschitz nonlinear systems [8–19]. For the purpose of designing this class of nonlinear estimator, a number of approaches have been developed, such as sliding mode observers [8, 9], H_∞ optimization techniques [10–13], adaptive observers [14, 15], high-gain observers [16], loop transfer recovery observers [17], proportional integral observers [18], and integral quadratic constraints approach [19]. All of the above results are obtained in the assumption that the Lipschitz nonlinear systems are delay free. However, time delay is an inherent characteristic of many physical systems, and it can result in instability and poor performances if it is ignored. The estimator design for time-delay Lipschitz nonlinear systems has become a substantial need. Unfortunately, compared with estimator design for delay-free Lipschitz nonlinear systems, less research has been carried out on the time-delay case. In [20], the linear matrix inequality-(LMI-) based full-order and reduced-order robust H_∞ observers are proposed for a class of Lipschitz nonlinear discrete-time systems with time delay. In [21], by using Lyapunov stability theory and LMI techniques, a delay-dependent approach to the H_∞ and L_2-L_∞ filtering is proposed for a class of uncertain Lipschitz nonlinear time-delay systems. In [22], by guaranteeing the asymptotic stability of the error dynamics, the robust observer is presented for a class of uncertain discrete-time Lipschitz nonlinear state delayed systems; In [23], based on the sliding mode techniques, a discontinuous observer is designed for a class of Lipschitz nonlinear systems with uncertainty. In [24], an LMI-based convex optimization approach to observer design is developed for both constant-delay and time-varying delay Lipschitz nonlinear systems.

In this paper, the H_∞ estimation problem is studied for a class of Lipschitz nonlinear discrete time-delay systems with disturbance input. Inspired by the recent study on H_∞ fault detection for linear discrete time-delay systems in [25], a recursive Kalman-like algorithm in an indefinite metric space, named the Krein space [26], will be developed to the design of H_∞ estimator for time-delay Lipschitz nonlinear systems. Unlike [20], the delay-free nonlinearities and the delayed nonlinearities in the presented systems are decoupling. For the case presented in [20], the H_∞ observer design problem, utilizing the technical line of this paper, can be solved by transforming it into a delay-free system through state augmentation. Indeed, the state augmentation results in a higher system dimension and, thus, a much more expensive computational cost. Therefore, this paper based on the presented time-delay Lipschitz nonlinear systems, focuses on the robust estimator design without state augmentation by employing innovation analysis approach in the Krein space. The major contribution of this paper can be summarized as follows: (i) it extends the Krein space linear estimation methodology [26] to the state estimation of the time-delay Lipschitz nonlinear systems and (ii) it develops a recursive Kalman-like robust estimator for time-delay Lipschitz nonlinear systems without state augmentation.

The remainder of this paper is arranged as follows. In Section 2, the interest system, the Lipschitz conditions, and the H_∞ estimation problem are introduced. In Section 3, a partially equivalent Krein space problem is constructed, the H_∞ estimator is obtained by computed Riccati-like difference equations, and sufficient existence condition is derived in terms of matrix inequalities. An example is given to show the effect of the proposed algorithm in Section 4. Finally, some concluding remarks are made in Section 5.

In the sequel, the following notation will be used: elements in the Krein space will be denoted by **boldface** letters, and elements in the Euclidean space of complex numbers

will be denoted by normal letters; $\mathbb{R}^n$ denotes the real n-dimensional Euclidean space; $\|\cdot\|$ denotes the Euclidean norm; $\theta(k) \in l_2[0,N]$ means $\sum_{k=0}^{N}(\theta^T(k)\theta(k)) < \infty$; the superscripts "$-1$" and "$T$" stand for the inverse and transpose of a matrix, resp.; I is the identity matrix with appropriate dimensions; For a real matrix, $P > 0$ ($P < 0$, resp.) means that P is symmetric and positive (negative, resp.) definite; $\langle *, * \rangle$ denotes the inner product in the Krein space; diag$\{\cdots\}$ denotes a block-diagonal matrix; $\mathcal{L}\{\cdots\}$ denotes the linear space spanned by sequence $\{\cdots\}$.

2. System Model and Problem Formulation

Consider a class of nonlinear systems described by the following equations:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k_d) + f(k, Fx(k), u(k)) \\ &\quad + h(k, Hx(k_d), u(k)) + Bw(k), \\ y(k) &= Cx(k) + v(k), \\ z(k) &= Lx(k), \end{aligned} \tag{2.1}$$

where $k_d = k - d$, and the positive integer d denotes the known state delay; $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is the measurable information, $w(k) \in \mathbb{R}^q$ and $v(k) \in \mathbb{R}^m$ are the disturbance input belonging to $l_2[0,N]$, $y(k) \in \mathbb{R}^m$ is the measurement output, and $z(k) \in \mathbb{R}^r$ is the signal to be estimated; the initial condition $x_0(s)$ $(s = -d, -d+1, \ldots, 0)$ is unknown; the matrices $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{r\times n}$, are real and known constant matrices.

In addition, $f(k, Fx(k), u(k))$ and $h(k, Hx(k_d), u(k))$ are assumed to satisfy the following Lipschitz conditions:

$$\begin{aligned} \|f(k, Fx(k), u(k)) - f(k, F\check{x}(k), u(k))\| &\le \alpha \|F(x(k) - \check{x}(k))\|, \\ \|h(k, Hx(k_d), u(k)) - h(k, H\check{x}(k_d), u(k))\| &\le \beta \|H(x(k_d) - \check{x}(k_d))\|, \end{aligned} \tag{2.2}$$

for all $k \in \{0, 1, \ldots, N\}$, $u(k) \in \mathbb{R}^p$ and $x(k), \check{x}(k), x(k_d), \check{x}(k_d) \in \mathbb{R}^n$. where $\alpha > 0$ and $\beta > 0$ are known Lipschitz constants, and F, H are real matrix with appropriate dimension.

The H_∞ estimation problem under investigation is stated as follows. Given the desired noise attenuation level $\gamma > 0$ and the observation $\{y(j)\}_{j=0}^{k}$, find an estimate $\check{z}(k \mid k)$ of the signal $z(k)$, if it exists, such that the following inequality is satisfied:

$$\sup_{(x_0, w, v) \neq 0} \frac{\sum_{k=0}^{N} \|\check{z}(k \mid k) - z(k)\|^2}{\sum_{k=-d}^{0} \|x_0(k)\|^2_{\Pi^{-1}(k)} + \sum_{k=0}^{N} \|w(k)\|^2 + \sum_{k=0}^{N} \|v(k)\|^2} < \gamma^2, \tag{2.3}$$

where $\Pi(k)$ $(k = -d, -d+1, \ldots, 0)$ is a given positive definite matrix function which reflects the relative uncertainty of the initial state $x_0(k)$ $(k = -d, -d+1, \ldots, 0)$ to the input and measurement noises.

Remark 2.1. For the sake of simplicity, the initial state estimate $\hat{x}_0(k)$ $(k = -d, -d+1, \ldots, 0)$ is assumed to be zero in inequality (2.3).

Remark 2.2. Although the system given in [20] is different from the one given in this paper, the problem mentioned in [20] can also be solved by using the presented approach. The resolvent first converts the system given in [20] into a delay-free one by using the classical system augmentation approach, and then designs estimator by employing the similar but easier technical line with our paper.

3. Main Results

In this section, the Krein space-based approach is proposed to design the H_∞ estimator for Lipschitz nonlinear systems. To begin with, the H_∞ estimation problem (2.3) and the Lipschitz conditions (2.2) are combined in an indefinite quadratic form, and the nonlinearities are assumed to be obtained by $\{y(i)\}_{i=0}^{k}$ at the time step k. Then, an equivalent Krein space problem is constructed by introducing an imaginary Krein space stochastic system. Finally, based on projection formula and innovation analysis approach in the Krein space, the recursive estimator is derived.

3.1. Construct a Partially Equivalent Krein Space Problem

It is proved in this subsection that the H_∞ estimation problem can be reduced to a positive minimum problem of indefinite quadratic form, and the minimum can be obtained by using the Krein space-based approach.

Since the denominator of the left side of (2.3) is positive, the inequality (2.3) is equivalent to

$$\underbrace{\sum_{k=-d}^{0}\|x_0(k)\|^2_{\Pi^{-1}(k)} + \sum_{k=0}^{N}\|w(k)\|^2 + \sum_{k=0}^{N}\|v(k)\|^2 - \gamma^{-2}\sum_{k=0}^{N}\|v_z(k)\|^2}_{\triangleq J_N^*} > 0, \quad \forall (x_0, w, v) \neq 0, \tag{3.1}$$

where $v_z(k) = \check{z}(k \mid k) - z(k)$.

Moreover, we denote

$$\begin{aligned} z_f(k) &= Fx(k), \qquad \check{z}_f(k \mid k) = F\check{x}(k \mid k), \\ z_h(k_d) &= Hx(k_d), \qquad \check{z}_h(k_d \mid k) = H\check{x}(k_d \mid k), \end{aligned} \tag{3.2}$$

where $\check{z}_f(k \mid k)$ and $\check{z}_h(k_d \mid k)$ denote the optimal estimation of $z_f(k)$ and $z_h(k_d)$ based on the observation $\{y(j)\}_{j=0}^{k}$, respectively. And, let

$$\begin{aligned} w_f(k) &= f\big(k, z_f(k), u(k)\big) - f\big(k, \check{z}_f(k \mid k), u(k)\big), \\ w_h(k_d) &= h(k, z_h(k_d), u(k)) - h(k, \check{z}_h(k_d \mid k), u(k)), \\ v_{z_f}(k) &= \check{z}_f(k \mid k) - z_f(k), \\ v_{z_h}(k_d) &= \check{z}_h(k_d \mid k) - z_h(k_d). \end{aligned} \tag{3.3}$$

From the Lipschitz conditions (2.2), we derive that

$$\underbrace{J_N^* + \sum_{k=0}^{N}\left\|w_f(k)\right\|^2 + \sum_{k=0}^{N}\|w_h(k_d)\|^2 - \alpha^2\sum_{k=0}^{N}\left\|v_{z_f}(k)\right\|^2 - \beta^2\sum_{k=0}^{N}\|v_{z_h}(k_d)\|^2}_{\triangleq J_N} \leq J_N^*. \tag{3.4}$$

Note that the left side of (3.1) and (3.4), J_N, can be recast into the form

$$\begin{aligned} J_N = &\sum_{k=-d}^{0}\|x_0(k)\|^2_{\overline{\Pi}^{-1}(k)} + \sum_{k=0}^{N}\|\overline{w}(k)\|^2 + \sum_{k=0}^{N}\|v(k)\|^2 \\ &- \gamma^{-2}\sum_{k=0}^{N}\|v_z(k)\|^2 - \alpha^2\sum_{k=0}^{N}\left\|v_{z_f}(k)\right\|^2 - \beta^2\sum_{k=d}^{N}\|v_{z_h}(k_d)\|^2, \end{aligned} \tag{3.5}$$

where

$$\begin{gathered} \overline{\Pi}(k) = \begin{cases} \left(\Pi^{-1}(k) - \beta^2 H^T H\right)^{-1}, & k = -d, \ldots, -1, \\ \Pi(k), & k = 0, \end{cases} \\ \overline{w}(k) = \left[w^T(k) \;\; w_f^T(k) \;\; w_h^T(k_d)\right]^T. \end{gathered} \tag{3.6}$$

Since $J_N \leq J_N^*$, it is natural to see that if $J_N > 0$ then the H_∞ estimation problem (2.3) is satisfied, that is, $J_N^* > 0$. Hence, the H_∞ estimation problem (2.3) can be converted into finding the estimate sequence $\{\{\check{z}(k \mid k)\}_{k=0}^N; \{\check{z}_f(k \mid k)\}_{k=0}^N; \{\check{z}_h(k_d \mid k)\}_{k=d}^N\}$ such that J_N has a minimum with respect to $\{x_0, \overline{w}\}$ and the minimum of J_N is positive. As mentioned in [25, 26], the formulated H_∞ estimation problem can be solved by employing the Krein space approach.

Introduce the following Krein space stochastic system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + A_d\mathbf{x}(k_d) + f\left(k, \check{\mathbf{z}}_f(k \mid k), \mathbf{u}(k)\right) \\ &\quad + h(k, \check{\mathbf{z}}_h(k_d \mid k), \mathbf{u}(k)) + \overline{B}\overline{\mathbf{w}}(k), \\ \mathbf{y}(k) &= C\mathbf{x}(k) + \mathbf{v}(k), \\ \check{\mathbf{z}}_f(k \mid k) &= F\mathbf{x}(k) + \mathbf{v}_{z_f}(k), \\ \check{\mathbf{z}}(k \mid k) &= L\mathbf{x}(k) + \mathbf{v}_z(k), \\ \check{\mathbf{z}}_h(k_d \mid k) &= H\mathbf{x}(k_d) + \mathbf{v}_{z_h}(k_d), \quad k \geq d, \end{aligned} \tag{3.7}$$

where $\overline{B} = [B \;\; I \;\; I]$; the initial state $\mathbf{x}_0(s)$ $(s = -d, -d+1, \ldots, 0)$ and $\overline{\mathbf{w}}(k)$, $\mathbf{v}(k)$, $\mathbf{v}_{z_f}(k)$, $\mathbf{v}_z(k)$ and $\mathbf{v}_{z_h}(k)$ are mutually uncorrelated white noises with zero means and known covariance matrices $\overline{\Pi}(s)$, $Q_{\overline{w}}(k) = I$, $Q_v(k) = I$, $Q_{v_{zf}}(k) = -\alpha^{-2}I$, $Q_{v_z}(k) = -\gamma^2 I$, and $Q_{v_{zh}}(k) = -\beta^{-2}I$; $\check{\mathbf{z}}_f(k \mid k)$, $\check{\mathbf{z}}(k \mid k)$ and $\check{\mathbf{z}}_h(k_d \mid k)$ are regarded as the imaginary measurement at time k for the linear combination $F\mathbf{x}(k)$, $L\mathbf{x}(k)$, and $H\mathbf{x}(k_d)$, respectively.

Let

$$\mathbf{y}_z(k) = \begin{cases} \left[\mathbf{y}^T(k) \;\; \check{\mathbf{z}}_m^T(k \mid k)\right]^T, & 0 \le k < d, \\ \left[\mathbf{y}^T(k) \;\; \check{\mathbf{z}}_m^T(k \mid k) \;\; \check{\mathbf{z}}_h^T(k_d \mid k)\right]^T, & k \ge d, \end{cases}$$

$$\mathbf{v}_{z,a}(k) = \begin{cases} \left[\mathbf{v}^T(k) \;\; \mathbf{v}_{z_f}^T(k) \;\; \mathbf{v}_z^T(k)\right]^T, & 0 \le k < d, \\ \left[\mathbf{v}^T(k) \;\; \mathbf{v}_{z_f}^T(k) \;\; \mathbf{v}_z^T(k) \;\; \mathbf{v}_{z_h}^T(k_d)\right]^T, & k \ge d, \end{cases} \tag{3.8}$$

$$\check{\mathbf{z}}_m(k \mid k) = \left[\check{\mathbf{z}}_f^T(k \mid k) \;\; \check{\mathbf{z}}^T(k \mid k)\right]^T.$$

Definition 3.1. The estimator $\hat{\mathbf{y}}(i \mid i-1)$ denotes the optimal estimation of $\mathbf{y}(i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}\}$; the estimator $\hat{\mathbf{z}}_m(i \mid i)$ denotes the optimal estimation of $\check{\mathbf{z}}_m(i \mid i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i)\}$; the estimator $\hat{\mathbf{z}}_h(i_d \mid i)$ denotes the optimal estimation of $\check{\mathbf{z}}_h(i_d \mid i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i), \check{\mathbf{z}}_m(i \mid i)\}$.

Furthermore, introduce the following stochastic vectors and the corresponding covariance matrices

$$\begin{aligned} \tilde{\mathbf{y}}(i \mid i-1) &= \mathbf{y}(i) - \hat{\mathbf{y}}(ii-1), & R_{\tilde{y}}(ii-1) &= \langle \tilde{\mathbf{y}}(ii-1), \tilde{\mathbf{y}}(ii-1) \rangle, \\ \tilde{\mathbf{z}}_m(i \mid i) &= \check{\mathbf{z}}_m(ii) - \hat{\mathbf{z}}_m(ii), & R_{\tilde{z}_m}(ii) &= \langle \tilde{\mathbf{z}}_m(ii), \tilde{\mathbf{z}}_m(ii) \rangle, \\ \tilde{\mathbf{z}}_h(i_d \mid i) &= \check{\mathbf{z}}_h(i_d i) - \hat{\mathbf{z}}_h(i_d i), & R_{\tilde{z}_h}(i_d i) &= \langle \tilde{\mathbf{z}}_h(i_d i), \tilde{\mathbf{z}}_h(i_d i) \rangle. \end{aligned} \tag{3.9}$$

And, denote

$$\tilde{\mathbf{y}}_z(i) = \begin{cases} \left[\tilde{\mathbf{y}}^T(i \mid i-1) \;\; \tilde{\mathbf{z}}_m^T(i \mid i)\right]^T, & 0 \le i < d, \\ \left[\tilde{\mathbf{y}}^T(i \mid i-1) \;\; \tilde{\mathbf{z}}_m^T(i \mid i) \;\; \tilde{\mathbf{z}}_h^T(i_d \mid i)\right]^T, & i \ge d, \end{cases} \tag{3.10}$$

$$R_{\tilde{y}_z}(i) = \langle \tilde{\mathbf{y}}_z(i), \tilde{\mathbf{y}}_z(i) \rangle.$$

For calculating the minimum of J_N, we present the following Lemma 3.2.

Lemma 3.2. $\{\{\tilde{\mathbf{y}}_z(i)\}_{i=0}^{k}\}$ *is the innovation sequence which spans the same linear space as that of* $\mathcal{L}\{\{\mathbf{y}_z(i)\}_{i=0}^{k}\}$.

Proof. From Definition 3.1 and (3.9), $\tilde{\mathbf{y}}(i \mid i-1)$, $\tilde{\mathbf{z}}_m(i \mid i)$ and $\tilde{\mathbf{z}}_h(i_d \mid i)$ are the linear combination of the observation sequence $\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i)\}$, $\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i), \check{\mathbf{z}}_m(i \mid i)\}$, and $\{\{\mathbf{y}_z(j)\}_{j=0}^{i}\}$, respectively. Conversely, $\mathbf{y}(i)$, $\check{\mathbf{z}}_m(i \mid i)$ and $\check{\mathbf{z}}_h(i_d \mid i)$ can be given by the linear combination of $\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1)\}$, $\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1), \tilde{\mathbf{z}}_m(i \mid i)\}$ and $\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i}\}$, respectively. Hence,

$$\mathcal{L}\left\{\{\tilde{\mathbf{y}}_z(i)\}_{i=0}^{k}\right\} = \mathcal{L}\left\{\{\mathbf{y}_z(i)\}_{i=0}^{k}\right\}. \tag{3.11}$$

It is also shown by (3.9) that $\tilde{\mathbf{y}}(i \mid i-1)$, $\tilde{\mathbf{z}}_m(i \mid i)$ and $\tilde{\mathbf{z}}_h(i_d \mid i)$ satisfy

$$\begin{aligned}
\tilde{\mathbf{y}}(i \mid i-1) &\perp \mathcal{L}\left\{\left\{\mathbf{y}_z(j)\right\}_{j=0}^{i-1}\right\}, \\
\tilde{\mathbf{z}}_m(i \mid i) &\perp \mathcal{L}\left\{\left\{\mathbf{y}_z(j)\right\}_{j=0}^{i-1}; \mathbf{y}(i)\right\}, \\
\tilde{\mathbf{z}}_h(i_d \mid i) &\perp \mathcal{L}\left\{\left\{\mathbf{y}_z(j)\right\}_{j=0}^{i-1}; \mathbf{y}(i), \tilde{\mathbf{z}}_m(i \mid i)\right\}.
\end{aligned} \tag{3.12}$$

Consequently,

$$\begin{aligned}
\tilde{\mathbf{y}}(i \mid i-1) &\perp \mathcal{L}\left\{\left\{\tilde{\mathbf{y}}_z(j)\right\}_{j=0}^{i-1}\right\}, \\
\tilde{\mathbf{z}}_m(i \mid i) &\perp \mathcal{L}\left\{\left\{\tilde{\mathbf{y}}_z(j)\right\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1)\right\}, \\
\tilde{\mathbf{z}}_h(i_d \mid i) &\perp \mathcal{L}\left\{\left\{\tilde{\mathbf{y}}_z(j)\right\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1), \tilde{\mathbf{z}}_m(i \mid i)\right\}.
\end{aligned} \tag{3.13}$$

This completes the proof. □

Now, an existence condition and a solution to the minimum of J_N are derived as follows.

Theorem 3.3. *Consider system* (2.1), *given a scalar* $\gamma > 0$ *and the positive definite matrix* $\Pi(k)$ $(k = -d, -d+1, \ldots, 0)$, *then* J_N *has the minimum if only if*

$$\begin{aligned}
R_{\tilde{y}}(k \mid k-1) &> 0, \quad 0 \le k \le N, \\
R_{\tilde{z}_m}(k \mid k) &< 0, \quad 0 \le k \le N, \\
R_{\tilde{z}_h}(k_d \mid k) &< 0, \quad d \le k \le N.
\end{aligned} \tag{3.14}$$

In this case the minimum value of J_N *is given by*

$$\begin{aligned}
\min J_N = &\sum_{k=0}^{N} \tilde{y}^T(k \mid k-1) R_{\tilde{y}}^{-1}(k \mid k-1) \tilde{y}(k \mid k-1) + \sum_{k=0}^{N} \tilde{z}_m^T(k \mid k) R_{\tilde{z}_m}^{-1}(k \mid k) \tilde{z}_m(k \mid k) \\
&+ \sum_{k=d}^{N} \tilde{z}_h^T(k_d \mid k) R_{\tilde{z}_h}^{-1}(k_d \mid k) \tilde{z}_h(k_d \mid k),
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\tilde{y}(k \mid k-1) &= y(k) - \hat{y}(k \mid k-1), \\
\tilde{z}_m(k \mid k) &= \breve{z}_m(k \mid k) - \hat{z}_m(k \mid k), \\
\tilde{z}_h(k_d \mid k) &= \breve{z}_h(k_d \mid k) - \hat{z}_h(k_d \mid k), \\
\breve{z}_m(k \mid k) &= \left[\breve{z}_f^T(k \mid k) \quad \breve{z}^T(k \mid k)\right]^T,
\end{aligned} \tag{3.16}$$

$\hat{y}(k \mid k-1)$ is obtained from the Krein space projection of $\mathbf{y}(k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}\}$, $\hat{z}_m(k \mid k)$ is obtained from the Krein space projection of $\check{\mathbf{z}}_m(k \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1};\ \mathbf{y}(k)\}$, and $\hat{z}_h(k_d \mid k)$ is obtained from the Krein space projection of $\check{\mathbf{z}}_h(k_d \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1};\ \mathbf{y}(k), \check{\mathbf{z}}_m(k \mid k)\}$.

Proof. Based on the definition (3.2) and (3.3), the state equation in system (2.1) can be rewritten as

$$\begin{aligned} x(k+1) = {} & Ax(k) + A_d x(k_d) + f\left(k, \check{z}_f(k \mid k), u(k)\right) \\ & + h(k, \check{z}_h(k_d \mid k), u(k)) + \overline{B}\overline{w}(k). \end{aligned} \tag{3.17}$$

In this case, it is assumed that $f(k, \check{z}_f(k \mid k), u(k))$ and $h(k, \check{z}_h(k_d \mid k), u(k))$ are known at time k. Then, we define

$$y_z(k) = \begin{cases} \left[y^T(k) \quad \check{z}_f^T(k \mid k) \quad \check{z}^T(k \mid k)\right]^T, & 0 \le k < d, \\ \left[y^T(k) \quad \check{z}_f^T(k \mid k) \quad \check{z}^T(k \mid k) \quad \check{z}_h^T(k_d \mid k)\right]^T, & k \ge d. \end{cases} \tag{3.18}$$

By introducing an augmented state

$$x_a(k) = \left[x^T(k) \quad x^T(k-1) \quad \cdots \quad x^T(k-d)\right]^T, \tag{3.19}$$

we obtain an augmented state-space model

$$\begin{aligned} x_a(k+1) &= A_a x_a(k) + B_{u,a}\overline{u}(k) + \overline{B}_a\overline{w}(k), \\ y_z(k) &= C_{z,a}(k) x_a(k) + v_{z,a}(k), \end{aligned} \tag{3.20}$$

where

$$A_a = \begin{bmatrix} A & 0 & \cdots & 0 & A_d \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \qquad B_{u,a} = \begin{bmatrix} I & I \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad \overline{B}_a = \begin{bmatrix} \overline{B} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$
\begin{aligned}
\mathrm{C}_{z,a}(k) &= \begin{cases} \begin{bmatrix} C & 0 & \cdots & 0 \\ F & 0 & \cdots & 0 \\ L & 0 & \cdots & 0 \end{bmatrix}, & 0 \le k < d, \\ \begin{bmatrix} C & 0 & \cdots & 0 \\ F & 0 & \cdots & 0 \\ L & 0 & \cdots & 0 \\ 0 & \cdots & 0 & H \end{bmatrix}, & k \ge d, \end{cases} \\
v_{z,a}(k) &= \begin{cases} \left[v^T(k) \;\; v_{z_f}^T(k) \;\; v_z^T(k) \right]^T, & 0 \le k < d, \\ \left[v^T(k) \;\; v_{z_f}^T(k) \;\; v_z^T(k) \;\; v_{z_h}^T(k_d) \right]^T, & k \ge d, \end{cases} \\
\overline{u}(k) &= \left[f^T\left(k, \breve{z}_f(k \mid k), u(k)\right) \;\; h^T(k, \breve{z}_h(k_d \mid k), u(k)) \right]^T.
\end{aligned}
\tag{3.21}
$$

Additionally, we can rewrite J_N as

$$
J_N = \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ v_{z,aN} \end{bmatrix}^T \begin{bmatrix} P_a(0) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{v_{z,aN}} \end{bmatrix}^{-1} \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ v_{z,aN} \end{bmatrix},
\tag{3.22}
$$

where

$$
\begin{aligned}
P_a(0) &= \operatorname{diag}\left\{ \overline{\Pi}(0), \overline{\Pi}(-1), \ldots, \overline{\Pi}(-d) \right\}, \\
\overline{w}_N &= \left[\overline{w}^T(0) \;\; \overline{w}^T(1) \;\; \cdots \;\; \overline{w}^T(N) \right]^T, \\
v_{z,aN} &= \left[v_{z,a}^T(0) \;\; v_{z,a}^T(1) \;\; \cdots \;\; v_{z,a}^T(N) \right]^T, \\
Q_{v_{z,aN}} &= \operatorname{diag}\left\{ Q_{v_{z,a}}(0), Q_{v_{z,a}}(1), \ldots, Q_{v_{z,a}}(N) \right\}, \\
Q_{v_{z,a}}(k) &= \begin{cases} \operatorname{diag}\left\{ I, -\gamma^2, -\alpha^{-2} \right\}, & 0 \le k < d, \\ \operatorname{diag}\left\{ I, -\gamma^2, -\alpha^{-2}, -\beta^{-2} \right\}, & k \ge d. \end{cases}
\end{aligned}
\tag{3.23}
$$

Define the following state transition matrix

$$
\begin{aligned}
\Phi(k+1, m) &= A_a \Phi(k, m), \\
\Phi(m, m) &= I,
\end{aligned}
\tag{3.24}
$$

and let

$$y_{zN} = \left[y_z^T(0) \;\; y_z^T(1) \;\; \cdots \;\; y_z^T(N)\right]^T,$$
$$\overline{u}_N = \left[\overline{u}^T(0) \;\; \overline{u}^T(1) \;\; \cdots \;\; \overline{u}^T(N)\right]^T. \tag{3.25}$$

Using (3.20) and (3.24), we have

$$y_{zN} = \Psi_{0N} x_a(0) + \Psi_{\overline{u}N}\overline{u}_N + \Psi_{\overline{w}N}\overline{w}_N + v_{z,aN}, \tag{3.26}$$

where

$$\Psi_{0N} = \begin{bmatrix} C_{z,a}(0)\Phi(0,0) \\ C_{z,a}(1)\Phi(1,0) \\ \vdots \\ C_{z,a}(N)\Phi(N,0) \end{bmatrix}, \quad \Psi_{\overline{u}N} = \begin{bmatrix} \varphi_{00} & \varphi_{01} & \cdots & \varphi_{0N} \\ \varphi_{10} & \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{N0} & \varphi_{N1} & \cdots & \varphi_{NN} \end{bmatrix},$$
$$\varphi_{ij} = \begin{cases} C_{z,a}(i)\Phi(i,j+1)B_{u,a}, & i > j, \\ 0, & i \le j. \end{cases} \tag{3.27}$$

The matrix $\Psi_{\overline{w}N}$ is derived by replacing $B_{u,a}$ in $\Psi_{\overline{u}N}$ with $\overline{B}_a$.

Thus, J_N can be reexpressed as

$$J_N = \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}^T \left\{ \Gamma_N \begin{bmatrix} P_a(0) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{v_{z,aN}} \end{bmatrix} \Gamma_N^T \right\}^{-1} \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}, \tag{3.28}$$

where

$$\overline{y}_{zN} = y_{zN} - \Psi_{\overline{u}N}\overline{u}_N,$$
$$\Gamma_N = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \Psi_{0N} & \Psi_{\overline{w}N} & I \end{bmatrix}. \tag{3.29}$$

Considering the Krein space stochastic system defined by (3.7) and state transition matrix (3.24), we have

$$\mathbf{y}_{zN} = \Psi_{0N}\mathbf{x}_a(0) + \Psi_{\overline{u}N}\overline{\mathbf{u}}_N + \Psi_{\overline{w}N}\overline{\mathbf{w}}_N + \mathbf{v}_{z,aN}, \tag{3.30}$$

where matrices Ψ_{0N}, $\Psi_{\overline{u}N}$, and $\Psi_{\overline{w}N}$ are the same as given in (3.26), vectors $\mathbf{y}_{zN}$ and $\overline{\mathbf{u}}_N$ are, respectively, defined by replacing Euclidean space element y_z and $\overline{u}$ in y_{zN} and $\overline{u}_N$ given

by (3.25) with the Krein space element $\mathbf{y}_z$ and $\overline{\mathbf{u}}$, vectors $\overline{\mathbf{w}}_N$ and $\mathbf{v}_{z,aN}$ are also defined by replacing Euclidean space element $\overline{w}$ and $v_{z,a}$ in $\overline{w}_N$ and $v_{z,aN}$ given by (3.23) with the Krein space element $\overline{\mathbf{w}}$ and $\mathbf{v}_{z,a}$, and vector $\mathbf{x}_a(0)$ is given by replacing Euclidean space element x in $x_a(k)$ given by (3.19) with the Krein space element $\mathbf{x}$ when $k = 0$.

Using the stochastic characteristic of $\mathbf{x}_a(0)$, $\overline{\mathbf{w}}_N$ and $\mathbf{v}_{z,a}$, we have

$$J_N = \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}^T \left\langle \begin{bmatrix} \mathbf{x}_a(0) \\ \overline{\mathbf{w}}_N \\ \overline{\mathbf{y}}_{zN} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_a(0) \\ \overline{\mathbf{w}}_N \\ \overline{\mathbf{y}}_{zN} \end{bmatrix} \right\rangle^{-1} \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}, \tag{3.31}$$

where $\overline{\mathbf{y}}_{zN} = \mathbf{y}_{zN} - \Psi_{\overline{u}N}\overline{\mathbf{u}}_N$.

In the light of Theorem 2.4.2 and Lemma 2.4.3 in [26], J_N has a minimum over $\{x_a(0), \overline{w}_N\}$ if and only if $R_{\overline{y}_{zN}} = \langle \overline{\mathbf{y}}_{zN}, \overline{\mathbf{y}}_{zN} \rangle$ and $Q_{v_{z,aN}} = \langle \mathbf{v}_{z,aN}, \mathbf{v}_{z,aN} \rangle$ have the same inertia. Moreover, the minimum of J_N is given by

$$\min J_N = \overline{y}_{zN}^T R_{\overline{y}_{zN}}^{-1} \overline{y}_{zN}. \tag{3.32}$$

On the other hand, applying the Krein space projection formula, we have

$$\overline{\mathbf{y}}_{zN} = \Theta_N \widetilde{\mathbf{y}}_{zN}, \tag{3.33}$$

where

$$\widetilde{\mathbf{y}}_{zN} = \left[\widetilde{\mathbf{y}}_z^T(0) \;\; \widetilde{\mathbf{y}}_z^T(1) \;\; \cdots \;\; \widetilde{\mathbf{y}}_z^T(N)\right]^T,$$

$$\Theta_N = \begin{bmatrix} \theta_{00} & \theta_{01} & \cdots & \theta_{0N} \\ \theta_{10} & \theta_{11} & \cdots & \theta_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N0} & \theta_{N1} & \cdots & \theta_{NN} \end{bmatrix},$$

$$\theta_{ij} = \begin{cases} \langle \overline{\mathbf{y}}_z(i), \widetilde{\mathbf{y}}_z(j) \rangle R_{\widetilde{y}_z}^{-1}(j), & i > j \geq 0, \\ \begin{bmatrix} I & 0 \\ m_1 & I \end{bmatrix}, & d > i = j \geq 0, \\ \begin{bmatrix} I & 0 & 0 \\ m_1 & I & 0 \\ m_2 & m_3 & I \end{bmatrix}, & i = j \geq d, \\ 0, & 0 \leq i < j, \end{cases}$$

$$m_1 = \left\langle \overline{\widetilde{\mathbf{z}}}_m(i \mid i), \widetilde{\mathbf{y}}(j \mid j-1) \right\rangle R_{\widetilde{y}}^{-1}(j \mid j-1),$$

$$m_2 = \left\langle \overline{\widetilde{\mathbf{z}}}_h(i_d \mid i), \widetilde{\mathbf{y}}(j \mid j-1) \right\rangle R_{\widetilde{y}}^{-1}(j \mid j-1),$$

$$m_3 = \left\langle \overline{\widetilde{\mathbf{z}}}_h(i_d \mid i), \widetilde{\mathbf{z}}_m(j \mid j) \right\rangle R_{\widetilde{z}_m}^{-1}(j \mid j),$$

$$\overline{\mathbf{y}}_z(i) = \mathbf{y}_z(i) - \sum_{j=0}^{N} \varphi_{ij} \overline{\mathbf{u}}(j),$$

$$\overline{\widetilde{\mathbf{z}}}_m(i \mid i) = \widetilde{\mathbf{z}}_m(i \mid i) - \sum_{j=0}^{N} \varphi_{m,ij} \overline{\mathbf{u}}(j),$$

$$\overline{\widetilde{\mathbf{z}}}_h(i_d \mid i) = \widetilde{\mathbf{z}}_h(i_d \mid i) - \sum_{j=0}^{N} \varphi_{h,ij} \overline{\mathbf{u}}(j), \tag{3.34}$$

where $\varphi_{m,ij}$ is derived by replacing $C_{z,a}$ in φ_{ij} with $\left[\begin{smallmatrix} F & 0 & \cdots & 0 \\ L & 0 & \cdots & 0 \end{smallmatrix}\right]$, $\varphi_{h,ij}$ is derived by replacing $C_{z,a}$ in φ_{ij} with $[0 \;\; 0 \;\; \cdots \;\; H]$ Furthermore, it follows from (3.33) that

$$R_{\overline{y}_{zN}} = \Theta_N R_{\widetilde{y}_{zN}} \Theta_N^T, \qquad \overline{y}_{zN} = \Theta_N \widetilde{y}_{zN}, \tag{3.35}$$

where

$$R_{\widetilde{y}_{zN}} = \langle \widetilde{\mathbf{y}}_{zN}, \widetilde{\mathbf{y}}_{zN} \rangle,$$

$$\widetilde{y}_{zN} = \left[\widetilde{y}_z^T(0) \;\; \widetilde{y}_z^T(1) \;\; \cdots \;\; \widetilde{y}_z^T(N) \right]^T,$$

$$\widetilde{y}_z(i) = \begin{cases} \left[\widetilde{y}^T(i \mid i-1) \;\; \widetilde{z}_m^T(i \mid i) \right]^T, & 0 \leq i < d, \\ \left[\widetilde{y}^T(i \mid i-1) \;\; \widetilde{z}_m^T(i \mid i) \;\; \widetilde{z}_h^T(i_d \mid i) \right]^T, & i \geq d. \end{cases} \tag{3.36}$$

Since matrix Θ_N is nonsingular, it follows from (3.35) that $R_{\overline{y}_{zN}}$ and $R_{\tilde{y}_{zN}}$ are congruent, which also means that $R_{\overline{y}_{zN}}$ and $R_{\tilde{y}_{zN}}$ have the same inertia. Note that both $R_{\tilde{y}_{zN}}$ and $Q_{v_{z,aN}}$ are block-diagonal matrices, and

$$R_{\tilde{y}_z}(k) = \begin{cases} \operatorname{diag}\{R_{\tilde{y}}(k \mid k-1), R_{\tilde{z}_m}(k \mid k)\}, & 0 \le k < d, \\ \operatorname{diag}\{R_{\tilde{y}}(k \mid k-1), R_{\tilde{z}_m}(k \mid k), R_{\tilde{z}_h}(k_d \mid k)\}, & k \le d, \end{cases} \tag{3.37}$$

$Q_{v_{z,a}}(k)$ is given by (3.23). It follows that $R_{\tilde{y}_{zN}}$ and $Q_{v_{z,aN}}$ have the same inertia if and only if $R_{\tilde{y}}(k \mid k-1) > 0$ $(0 \le k \le N)$, $R_{\tilde{z}_m}(k \mid k) < 0$ $(0 \le k \le N)$ and $R_{\tilde{z}_h}(k_d \mid k) < 0$ $(d \le k \le N)$.

Therefore, J_N subject to system (2.1) with Lipschitz conditions (2.2) has the minimum if and only if $R_{\tilde{y}}(k \mid k-1) > 0$ $(0 \le k \le N)$, $R_{\tilde{z}_m}(k \mid k) < 0$ $(0 \le k \le N)$ and $R_{\tilde{z}_h}(k_d \mid k) < 0$ $(d \le k \le N)$. Moreover, the minimum value of J_N can be rewritten as

$$\begin{aligned} \min J_N &= \overline{y}_{zN}^T R_{\overline{y}_{zN}}^{-1} \overline{y}_{zN} = \tilde{y}_{zN}^T R_{\tilde{y}_{zN}}^{-1} \tilde{y}_{zN} \\ &= \sum_{k=0}^{N} \tilde{y}^T(k \mid k-1) R_{\tilde{y}}^{-1}(k \mid k-1) \tilde{y}(k \mid k-1) + \sum_{k=0}^{N} \tilde{z}_m^T(k \mid k) R_{\tilde{z}_m}^{-1}(k \mid k) \tilde{z}_m(k \mid k) \\ &\quad + \sum_{k=d}^{N} \tilde{z}_h^T(k_d \mid k) R_{\tilde{z}_h}^{-1}(k_d \mid k) \tilde{z}_h(k_d \mid k). \end{aligned} \tag{3.38}$$

The proof is completed. □

Remark 3.4. Due to the built innovation sequence $\{\{\tilde{\mathbf{y}}_z(i)\}_{i=0}^{k}\}$ in Lemma 3.2, the form of the minimum on indefinite quadratic form J_N is different from the one given in [26–28]. It is shown from (3.15) that the estimation errors $\tilde{y}(k \mid k-1)$, $\tilde{z}_m(k \mid k)$ and $\tilde{z}_h(k_d \mid k)$ are mutually uncorrelated, which will make the design of H_∞ estimator much easier than the one given in [26–28].

3.2. Solution of the H_∞ Estimation Problem

In this subsection, the Kalman-like recursive H_∞ estimator is presented by using orthogonal projection in the Krein space.

Denote

$$\begin{aligned} \mathbf{y}_0(i) &= \mathbf{y}(i), \\ \mathbf{y}_1(i) &= \left[\mathbf{y}^T(i) \;\; \check{\mathbf{z}}_m^T(i \mid i)\right]^T, \\ \mathbf{y}_2(i) &= \left[\mathbf{y}^T(i) \;\; \check{\mathbf{z}}_m^T(i \mid i) \;\; \check{\mathbf{z}}_h^T(i \mid i+d)\right]^T. \end{aligned} \tag{3.39}$$

Observe from (3.8), we have

$$\begin{aligned}
\mathcal{L}\left\{\{\mathbf{y}_z(i)\}_{i=0}^{j}\right\} &= \mathcal{L}\left\{\{\mathbf{y}_1(i)\}_{i=0}^{j}\right\}, \quad 0 \le j < d, \\
\mathcal{L}\left\{\{\mathbf{y}_z(i)\}_{i=0}^{j}\right\} &= \mathcal{L}\left\{\{\mathbf{y}_2(i)\}_{i=0}^{j_d}\right\}; \left\{\{\mathbf{y}_1(i)\}_{i=j_d+1}^{j}\right\}, \quad j \ge d.
\end{aligned} \tag{3.40}$$

Definition 3.5. Given $k \ge d$, the estimator $\widehat{\xi}(i \mid j, 2)$ for $0 \le j < k_d$ denotes the optimal estimate of $\xi(i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_2(s)\}_{s=0}^{j}\}$, and the estimator $\widehat{\xi}(i \mid j, 1)$ for $k_d \le j \le k$ denotes the optimal estimate of $\xi(i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_2(s)\}_{s=0}^{k_d-1}; \{\mathbf{y}_1(\tau)\}_{\tau=k_d}^{j}\}$. For simplicity, we use $\widehat{\xi}(i, 2)$ to denote $\widehat{\xi}(i \mid i-1, 2)$, and use $\widehat{\xi}(i, 1)$ to denote $\widehat{\xi}(i \mid i-1, 1)$ throughout the paper.

Based on the above definition, we introduce the following stochastic sequence and the corresponding covariance matrices

$$\begin{aligned}
\widetilde{\mathbf{y}}_2(i,2) &= \mathbf{y}_2(i) - \widehat{\mathbf{y}}_2(i,2), \qquad R_{\widetilde{y}_2}(i,2) = \langle \widetilde{\mathbf{y}}_2(i,2), \widetilde{\mathbf{y}}_2(i,2) \rangle, \\
\widetilde{\mathbf{y}}_1(i,1) &= \mathbf{y}_1(i) - \widehat{\mathbf{y}}_1(i,1), \qquad R_{\widetilde{y}_1}(i,1) = \langle \widetilde{\mathbf{y}}_1(i,1), \widetilde{\mathbf{y}}_1(i,1) \rangle, \\
\widetilde{\mathbf{y}}_0(i,0) &= \mathbf{y}_0(i) - \widehat{\mathbf{y}}_0(i,1), \qquad R_{\widetilde{y}_0}(i,0) = \langle \widetilde{\mathbf{y}}_0(\mathrm{i},0), \widetilde{\mathbf{y}}_0(i,0) \rangle.
\end{aligned} \tag{3.41}$$

Similar to the proof of Lemma 2.2.1 in [27], we can obtain that $\{\widetilde{\mathbf{y}}_2(0,2), \ldots, \widetilde{\mathbf{y}}_2(k_d - 1,2); \widetilde{\mathbf{y}}_1(k_d,1), \ldots, \widetilde{\mathbf{y}}_1(k-1,1)\}$ is the innovation sequence which is a mutually uncorrelated white noise sequence and spans the same linear space as $\mathcal{L}\{\mathbf{y}_2(0), \ldots, \mathbf{y}_2(k_d - 1); \mathbf{y}_1(k_l), \ldots, \mathbf{y}_1(k-1)\}$ or equivalently $\mathcal{L}\{\mathbf{y}_z(0), \ldots, \mathbf{y}_z(k-1)\}$.

Applying projection formula in the Krein space, $\widehat{\mathbf{x}}(i,2)$ $(i = 0, 1, \ldots, k_d)$ is computed recursively as

$$\begin{aligned}
\widehat{\mathbf{x}}(i+1,2) &= \sum_{j=0}^{i} \langle \mathbf{x}(i+1), \widetilde{\mathbf{y}}_2(j,2) \rangle R_{\widetilde{y}_2}^{-1}(j,2) \widetilde{\mathbf{y}}_2(j,2) \\
&= A\widehat{\mathbf{x}}(i \mid i,2) + A_d \widehat{\mathbf{x}}(i_d \mid i,2) + f\left(i, \check{\mathbf{z}}_f(i \mid i), \mathbf{u}(i)\right) \\
&\quad + h(i, \check{\mathbf{z}}_h(i_d \mid i), \mathbf{u}(i)), \quad i = 0, 1, \ldots, k_d - 1, \\
\widehat{\mathbf{x}}(\tau,2) &= 0, \quad (\tau = -d, -d+1, \ldots, 0).
\end{aligned} \tag{3.42, 3.43}$$

Note that

$$\begin{aligned}
\widehat{\mathbf{x}}(i \mid i,2) &= \widehat{\mathbf{x}}(i,2) + P_2(i,i) C_2^T R_{\widetilde{y}_2}^{-1}(i,2) \widetilde{\mathbf{y}}_2(i,2), \\
\widehat{\mathbf{x}}(i_d \mid i,2) &= \widehat{\mathbf{x}}(i_d,2) + \sum_{j=i_d}^{i} P_2(i_d, j) C_2^T R_{\widetilde{y}_2}^{-1}(j,2) \widetilde{\mathbf{y}}_2(j,2),
\end{aligned} \tag{3.44}$$

where

$$\begin{aligned}
C_2 &= \left[C^T \quad F^T \quad L^T \quad H^T\right]^T, \\
P_2(i,j) &= \langle \mathbf{e}(i,2), \mathbf{e}(j,2) \rangle, \\
\mathbf{e}(i,2) &= \mathbf{x}(i) - \hat{\mathbf{x}}(i,2), \\
R_{\tilde{y}_2}(i,2) &= C_2 P_2(i,i) C_2^T + Q_{v_2}(i), \\
Q_{v_2}(i) &= \operatorname{diag}\left\{I, -\alpha^{-2} I, -\gamma^2 I, -\beta^{-2} I\right\}.
\end{aligned} \tag{3.45}$$

Substituting (3.44) into (3.43), we have

$$\begin{aligned}
\hat{\mathbf{x}}(i+1,2) &= A\hat{\mathbf{x}}(i,2) + A_d \hat{\mathbf{x}}(i_d,2) + f\left(i, \check{\mathbf{z}}_f(i \mid i), \mathbf{u}(i)\right) + h(i, \check{\mathbf{z}}_h(i_d \mid i), \mathbf{u}(i)) \\
&\quad + A_d \sum_{j=i_d}^{i-1} P_2(i_d, j) C_2^T R_{\tilde{y}_2}^{-1}(j,2) \tilde{\mathbf{y}}_2(j,2) + K_2(i) \tilde{\mathbf{y}}_2(i,2), \\
K_2(i) &= A_d P_2(i_d, i) C_2^T R_{\tilde{y}_2}^{-1}(i,2) + A P_2(i,i) C_2^T R_{\tilde{y}_2}^{-1}(i,2).
\end{aligned} \tag{3.46}$$

Moreover, taking into account (3.7) and (3.46), we obtain

$$\begin{aligned}
\mathbf{e}(i+1,2) &= A\mathbf{e}(i,2) + A_d \mathbf{e}(i_d,2) + \overline{B}\overline{\mathbf{w}}(i) - K_2(i)\tilde{\mathbf{y}}_2(i,2) \\
&\quad - A_d \sum_{j=i_d}^{i-1} P_2(i_d, j) C_2^T R_{\tilde{y}_2}^{-1}(j,2) \tilde{\mathbf{y}}_2(j,2), \quad i = 0, 1, \ldots, k_d - 1.
\end{aligned} \tag{3.47}$$

Consequently,

$$\begin{aligned}
P_2(i-j, i+1) &= \langle \mathbf{e}(i-j,2), \mathbf{e}(i+1,2) \rangle \\
&= P_2(i-j,i) A^T + P_2^T(i_d, i-j) A_d^T - P_2(i-j,i) C_2^T K_2^T(i) \\
&\quad - \sum_{t=i-j}^{i-1} P_2(i-j,t) C_2^T R_{\tilde{y}_2}^{-1}(t,2) C_2 P_2^T(i_d,t) A_d^T, \quad j = 0, 1, \ldots, d, \\
P_2(i+1, i+1) &= \langle \mathbf{e}(i+1,2), \mathbf{e}(i+1,2) \rangle \\
&= A P_2(i, i+1) + A_d P_2(i_d, i+1) + \overline{B} Q_{\overline{w}}(i) \overline{B}^T,
\end{aligned} \tag{3.48}$$

where $Q_{\overline{w}}(i) = I$. Thus, $P_2(i,i)$ $(i = 0,1,\ldots,k_d)$ can be computed recursively as

$$\begin{aligned} P_2(i-j,i+1) &= P_2(i-j,i)A^T + P_2^T(i_d,i-j)A_d^T - P_2(i-j,i)C_2^T K_2^T(i) \\ &\quad - \sum_{t=i-j}^{i-1} P_2(i-j,t)C_2^T R_{\tilde{y}_2}^{-1}(t,2)C_2 P_2^T(i_d,t)A_d^T, \\ P_2(i+1,i+1) &= AP_2(i,i+1) + A_d P_2(i_d,i+1) + \overline{B}Q_{\overline{w}}(i)\overline{B}^T, \quad j = 0,1,\ldots,d. \end{aligned} \tag{3.49}$$

Similarly, employing the projection formula in the Krein space, the optimal estimator $\hat{\mathbf{x}}(i,1)$ $(i = k_d+1,\ldots,k)$ can be computed by

$$\begin{aligned} \hat{\mathbf{x}}(i+1,1) &= A\hat{\mathbf{x}}(i,1) + A_d\hat{\mathbf{x}}(i_d,2) + f(i,\check{\mathbf{z}}_f(i \mid i),\mathbf{u}(i)) + h(i,\check{\mathbf{z}}_h(i_d \mid i),\mathbf{u}(i)) \\ &\quad + K_1(i)\tilde{\mathbf{y}}_1(i,1) + A_d\sum_{j=i_d}^{k_d-1} P_2(i_d,j)C_2^T R_{\tilde{y}_2}^{-1}(j,2)\tilde{\mathbf{y}}_2(j,2) \\ &\quad + A_d\sum_{j=k_d}^{i-1} P_1(i_d,j)C_1^T R_{\tilde{y}_1}^{-1}(j,1)\tilde{\mathbf{y}}_1(j,1), \\ \hat{\mathbf{x}}(k_d,1) &= \hat{\mathbf{x}}(k_d,2), \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} C_1 &= \begin{bmatrix} C^T & F^T & L^T \end{bmatrix}^T, \\ P_1(i,j) &= \begin{cases} \langle \mathbf{e}(i,2),\mathbf{e}(j,1)\rangle, & i < k_d, \\ \langle \mathbf{e}(i,1),\mathbf{e}(j,1)\rangle, & i \geq k_d, \end{cases} \\ \mathbf{e}(i,1) &= \mathbf{x}(i) - \hat{\mathbf{x}}(i,1), \\ R_{\tilde{y}_1}(i,1) &= C_1P_1(i,i)C_1^T + Q_{v_1}(i), \\ Q_{v_1}(i) &= \operatorname{diag}\left\{I, -\alpha^{-2}I, -\gamma^2 I\right\}, \\ K_1(i) &= AP_1(i,i)C_1^T R_{\tilde{y}_1}^{-1}(i,1) + A_dP_1(i_d,i)C_1^T R_{\tilde{y}_1}^{-1}(i,1). \end{aligned} \tag{3.51}$$

Then, from (3.7) and (3.50), we can yield

$$\begin{aligned} \mathbf{e}(i+1,1) &= A\mathbf{e}(i,1) + A_d\mathbf{e}(i_d,2) + \overline{B}\overline{\mathbf{w}}(i) - K_1(i)\tilde{\mathbf{y}}_1(i,1) \\ &\quad - A_d\sum_{j=i_d}^{k_d-1} P_2(i_d,j)C_2^T R_{\tilde{y}_2}^{-1}(j,2)\tilde{\mathbf{y}}_2(j,2) \\ &\quad - A_d\sum_{j=k_d}^{i-1} P_1(i_d,j)C_1^T R_{\tilde{y}_1}^{-1}(j,1)\tilde{\mathbf{y}}_1(j,1). \end{aligned} \tag{3.52}$$

Thus, we obtain that

(1) if $i-j \geq k_d$, we have

$$\begin{aligned} P_1(i-j,i+1) &= \langle \mathbf{e}(i-j,1), \mathbf{e}(i+1,1)\rangle \\ &= P_1(i-j,i)A^T + P_1^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \end{aligned} \tag{3.53}$$

(2) if $i-j < k_d$, we have

$$\begin{aligned} P_1(i-j,i+1) &= \langle \mathbf{e}(i-j,2), \mathbf{e}(i+1,1)\rangle \\ &= P_1(i-j,i)A^T + P_2^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{k_d-1} P_2(i-j,t)C_2^T R_{\tilde{y}_2}^{-1}(t,2)C_2 P_2^T(i_d,t)A_d^T \\ &\quad - \sum_{t=k_d}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \end{aligned} \tag{3.54}$$

$$\begin{aligned} P_1(i+1,i+1) &= \langle \mathbf{e}(i-j,2), \mathbf{e}(i+1,1)\rangle \\ &= AP_1(i,i+1) + A_d P_1(i_d,i+1) + \overline{B}Q_{\overline{w}}(i)\overline{B}^T. \end{aligned} \tag{3.55}$$

It follows from (3.53), (3.54), and (3.55) that $P_1(i,i)$ $(i = k_d+1,\ldots,k)$ can be computed by

$$\begin{aligned} P_1(i-j,i+1) &= P_1(i-j,i)A^T + P_2^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{k_d-1} P_2(i-j,t)C_2^T R_{\tilde{y}_2}^{-1}(t,2)C_2 P_2^T(i_d,t)A_d^T \\ &\quad - \sum_{t=k_d}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \quad i-j < k_d, \\ P_1(i-j,i+1) &= P_1(i-j,i)A^T + P_1^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^{\mathrm{T}}, \quad i-j \geq k_d, \\ P_1(i+1,i+1) &= AP_1(i,i+1) + A_d P_1(i_d,i+1) + \overline{B}Q_{\overline{w}}(i)\overline{B}^T, \quad j = 0,1,\ldots,d. \end{aligned} \tag{3.56}$$

Next, according to the above analysis, $\hat{\mathbf{z}}_m(k \mid k)$ as the Krein space projections of $\check{\mathbf{z}}_m(k \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}; \mathbf{y}_0(k)\}$ can be computed by the following formula

$$\hat{\mathbf{z}}_m(k \mid k) = C_m \hat{\mathbf{x}}(k,1) + C_m P_1(k,k) C^T R_{\tilde{y}_0}^{-1}(k,0) \tilde{\mathbf{y}}_0(k,0), \tag{3.57}$$

where

$$\begin{gathered} C_m = \begin{bmatrix} F^T & L^T \end{bmatrix}^T, \\ R_{\tilde{y}_0}(k,0) = C P_1(k,k) C^T + Q_v(k). \end{gathered} \tag{3.58}$$

And, $\hat{\mathbf{z}}_h(k_d \mid k)$ as the Krein space projections of $\check{\mathbf{z}}_h(k_d \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}; \mathbf{y}_1(k)\}$ can be computed by the following formula

$$\hat{\mathbf{z}}(k_d \mid k) = H \hat{\mathbf{x}}(k_d,1) + \sum_{j=k_d}^{k} H P_1(k_d,j) C_1^T R_{\tilde{y}_1}^{-1}(j,1) \tilde{\mathbf{y}}_1(j,1). \tag{3.59}$$

Based on Theorem 3.3 and the above discussion, we propose the following results.

Theorem 3.6. *Consider system* (2.1) *with Lipschitz conditions* (2.2), *given a scalar* $\gamma > 0$ *and matrix* $\Pi(k)$ $(k = -d, \ldots, 0)$, *then the* H_∞ *estimator that achieves* (2.3) *if*

$$\begin{gathered} R_{\tilde{y}}(k \mid k-1) > 0, \quad 0 \le k \le N, \\ R_{\tilde{z}_m}(k \mid k) < 0, \quad 0 \le k \le N, \\ R_{\tilde{z}_h}(k_d \mid k) < 0, \quad d \le k \le N, \end{gathered} \tag{3.60}$$

where

$$\begin{gathered} R_{\tilde{y}}(k \mid k-1) = R_{\tilde{y}_0}(k,0), \\ R_{\tilde{z}_m}(k \mid k) = C_m P_1(k,k) C_m^T - C_m P_1(k,k) C^T R_{\tilde{y}_0}^{-1}(k,0) C P_1(k,k) C_m^T + Q_{v_m}(k), \\ R_{\tilde{z}_h}(k_d \mid k) = H P_1(k_d,k_d) H^T - \sum_{j=k_d}^{k} H P_1(k_d,j) C_1^T R_{\tilde{y}_1}^{-1}(j,1) C_1 P_1^T(k_d,j) H^T - \beta^{-2} I, \\ Q_{v_m}(k) = \operatorname{diag}\left\{-\alpha^{-2} I, -\gamma^2 I\right\}, \end{gathered} \tag{3.61}$$

$R_{\tilde{y}_0}(k,0)$, $P_1(i,j)$, *and* $R_{\tilde{y}_1}(j,1)$ *are calculated by* (3.58), (3.56), *and* (3.51), *respectively.*

Moreover, one possible level-γ H_∞ *estimator is given by*

$$\check{z}(k \mid k) = E \hat{z}_m(k \mid k), \tag{3.62}$$

where $E = [0 \;\; I]$, *and* $\hat{z}_m(k \mid k)$ *is computed by* (3.57).

Proof. In view of Definitions 3.1 and 3.5, it follows from (3.9) and (3.41) that $R_{\tilde{y}}(k \mid k-1) = R_{\tilde{y}_0}(k,0)$. In addition, according to (3.7), (3.9), and (3.57), the covariance matrix $R_{\tilde{z}_m}(k \mid k)$ can be given by the second equality in (3.61). Similarly, based on (3.7), (3.9), and (3.59), the covariance matrix $R_{\tilde{z}_h}(k_d \mid k)$ can be obtained by the third equality in (3.61). Thus, from Theorem 3.3, it follows that J_N has a minimum if (3.60) holds.

On the other hand, note that the minimum value of J_N is given by (3.15) in Theorem 3.3 and any choice of estimator satisfying $\min J_N > 0$ is an acceptable one. Therefore, Taking into account (3.60), one possible estimator can be obtained by setting $\check{z}_m(k \mid k) = \hat{z}_m(k \mid k)$ and $\check{z}_h(k_d \mid k) = \hat{z}_h(k_d \mid k)$. This completes the proof. □

Remark 3.7. It is shown from (3.57) and (3.59) that $\hat{\mathbf{z}}_m(k \mid k)$ and $\hat{\mathbf{z}}(k_d \mid k)$ are, respectively, the filtering estimate and fixed-lag smoothing of $\check{\mathbf{z}}_m(k \mid k)$ and $\check{\mathbf{z}}(k_d \mid k)$ in the Krein space. Additionally, it follows from Theorem 3.6 that $\check{z}_m(k \mid k)$ and $\check{z}_h(k_d \mid k)$ achieving the H_∞ estimation problem (2.3) can be, respectively, computed by the right side of (3.57) and (3.59). Thus, it can be concluded that the proposed results in this paper are related with both the H_2 filtering and H_2 fixed-lag smoothing in the Krein space.

Remark 3.8. Recently, the robust H_∞ observers for Lipschitz nonlinear delay-free systems with Lipschitz nonlinear additive uncertainties and time-varying parametric uncertainties have been studied in [10, 11], where the optimization of the admissible Lipschitz constant and the disturbance attenuation level are discussed simultaneously by using the multiobjective optimization technique. In addition, the sliding mode observers with H_∞ performance have been designed for Lipschitz nonlinear delay-free systems with faults (matched uncertainties) and disturbances in [8]. Although the Krein space-based robust H_∞ filter has been proposed for discrete-time uncertain linear systems in [28], it cannot be applied to solving the H_∞ estimation problem given in [10] since the considered system contains Lipschitz nonlinearity and Lipschitz nonlinear additive uncertainty. However, it is meaningful and promising in the future, by combining the algorithm given in [28] with our proposed method in this paper, to construct a Krein space-based robust H_∞ filter for discrete-time Lipschitz nonlinear systems with nonlinear additive uncertainties and time-varying parametric uncertainties.

4. A Numerical Example

Consider the system (2.1) with time delay $d = 3$ and the parameters

$$\begin{gathered} A = \begin{bmatrix} 0.7 & 0 \\ 0 & -0.4 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \\ H = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.2 \\ 0.7 \end{bmatrix}, \qquad C = \begin{bmatrix} 1.7 & 0.9 \end{bmatrix}, \qquad L = \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, \\ f(k, Fx(k), u(k)) = \sin(Fx(k)), \qquad h(k, Hx(k_d), u(k)) = \cos(Hx(k_d)). \end{gathered} \tag{4.1}$$

Then we have $\alpha = \beta = 1$. Set $x(k) = [-0.2k \;\; 0.1k]^T$ $(k = -3, -2, -1, 0)$, and $\Pi(k) = I$ $(k = -3, -2, -1, 0)$. Both the system noise $w(k)$ and the measurement noise $v(k)$ are supposed to be band-limited white noise with power 0.01. By applying Theorem 3.1 in [20], we obtain the minimum disturbance attenuation level $\gamma_{\min} = 1.6164$ and the observer

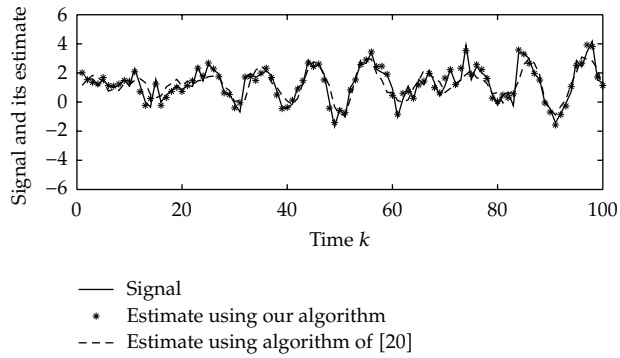

Figure 1: Signal $z(k)$ (solid), its estimate using our algorithm (star), and its estimate using algorithm in [20] (dashed).

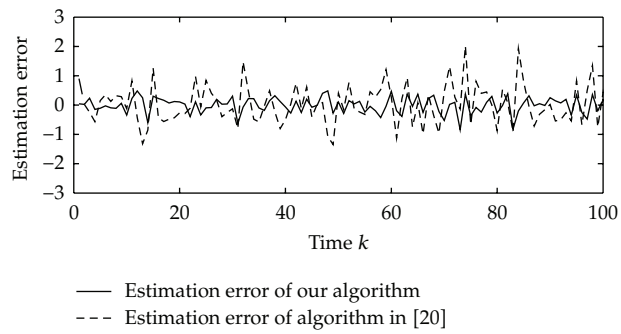

Figure 2: Estimation error of our algorithm (solid) and estimation error of algorithm in [20] (dashed).

parameter $L = [-0.3243 \;\; 0.0945]^T$ of (5) in [20]. In this numerical example, we compare our algorithm with the one given in [20] in case of $\gamma = 1.6164$. Figure 1 shows the true value of signal $z(k)$, the estimate using our algorithm, and the estimate using the algorithm given in [20]. Figure 2 shows the estimation error of our approach and the estimation error of the approach in [20]. It is shown in Figures 1 and 2 that the proposed algorithm is better than the one given in [20]. Figure 3 shows the ratios between the energy of the estimation error and input noises for the proposed H_∞ estimation algorithm. It is shown that the maximum energy ratio from the input noises to the estimation error is less than γ^2 by using our approach. Figure 4 shows the value of indefinite quadratic form J_N for the given estimation algorithm. It is shown that the value of indefinite quadratic form J_N is positive by employing the proposed algorithm in Theorem 3.6.

5. Conclusions

A recursive H_∞ filtering estimate algorithm for discrete-time Lipschitz nonlinear systems with time-delay and disturbance input is proposed. By combining the H_∞-norm estimation condition with the Lipschitz conditions on nonlinearity, the H_∞ estimation problem is converted to the positive minimum problem of indefinite quadratic form. Motivated by the observation that the minimum problem of indefinite quadratic form coincides with Kalman filtering in the Krein space, a novel Krein space-based H_∞ filtering estimate algorithm is

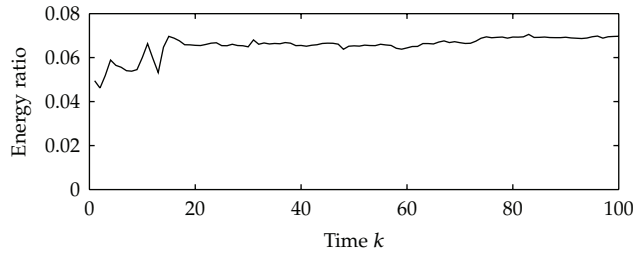

Figure 3: The energy ratio between estimation error and all input noises for the proposed H_∞ estimation algorithm.

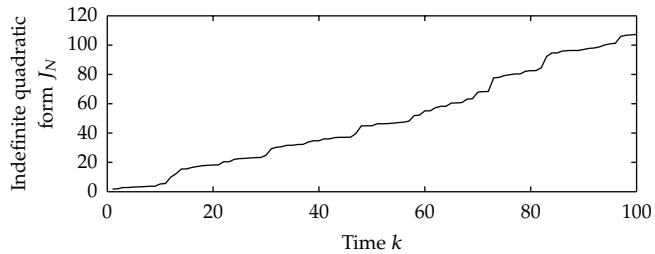

Figure 4: The value of indefinite quadratic form J_N for the given estimation algorithm.

developed. Employing projection formula and innovation analysis technology in the Krein space, the H_∞ estimator and its sufficient existence condition are presented based on Riccati-like difference equations. A numerical example is provided in order to demonstrate the performances of the proposed approach.

Future research work will extend the proposed method to investigate more general nonlinear system models with nonlinearity in observation equations. Another interesting research topic is the H_∞ multistep prediction and fixed-lag smoothing problem for time-delay Lipschitz nonlinear systems.

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