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Research Article

Exponential Stabilization of Neutral-Type Neural Networks with Interval Nondifferentiable and Distributed Time-Varying Delays

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The problem of exponential stabilization of neutral-type neural networks with various activation functions and interval nondifferentiable and distributed time-varying delays is considered. The interval time-varying delay function is not required to be differentiable. By employing new and improved Lyapunov-Krasovskii functional combined with Leibniz-Newton's formula, the stabilizability criteria are formulated in terms of a linear matrix inequalities. Numerical examples are given to illustrate and show the effectiveness of the obtained results.

1. Introduction

In recent years, there have been great attentions on the stability analysis of neural networks due to its real world application to various systems such as signal processing, pattern recognition, content-addressable memory, and optimization [1–5]. In performing a periodicity or stability analysis of a neural network, the conditions to be imposed on the neural network are determined by the characteristics of various activation functions and network parameters. When neural networks are designed for problem solving, it is desirable that their activation functions are not too restrictive, [3, 6, 7]. It is known that time delays cannot be avoided in the hardware implementation of neural networks due to the finite switching speed of amplifies in electronic neural networks or the finite signal propagation time in biological networks. The existence of time delays may result in instability or oscillation of a neural network. Therefore, many researchers have focused on the study of stability analysis of delayed neural networks with various activation functions with more general conditions during the last decades [2, 8–11].

The stability criteria for system with time delays can be classified into two categories: delay independent and delay dependent. Delay-independent criteria do not employ any information on the size of the delay; while delay-dependent criteria make use of such information at different levels. Delay-dependent stability conditions are generally less conservative than delay-independent ones especially when the delay is small. In many situations, time delays are time-varying continuous functions which vary from 0 to a given upper bound. In addition, the range of time delays may vary in a range for which the lower bound is not restricted to be 0; in which case time delays are called interval time-varying delay. A typical example with interval time delay is the networked control system, which has been widely studied in the recent literature (see, e.g., [2, 11–14]). Therefore, it is of great significance to investigate the stability of system with interval time-varying delay. Another important type of time delay is distributed delay where stability analysis of neural networks with distributed delayed has been studied extensively recently, see [2, 5, 8–11, 15–17].

It is know that exponential stability is more important than asymptotic stability since it provides information on convergence rate of solutions of systems to equilibrium points. It is particularly important for neural networks where the exponential convergence rate is used to determine the speed of neural computations. Therefore, it is important to determine the exponential stability and to estimate the exponential convergence rate for delayed neural networks. Consequently, many researchers have considered the exponential stability analysis problem for delayed neural networks and several results on this topic that have been reported in the literatures [3, 9, 13, 14, 17].

In practical control designs, due to systems uncertainty, failure modes, or systems with various modes of operation, the simultaneous stabilization problem has often to be taken into account. The problem is concerned with designing a single controller which can simultaneously stabilize a set of systems. Recently, the exponential stability and stabilization problems for time-delay systems have been studied by many researchers, see [8, 12, 18, 19]. Among the usual approach, there are many results on the stabilization problem of neural networks being reported in the literature (see [2, 9, 15, 16, 20, 21]). In [21], robust stabilization criterion are provided via designing a memoryless state feedback controller for the time-delay dynamical neural networks with nonlinear perturbation. However, time-delay is required to be constant. In [16], global robust stabilizing control are presented for neural network with time-varying delay with the lower bound restricted to be 0. For neural network with interval time-varying delay, global stability analysis is considered with control input in [2, 9, 12]. Nonetheless, in most studies, the time-varying delays are required to be differentiable [2, 9, 15, 16]. Therefore, their methods have a conservatism which can be improved upon.

It is noted that these stability conditions are either with testing difficulty or with conservatism to some extent. It is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions. Such systems are called neutral-type systems in which the systems contain both state delay as well as state derivative delay, so called neutral delay. The phenomena on neutral delay often appears in the study of heat exchanges, distributed networks containing lossless transmission lines, partial element equivalent circuits and population ecology are examples of neutral systems see [1, 10, 11, 22, 23] and references cited therein.

Based on the above discussions, we consider the problem of exponential stabilization of neutral-type neural networks with interval and distributed time-varying delays. There are various activation functions which are considered in the system and the restriction on differentiability of interval time-varying delays is removed, which means that a fast interval time-varying delay is allowed. Based on the construction of improved Lyapunov-Krasovskii

functionals combined with Liebniz-Newton's formula and some appropriate estimation of integral terms, new delay-dependent sufficient conditions for the exponential stabilization of the system are derived in terms of LMIs without introducing any free-weighting matrices. The new stability conditions are much less conservative and more general than some existing results. Numerical examples are given to illustrate the effectiveness and less conservativeness of our theoretical results. To the best of our knowledge, our results are among the first on investigation of exponential stabilization for neutral-type neural networks with discrete, neutral, and distributed delays.

The rest of this paper is organized as follows. In Section 2, we give notations, definitions, propositions, and lemmas which will be used in the proof of the main results. Delay-dependent sufficient conditions for the exponential stabilization of neutral-type neural networks with various activation functions, interval and distributed time-varying delays, and designs of memoryless feedback controls are presented in Section 3. Numerical examples illustrated the obtained results are given in Section 4. The paper ends with conclusions in Section 5 and cited references.

2. Preliminaries

The following notation will be used in this paper: $\mathbb{R}^+$ denotes the set of all real nonnegative numbers; $\mathbb{R}^n$ denotes the n-dimensional space and the vector norm $\|\cdot\|$; $M^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions.

A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda; \lambda \in \lambda(A)\}$.

$x_t := \{x(t+s) : s \in [-h,0]\}$, $\|x_t\| = \sup_{s\in[-h,0]}\|x(t+s)\|$; $C([0,t],\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuous functions on $[0,t]$; $L_2([0,t],\mathbb{R}^m)$ denotes the set of all the $\mathbb{R}^m$-valued square integrable functions on $[0,t]$.

Matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. The symmetric term in a matrix is denoted by $*$.

Consider the following neural networks with mixed time-varying delays and control input

$$\begin{aligned}
\dot{x}(t) &= -(A + \Delta A(t))x(t) + (W_0 + \Delta W_0)f(x(t)) + (W_1 + \Delta W_1)g(x(t-h(t))) \\
&\quad + (W_2 + \Delta W_2)\int_{t-k(t)}^{t} h(x(s))ds + B_0\dot{x}\big(t-\eta(t)\big) + Bu(t), \\
x(t) &= \phi(t), \quad t \in [-d,0], \quad d = \max\{h_2, k, \eta\},
\end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the neural networks, $u(\cdot) \in L_2([0,t],\mathbb{R}^m)$ is the control, n is the number of neurals, and

$$\begin{aligned}
f(x(t)) &= \big[f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))\big]^T, \\
g(x(t)) &= \big[g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))\big]^T, \\
h(x(t)) &= [h_1(x_1(t)), h_2(x_2(t)), \ldots h_n(x_n(t))]^T
\end{aligned} \tag{2.2}$$

are the activation functions, $A = \operatorname{diag}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)$, $\overline{a}_i > 0$ represents the self-feedback term and W_0, W_1, W_2 denote the connection weights, the discretely delayed connection weights, and the distributively delayed connection weight, respectively. In this paper, we consider various activation functions and assume that the activation functions $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are Lipschitzian with the Lipschitz constants $a_i, b_i, c_i > 0$

$$\begin{aligned} &\left|f_i(\xi_1) - f_i(\xi_2)\right| \le a_i|\xi_1 - \xi_2|, \quad i = 1, 2, \ldots, n, \ \forall \xi_1, \xi_2 \in \mathbb{R}, \\ &\left|g_i(\xi_1) - g_i(\xi_2)\right| \le b_i|\xi_1 - \xi_2|, \quad i = 1, 2, \ldots, n, \ \forall \xi_1, \xi_2 \in \mathbb{R}, \\ &|h_i(\xi_1) - h_i(\xi_2)| \le c_i|\xi_1 - \xi_2|, \quad i = 1, 2, \ldots, n, \ \forall \xi_1, \xi_2 \in \mathbb{R}. \end{aligned} \tag{2.3}$$

The time-varying delay functions $h(t), k(t), \eta(t)$ satisfy the condition

$$\begin{gathered} 0 \le h_1 \le h(t) \le h_2, \\ 0 \le k(t) \le k, \\ 0 \le \eta(t) \le \eta, \qquad \dot{\eta}(t) \le \delta < 1. \end{gathered} \tag{2.4}$$

It is worth noting that the time delay is assumed to be a continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay function is bounded but not restricted to being zero. The initial functions $\phi(t) \in C^1([-d, 0], \mathbb{R}^n)$, with the norm

$$\|\phi\| = \sup_{t \in [-d,0]} \sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}. \tag{2.5}$$

The uncertainties satisfy the following condition:

$$\begin{aligned} \Delta A(t) &= E_a F_a(t) H_a, \qquad \Delta W_0(t) = E_0 F_0(t) H_0, \\ \Delta W_1(t) &= E_1 F_1(t) H_1, \qquad \Delta W_2(t) = E_2 F_2(t) H_2, \end{aligned} \tag{2.6}$$

where E_i, H_i, $i = a, 0, 1, 2$ are given constant matrices with appropriate dimensions, $F_i(t)$, $i = a, 0, 1, 2$ are unknown, real matrices with Lebesgue measurable elements satisfying

$$F_i^T(t) F_i(t) \le I, \quad i = a, 0, 1, 2 \ \forall t \ge 0. \tag{2.7}$$

Definition 2.1. The zero solution of system (2.8) is exponentially stabilizable if there exists a feedback control $u(t) = Kx(t)$, $K \in \mathbb{R}^{m \times n}$ such that the resulting closed-loop system

$$\begin{aligned} \dot{x}(t) = &-[A - BK]x(t) + W_0 f(x(t)) + W_1 g(x(t - h(t))) + W_2 \int_{t-k(t)}^{t} h(x(s))ds \\ &+ B_0 \dot{x}\big(t - \eta(t)\big) \end{aligned} \tag{2.8}$$

is α-stable.

Definition 2.2. Given $\alpha > 0$. The zero solution of system (2.1) with $u(t) = 0$ is α-stable if there exists a positive number $N > 0$ such that every solution $x(t,\phi)$ satisfies the following condition:

$$\|x(t,\phi)\| \leq Ne^{-\alpha t}\|\phi\|, \quad \forall t \geq 0. \tag{2.9}$$

We introduce the following technical well-known propositions and lemma, which will be used in the proof of our results.

Proposition 2.3 (Cauchy inequality). *For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathbb{R}^n$ one has*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y. \tag{2.10}$$

Proposition 2.4 (see [24]). *For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:*

$$\left(\int_0^\gamma \omega(s)ds\right)^T M\left(\int_0^\gamma \omega(s)ds\right) \leq \gamma\left(\int_0^\gamma \omega^T(s)M\omega(s)ds\right). \tag{2.11}$$

Proposition 2.5 ([24, Schur complement lemma]). *Given constant symmetric matrices X, Y, Z with appropriate dimensions satisfying $X = X^T$, $Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0. \tag{2.12}$$

Lemma 2.6 (see [25]). *Given matrices $Q = Q^T, H, E$, and $R = R^T > 0$ with appropriate dimensions. Then*

$$Q + HFE + E^T F^T H^T < 0, \tag{2.13}$$

for all F satisfying $F^T F \leq R$, if and only if there exists an $\epsilon > 0$ such that

$$Q + \epsilon HH^T + \epsilon^{-1} E^T RE < 0. \tag{2.14}$$

3. Main Results

3.1. Exponential Stabilization for Nominal Interval Time-Varying Delay Systems

The nominal system is given by

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2 \int_{t-k(t)}^{t} h(x(s))ds + B_0 \dot{x}(t-\eta(t)) + Bu(t)$$
$$x(t) = \phi(t), \quad t \in [-d, 0]. \tag{3.1}$$

First, we present a delay-dependent exponential stabilizability analysis conditions for the given nominal interval time-varying delay system (3.1) with $\Delta A(t) = \Delta W_0(t) = \Delta W_1(t) = \Delta W_2(t) = 0$. Let us set

$$\begin{aligned}
\lambda_1 &= \lambda_{\min}\left(P^{-1}\right), \\
\lambda_2 &= \lambda_{\max}\left(P^{-1}\right) + 2h_2\lambda_{\max}\left(P^{-1}QP^{-1}\right) + 2h_2^2\lambda_{\max}\left(P^{-1}RP^{-1}\right) \\
&\quad + \eta\lambda_{\max}\left(P^{-1}Q_1P^{-1}\right) + 2\lambda_{\max}\left(HD_2^{-1}H\right) + (h_2-h_1)^2\lambda_{\max}\left(P^{-1}UP^{-1}\right).
\end{aligned} \tag{3.2}$$

Assumption 3.1. All the eigenvalues of matrix B_0 are inside the unit circle.

Theorem 3.2. *Given $\alpha > 0$. The system (3.1) is α-exponentially stabilizable if there exist symmetric positive definite matrices $P, Q, R, U, Q1$, three diagonal matrices D_i, $i = 0, 1, 2$ such that the following LMI holds:*

$$\mathcal{M}_1 = \mathcal{M} - \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix} < 0, \tag{3.3}$$

$$\mathcal{M}_2 = \mathcal{M} - \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.4}$$

$$\mathcal{M}_3 = \begin{bmatrix} -0.1BB^T - 0.1\left(e^{-2\alpha h_1} + e^{-2\alpha h_2}\right)R & 2kPH & 2PF \\ * & -2kD_2 & 0 \\ * & * & -2D_0 \end{bmatrix} < 0, \tag{3.5}$$

$$\mathcal{M}_4 = \begin{bmatrix} -0.1e^{-2\alpha h_2}U & 2PG \\ * & -2D_1 \end{bmatrix} < 0, \tag{3.6}$$

$$\mathcal{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & 0 \\ * & M_{22} & 0 & 0 & M_{25} & 0 \\ * & * & M_{33} & 0 & 0 & M_{36} \\ * & * & * & M_{44} & 0 & M_{46} \\ * & * & * & * & M_{55} & 0 \\ * & * & * & * & * & M_{66} \end{bmatrix}, \tag{3.7}$$

where

$$\begin{aligned}
M_{11} &= [-A+\alpha I]P + P[-A+\alpha I]^T - 0.9BB^T + 2Q + W_0D_0W_0^T + W_1D_1W_1^T \\
&\quad + ke^{2\alpha k}W_2D_2W_2^T - 0.9e^{-2\alpha h_1}R - 0.9e^{-2\alpha h_2}R\Big] \\
M_{12} &= B_0P, \qquad M_{13} = e^{-2\alpha h_1}R, \qquad M_{14} = e^{-2\alpha h_2}R, \qquad M_{15} = -PA^T - 0.5BB^T, \\
M_{22} &= -(1-\delta)e^{-2\alpha\eta}Q_1, \qquad M_{25} = PB_0^T, \qquad M_{33} = -e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U, \\
M_{36} &= e^{-2\alpha h_2}U, \qquad M_{44} = -e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U, \qquad M_{46} = e^{-2\alpha h_2}U, \\
M_{55} &= h_1^2R + h_2^2R + (h_2-h_1)^2U + Q_1 - 2P + W_0D_0W_0^T + W_1D_1W_1^T \\
&\quad + ke^{2\alpha k}W_2D_2W_2^T\Big] \\
M_{66} &= -1.9e^{-2\alpha h_2}U.
\end{aligned} \tag{3.8}$$

Moreover, the memoryless feedback control is

$$u(t) = -0.5B^TP^{-1}x(t), \quad t \geq 0, \tag{3.9}$$

and the solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|, \quad \forall t \geq 0. \tag{3.10}$$

Proof. Let $Y = P^{-1}$, $y(t) = Yx(t)$. Using the feedback control (3.9) we consider the following Lyapunov-Krasovskii functional:

$$V(t,x_t) = \sum_{i=1}^{8} V_i, \tag{3.11}$$

where

$$\begin{aligned}
V_1 &= x^T(t)Yx(t), \\
V_2 &= \int_{t-h_1}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)ds, \\
V_3 &= \int_{t-h_2}^{t} e^{2\alpha(s-t)}x^T(s)YQYx(s)ds, \\
V_4 &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YRY\dot{x}(\tau)d\tau\, ds, \\
V_5 &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YRY\dot{x}(\tau)d\tau\, ds,
\end{aligned}$$

$$V_6 = (h_2 - h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} e^{2\alpha(\tau-t)}\dot{x}^T(\tau)YUY\dot{x}(\tau)d\tau\, ds,$$

$$V_7 = \int_{t-\eta(t)}^{t} e^{2\alpha(s-t)}\dot{x}^T(s)YQ_1Y\dot{x}(s)ds,$$

$$V_8 = 2\int_{-k}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)}h^T(x(\tau))D_2^{-1}h(x(\tau))d\tau\, ds. \tag{3.12}$$

It easy to check that

$$\lambda_1\|x(t)\|^2 \le V(t,x_t) \le \lambda_2\|x_t\|^2, \quad \forall t \ge 0. \tag{3.13}$$

Taking the derivative of $V(x_t)$ along the solution of system (3.1) we have

$$\begin{aligned}
\dot{V}_1 &= 2x^T(t)Y\dot{x}(t),\\
&= 2y^T(t)\Big[-Ax(t) + W_0f(x(t)) + W_1g(x(t-h(t))) + W_2\int_{t-k(t)}^{t} h(x(s))ds\\
&\qquad +B_0\dot{x}\big(t-\eta(t)\big) - 0.5BB^TP^{-1}x(t)\Big]\\
&= 2y^T(t)\Big[-APy(t) + W_0f(x(t)) + W_1g(x(t-h(t))) + W_2\int_{t-k(t)}^{t} h(x(s))ds\\
&\qquad +B_0P\dot{y}\big(t-\eta(t)\big) - 0.5BB^Ty(t)\Big]
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&= y^T(t)\Big[-AP - PA^T\Big]y(t) + 2y^T(t)W_0f(x(t)) + 2y^T(t)W_1g(x(t-h(t)))\\
&\quad + 2y^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + 2y^T(t)B_0P\dot{y}\big(t-\eta(t)\big) - y^T(t)BB^Ty(t)\\
&\quad + 2\alpha y^T(t)Py(t) - 2\alpha V_1,\\
\dot{V}_2 &= y^T(t)Qy(t) - e^{-2\alpha h_1}y^T(t-h_1)Qy(t-h_1) - 2\alpha V_2,\\
\dot{V}_3 &= y^T(t)Qy(t) - e^{-2\alpha h_2}y^T(t-h_2)Qy(t-h_2) - 2\alpha V_3,\\
\dot{V}_4 &\le h_1^2\dot{y}^T(t)R\dot{y}(t) - h_1e^{-2\alpha h_1}\int_{t-h_1}^{t}\dot{y}^T(s)R\dot{y}(s)ds - 2\alpha V_4,\\
\dot{V}_5 &\le h_2^2\dot{y}^T(t)R\dot{y}(t) - h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot{y}^T(s)R\dot{y}(s)ds - 2\alpha V_5,
\end{aligned}$$

$$\begin{aligned}
\dot{V}_6 &\le (h_2-h_1)^2\dot{y}^T(t)U\dot{y}(t)-(h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds-2\alpha V_6,\\
\dot{V}_7 &\le \dot{y}^T(t)Q_1\dot{y}(t)-(1-\delta)e^{-2\alpha\eta}\dot{y}^T(t-\eta(t))Q_1\dot{y}(t-\eta(t))-2\alpha V_7,\\
\dot{V}_8 &\le 2kh^T(x(t))D_2^{-1}h(x(t))-2e^{-2\alpha k}\int_{t-k}^{t}h^T(x(s))D_2^{-1}h(x(s))ds-2\alpha V_8.
\end{aligned} \tag{3.15}$$

Using the condition (2.3) and since the matrices $D_i^{-1}>0,\ i=0,1,2$ are diagonal, we have

$$\begin{aligned}
kh^T(x(t))D_2^{-1}h(x(t)) &\le kx^T(t)HD_2^{-1}Hx(t)=ky^T(t)PHD_2^{-1}HPy(t),\\
f^T(x(t))D_0^{-1}f(x(t)) &\le x^T(t)FD_0^{-1}Fx(t)=y^T(t)PFD_0^{-1}FPy(t),\\
g^T(x(t-h(t)))D_1^{-1}g(x(t-h(t))) &\le x^T(t-h(t))GD_1^{-1}Gx(t-h(t)),\\
&=y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)),
\end{aligned} \tag{3.16}$$

and using (2.3) and the Proposition (2.3) for the following estimations:

$$\begin{aligned}
2y^T(t)W_0f(x(t)) &\le y^T(t)W_0D_0W_0^Ty(t)+f^T(x(t))D_0^{-1}f(x(t))\\
&\le y^T(t)W_0D_0W_0^Ty(t)+x^T(t)FD_0^{-1}Fx(t)\\
&\le y^T(t)W_0D_0W_0^Ty(t)+y^T(t)PFD_0^{-1}FPy(t),\\
2y^T(t)W_1g(x(t-h(t))) &\le y^T(t)W_1D_1W_1^Ty(t)+g^T(x(t-h(t)))D_1^{-1}g(x(t-h(t)))\\
&\le y^T(t)W_1D_1W_1^Ty(t)+x^T(t-h(t))GD_1^{-1}Gx(t-h(t))\\
&\le y^T(t)W_1D_1W_1^Ty(t)+y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)),\\
2y^T(t)W_2\int_{t-k(t)}^{t}h(x(s))ds &\le ke^{2\alpha k}y^T(t)W_2D_2W_2^Ty(t)\\
&\quad+k^{-1}e^{-2\alpha k}\left(\int_{t-k(t)}^{t}h(x(s)ds)\right)^TD_2^{-1}\left(\int_{t-k(t)}^{t}h(x(s))ds\right)\\
&\le ke^{2\alpha k}y^T(t)W_2D_2W_2^Ty(t)+e^{-2\alpha k}\int_{t-k}^{t}h^T(x(s))D_2^{-1}h(x(s))ds.
\end{aligned} \tag{3.17}$$

Applying Proposition 2.4 and the Leibniz-Newton formula, we have

$$\begin{aligned}
-h_1\int_{t-h_1}^{t}\dot{y}^T(s)R\dot{y}(s)ds &\le -\left[\int_{t-h_1}^{t}\dot{y}(s)\right]^T R\left[\int_{t-h_1}^{t}\dot{y}(s)\right]\\
&\le -\left[y(t)-y(t-h_1)\right]^T R\left[y(t)-y(t-h_1)\right]\\
&= -y^T(t)Ry(t)+2y^T(t)Ry(t-h_1)-y^T(t-h_1)Ry(t-h_1),\\
-h_2\int_{t-h_2}^{t}\dot{y}^T(s)R\dot{y}(s)ds &\le -\left[\int_{t-h_2}^{t}\dot{y}(s)\right]^T R\left[\int_{t-h_2}^{t}\dot{y}(s)\right]\\
&\le -\left[y(t)-y(t-h_2)\right]^T R\left[y(t)-y(t-h_2)\right]\\
&= -y^T(t)Ry(t)+2y^T(t)Ry(t-h_2)-y^T(t-h_2)Ry(t-h_2).
\end{aligned} \tag{3.18}$$

Note that

$$\begin{aligned}
-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds &= -(h_2-h_1)\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)ds\\
&\quad -(h_2-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds\\
&= -(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)ds\\
&\quad -(h(t)-h_1)\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)ds\\
&\quad -(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds\\
&\quad -(h_2-h(t))\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds.
\end{aligned} \tag{3.19}$$

Using Proposition 2.4 gives

$$\begin{aligned}
-(h_2-h(t))\int_{t-h_2}^{t-h(t)}\dot{y}^T(s)U\dot{y}(s)ds &\le -\left[\int_{t-h_2}^{t-h(t)}\dot{y}(s)ds\right]^T U\left[\int_{t-h_2}^{t-h(t)}\dot{y}(s)ds\right]\\
&\le -\left[y(t-h(t))-y(t-h_2)\right]^T U\left[y(t-h(t))-y(t-h_2)\right],
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
-(h(t)-h_1)\int_{t-h(t)}^{t-h_1}\dot{y}^T(s)U\dot{y}(s)ds &\le -\left[\int_{t-h(t)}^{t-h_1}\dot{y}(s)ds\right]^T U\left[\int_{t-h(t)}^{t-h_1}\dot{y}(s)ds\right]\\
&\le -\left[y(t-h_1)-y(t-h(t))\right]^T U\left[y(t-h_1)-y(t-h(t))\right].
\end{aligned} \tag{3.21}$$

Let $\beta = (h_2 - h(t))/(h_2 - h_1) \leq 1$. Then

$$\begin{aligned}
-(h_2 - h(t)) \int_{t-h(t)}^{t-h_1} \dot{y}^T(s) U \dot{y}(s) ds &= -\beta \int_{t-h(t)}^{t-h_1} (h_2 - h_1) \dot{y}^T(s) U \dot{y}(s) ds \\
&\leq -\beta \int_{t-h(t)}^{t-h_1} (h(t) - h_1) \dot{y}^T(s) U \dot{y}(s) ds \\
&\leq -\beta \left[y(t-h_1) - y(t-h(t)) \right]^T U \left[y(t-h_1) - y(t-h(t)) \right],
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
-(h(t) - h_1) \int_{t-h_2}^{t-h(t)} \dot{y}^T(s) U \dot{y}(s) ds &= -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h_1) \dot{y}^T(s) U \dot{y}(s) ds \\
&\leq -(1-\beta) \int_{t-h_2}^{t-h(t)} (h_2 - h(t)) \dot{y}^T(s) U \dot{y}(s) ds \\
&\leq -(1-\beta) \left[y(t-h(t)) - y(t-h_2) \right]^T \\
&\quad \times U \left[y(t-h(t)) - y(t-h_2) \right].
\end{aligned} \tag{3.23}$$

Therefore from (3.20)–(3.23), we obtain

$$\begin{aligned}
-(h_2 - h_1) \int_{t-h_2}^{t-h_1} \dot{y}^T(s) U \dot{y}(s) ds \leq & -\left[y(t-h(t)) - y(t-h_2) \right]^T U \left[y(t-h(t)) - y(t-h_2) \right] \\
& - \left[y(t-h_1) - y(t-h(t)) \right]^T U \left[y(t-h_1) - y(t-h(t)) \right] \\
& - \beta \left[y(t-h_1) - y(t-h(t)) \right]^T U \left[y(t-h_1) - y(t-h(t)) \right] \\
& - (1-\beta) \left[y(t-h(t)) - y(t-h_2) \right]^T \\
& \times U \left[y(t-h(t)) - y(t-h_2) \right].
\end{aligned} \tag{3.24}$$

By using the following identity relation:

$$\begin{aligned}
& - P\dot{y}(t) - APy(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_0 \int_{t-k(t)}^{t} h(x(s)) ds \\
& \quad + B_0 P \dot{y}(t - \eta(t)) - 0.5 B B^T y(t) = 0,
\end{aligned} \tag{3.25}$$

we have

$$\begin{aligned}
& -2\dot{y}^T(t) P \dot{y}(t) - 2\dot{y}^T(t) APy(t) + 2\dot{y}^T(t) W_0 f(x(t)) + 2\dot{y}^T(t) W_1 g(x(t-h(t))) \\
& \quad + 2\dot{y}^T(t) W_0 \int_{t-k(t)}^{t} h(x(s)) ds + 2\dot{y}^T(t) B_0 P \dot{y}(t - \eta(t)) - \dot{y}^T(t) B B^T y(t) = 0.
\end{aligned} \tag{3.26}$$

By using Propositions 2.3 and 2.4, we have

$$\begin{aligned} 2\dot{y}^T(t)W_0 f(x(t)) &\le \dot{y}^T(t)W_0D_0W_0^T\dot{y}(t) + f^T(x(t))D_0^{-1}f(x(t)) \\ &\le \dot{y}^T(t)W_0D_0W_0^T\dot{y}(t) + y^T(t)PFD_0^{-1}FPy(t), \end{aligned} \tag{3.27}$$

$$\begin{aligned} 2\dot{y}^T(t)W_1 g(x(t-h(t))) &\le \dot{y}^T(t)W_1D_1W_1^T\dot{y}(t) + g^T(x(t-h(t)))D_1^{-1}g(x(t-h(t))) \\ &\le \dot{y}^T(t)W_1D_1W_1^T\dot{y}(t) + y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)), \end{aligned} \tag{3.28}$$

$$\begin{aligned} 2\dot{y}^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds &\le ke^{2\alpha k}\dot{y}^T(t)W_2D_2W_2^T\dot{y}(t) \\ &\quad + e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds. \end{aligned} \tag{3.29}$$

From (3.15)–(3.29), we obtain

$$\begin{aligned} \dot{V}(t,x_t) + 2\alpha V(t,x_t) \le{}& y^T(t)\Big[-AP - PA^T + 2\alpha P - BB^T + 2Q + 2kPHD_2^{-1}HP \\ &+ W_0D_0W_0^T + W_1D_1W_1^T + ke^{2\alpha k}W_2D_2W_2^T \\ &-e^{-2\alpha h_1}R - e^{-2\alpha h_2}R + PFD_1^{-1}FP\Big]y(t) + 2y^T(t)B_0P\dot{y}(t-\eta(t)) \\ &+y^T(t-h_1)\Big[-e^{-2\alpha h_1}Q - e^{-2\alpha h_1}R - e^{-2\alpha h_2}U\Big]y(t-h_1), \\ &+(t-h_2)\Big[-e^{-2\alpha h_2}Q - e^{-2\alpha h_2}R - e^{-2\alpha h_2}U\Big]y(t-h_2), \\ &+\dot{y}^T(t)\Big[h_1^2R + h_2^2R + (h_2-h_1)^2U + Q_1 - 2P + W_0D_0W_0^T \\ &\quad +W_1D_1W_1^T + ke^{2\alpha k}W_2D_2W_2^T\Big]\dot{y}(t) \\ &-(1-\delta)e^{-2\alpha\eta}\dot{y}^T(t-\eta(t))Q_1\dot{y}(t-\eta(t)) \\ &+y^T(t-h(t))\Big[2PGD_1^{-1}GP - 2e^{-2\alpha h_2}U\Big]y(t-h(t)) \\ &+2e^{-2\alpha h_1}y^T(t)Ry(t-h_1) + 2e^{-2\alpha h_2}y^T(t)Ry(t-h_2) \\ &+2e^{-2\alpha h_2}y^T(t-h(t))Uy(t-h_2) \\ &+2e^{-2\alpha h_2}y^T(t-h(t))Uy(t-h_1) - 2\dot{y}^T(t)APy(t) \\ &+2\dot{y}^T(t)B_0P\dot{y}(t-\eta(t)) - \dot{y}^T(t)BB^Ty(t) \\ &-e^{-2\alpha h_2}\beta[y(t-h_1)-y(t-h(t))]^TU[y(t-h_1)-y(t-h(t))] \\ &-e^{-2\alpha h_2}(1-\beta)[y(t-h(t))-y(t-h_2)]^TU[y(t-h(t))-y(t-h_2)] \end{aligned}$$

$$
\begin{aligned}
&= \xi^T(t)\Omega\xi(t) - e^{-2\alpha h_2}\beta\left[y(t-h_1) - y(t-h(t))\right]^T U \\
&\quad \times \left[y(t-h_1) - y(t-h(t))\right] - e^{-2\alpha h_2}(1-\beta)\left[y(t-h(t)) - y(t-h_2)\right]^T \\
&\quad \times U\left[y(t-h(t)) - y(t-h_2)\right] \\
&= \xi^T(t)\left[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2\right]\xi(t) + y^T(t)M_3 y(t) \\
&\qquad + y^T(t-h(t))M_4 y(t-h(t)),
\end{aligned}
\tag{3.30}
$$

where

$$
\begin{gathered}
M_3 = -0.1BB^T - 0.1\left(e^{-2\alpha h_1} + e^{-2\alpha h_2}\right)R + 2kPHD_2^{-1}HP + 2PFD_0^{-1}FP, \\
M_4 = -0.1e^{-2\alpha h_2}U + 2PGD_1^{-1}GP, \\
\zeta(t) = \left[y(t), \dot{y}(t-\eta(t)), y(t-h_1), y(t-h_2), \dot{y}(t), y(t-h(t))\right].
\end{gathered}
\tag{3.31}
$$

Since $0 \le \beta \le 1$, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2$ is a convex combination of $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore, $(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2 < 0$ is equivalent to $\mathcal{M}_1 < 0$ and $\mathcal{M}_2 < 0$. Applying Schur complement lemma Proposition 2.5, the inequalities $M_3 < 0$ and $M_4 < 0$ are equivalent to $\mathcal{M}_3 < 0$ and $\mathcal{M}_4 < 0$, respectively. Thus, it follows from (3.3)–(3.6) and (3.30), we obtain

$$
\dot{V}(t, x_t) \le -2\alpha V(t, x_t), \quad \forall t \ge 0. \tag{3.32}
$$

Integrating both sides of (3.32) from 0 to t, we obtain

$$
V(t, x_t) \le V(\phi)e^{-2\alpha t}, \quad \forall t \ge 0. \tag{3.33}
$$

Furthermore, taking condition (3.13) into account, we have

$$
\lambda_1\|x(t,\phi)\|^2 \le V(x_t) \le V(\phi)e^{-2\alpha t} \le \lambda_2 e^{-2\alpha t}\|\phi\|^2, \tag{3.34}
$$

then

$$
\|x(t,\phi)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}\|\phi\|, \quad t \ge 0. \tag{3.35}
$$

Therefore, nominal system (3.1) is α-exponentially stabilizable. The proof is completed. □

3.2. Exponential Stabilization for Interval Time-Varying Delay Systems

Based on Theorem 3.2, we derive robustly α-exponential stabilizability conditions of uncertain linear control systems with interval time-varying delay (2.1) in terms of LMIs.

Theorem 3.3. *Given $\alpha > 0$. The system (2.1) is α-exponentially stabilizable if there exist symmetric positive definite matrices P, Q, R, U, $Q1$, three diagonal matrices D_i, $i = 0,1,2$ and $\epsilon_i > 0$, $i = 1,2,\ldots,6$ such that the following LMI holds:*

$$\mathcal{W}_1 = \mathcal{W} - \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & 0 & -I & 0 & I \end{bmatrix} < 0, \tag{3.36}$$

$$\mathcal{W}_2 = \mathcal{W} - \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix}^T \times e^{-2\alpha h_2} U \begin{bmatrix} 0 & 0 & I & 0 & 0 & -I \end{bmatrix} < 0, \tag{3.37}$$

$$\mathcal{W}_3 = \begin{bmatrix} \Delta & 4kPH & 2PF & PH_a^T & PH_a^T & PFH_0^T & PFH_0^T \\ * & -4kD_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & -2D_0 & 0 & 0 & 0 & 0 \\ * & * & * & -\epsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & -\epsilon_4 I & 0 & 0 \\ * & * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & * & -\epsilon_5 I \end{bmatrix} < 0, \tag{3.38}$$

$$\mathcal{W}_4 = \begin{bmatrix} -0.1e^{-2\alpha h_2} U & 2PG & PGH_1^T & PGH_1^T \\ * & -2D_1 & 0 & 0 \\ * & * & -\epsilon_3 I & 0 \\ * & * & * & -\epsilon_6 I \end{bmatrix} < 0,$$

$$\mathcal{W} = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & 0 \\ * & W_{22} & 0 & 0 & W_{25} & 0 \\ * & * & W_{33} & 0 & 0 & W_{36} \\ * & * & * & W_{44} & 0 & W_{46} \\ * & * & * & * & W_{55} & 0 \\ * & * & * & * & * & W_{66} \end{bmatrix}, \tag{3.39}$$

where

$$\begin{aligned}
\Delta &= -0.1BB^T - 0.1\left(e^{-2\alpha h_1} + e^{-2\alpha h_2}\right)R, \\
W_{11} &= [-A + \alpha I]P + P[-A + \alpha I]^T - 0.9BB^T + 2Q + W_0 D_0 W_0^T + W_1 D_1 W_1^T \\
&\quad + ke^{2\alpha k} W_2 D_2 W_2^T - 0.9e^{-2\alpha h_1} R - 0.9e^{-2\alpha h_2} R + \epsilon_1 E_a^T E_a + \epsilon_3 E_1^T E_1 + \epsilon_2 E_0^T E_0 \\
&\quad + ke^{2\alpha k} E_2^T H_2^T D_2 H_2 E_2, \\
W_{12} &= B_0 P, \qquad W_{13} = e^{-2\alpha h_1} R, \qquad W_{14} = e^{-2\alpha h_2} R, \qquad W_{15} = -PA^T - 0.5BB^T, \\
W_{22} &= -(1-\delta)e^{-2\alpha\eta} Q_1, \qquad W_{25} = PB_0^T, \qquad W_{33} = -e^{-2\alpha h_1} Q - e^{-2\alpha h_1} R - e^{-2\alpha h_2} U, \\
W_{36} &= e^{-2\alpha h_2} U, \qquad W_{44} = -e^{-2\alpha h_2} Q - e^{-2\alpha h_2} R - e^{-2\alpha h_2} U, \qquad W_{46} = e^{-2\alpha h_2} U,
\end{aligned}$$

$$
\begin{aligned}
W_{55} &= h_1^2 R + h_2^2 R + (h_2 - h_1)^2 U + Q_1 - 2P + W_0 D_0 W_0^T + W_1 D_1 W_1^T \\
&\quad + k e^{2\alpha k} W_2 D_2 W_2^T + \epsilon_4 E_a^T E_a + \epsilon_5 E_0^T E_0 + \epsilon_6 E_1^T E_1 + k e^{2\alpha k} E_2^T H_2^T D_2 H_2 E_2, \\
W_{66} &= -1.9 e^{-2\alpha h_2} U.
\end{aligned}
\tag{3.40}
$$

Proof. Choose Lyapunov-Krasovskii functional as in (3.11) but change V_8 to $V_8 = 4\int_{-k}^{0}\int_{t+s}^{t} e^{2\alpha(\tau-t)} h^T(x(\tau)) D_2^{-1} h(x(\tau)) d\tau\, ds$, we may proof the Theorem by using a similar argument as in the proof of Theorem 3.2. By replacing A, W_0, W_1, and W_2 with $A + E_a F_a(t) H_a$, $W_0 + E_0 F_0(t) H_0$, $W_1 + E_1 F_1(t) H_1$, and $W_2 + E_2 F_2(t) H_2$, respectively. We have the following:

$$
\begin{aligned}
\dot{V}(t, x_t) + 2\alpha V(t, x_t) \le{}& y^T(t)\Big[(-A + E_a F_a(t) H_a) P + P(-A + E_a F_a(t) H_a)^T - BB^T + 2\alpha P \\
&\quad + 2Q - e^{-2\alpha h_1} R - e^{-2\alpha h_2} R\Big] y(t) + 2y^T(t)(W_0 + E_0 F_0(t) H_0) f(x(t)) \\
&+ 2y^T(t)(W_1 + E_1 F_1(t) H_1) g(x(t - h(t))) + 2y^T(t)(W_2 + E_2 F_2(t) H_2) \\
&\times \int_{t-k(t)}^{t} h(x(s)) ds + 2y^T(t) B_0 P \dot{y}(t - \eta(t)) \\
&- e^{-2\alpha h_1} y^T(t - h_1) Q y(t - h_1) - e^{-2\alpha h_2} y^T(t - h_2) Q y(t - h_2) \\
&+ h_1^2 \dot{y}^T(t) R \dot{y}(t) + h_2^2 \dot{y}^T(t) R \dot{y}(t) + (h_2 - h_1)^2 \dot{y}^T(t) U \dot{y}(t) \\
&+ e^{-2\alpha h_1} 2y^T(t) R y(t - h_1) - e^{-2\alpha h_1} y^T(t - h_1) R y(t - h_1) \\
&+ e^{-2\alpha h_2} 2y^T(t) R y(t - h_2) - e^{-2\alpha h_2} y^T(t - h_2) R y(t - h_2) \\
&- e^{-2\alpha h_2} \left[y(t - h(t)) - y(t - h_2)\right]^T U \left[y(t - h(t) - y(t - h_2))\right]^T \\
&- e^{-2\alpha h_2} \left[y(t - h_1) - y(t - h(t))\right]^T U \left[y(t - h_1) - y(t - h(t))\right]^T \\
&- \beta e^{-2\alpha h_2} \left[y(t - h_1) - y(t - h(t))\right]^T U \left[y(t - h_1) - y(t - h(t))\right]^T \\
&- e^{-2\alpha h_2} (1 - \beta) \left[y(t - h(t)) - y(t - h_2)\right]^T U \left[y(t - h(t)) - y(t - h_2)\right]^T \\
&+ \dot{y}^T(t) Q_1 \dot{y}(t) - (1 - \delta) e^{-2\alpha\eta} \dot{y}^T(t - \eta(t)) Q_1 \dot{y}(t - \eta(t)) \\
&+ 4k y^T(t) P H D_2^{-1} H P y(t) - 4e^{-2\alpha k} \int_{t-k}^{t} h^T(x(s)) D_2^{-1} h(x(s)) ds \\
&- 2\dot{y}^T(t) P \dot{y}(t) - 2\dot{y}^T(t)(A + E_a F_a(t) H_a) P y(t) \\
&+ 2\dot{y}^T(t)(W_0 + E_0 F_0(t) H_0) f(x(t)) \\
&+ 2\dot{y}^T(t)(W_1 + E_1 F_1(t) H_1) g(x(t - h(t)))
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\dot{y}^T(t)(W_2 + E_2F_2(t)H_2)\int_{t-k(t)}^{t} h(x(s))ds + 2\dot{y}^T(t)B_0P\dot{y}\big(t-\eta(t)\big) \\
&- \dot{y}^T(t)BB^Ty(t) \\
&= \xi^T(t)\big[(1-\beta)\mathcal{M}_1 + \beta\mathcal{M}_2\big]\xi(t) + y^T(t)M_3y(t) \\
&+ y^T(t-h(t))M_4y(t-h(t)).
\end{aligned} \tag{3.41}
$$

Applying Proposition 2.3 and Lemma 2.6, the following estimations hold:

$$
\begin{aligned}
&y^T(t)\Big[(-A + E_aF_a(t)H)_a)P + P\Big(-A^T + H_a^TF_a^T(t)E_a^T\Big)\Big]y(t) \\
&\quad \le y^T(t)\Big[-PA^T - AP\Big]y(t) + \epsilon_1y^T(t)E_a^TE_ay(t) + \epsilon_1^{-1}y^T(t)PH_a^TH_aPy(t), \\
&2y^T(t)[W_0 + E_0F_0(t)H_0]f(x(t)) \\
&\quad = 2y^T(t)W_0f(x(t)) + 2y^T(t)E_0F_0(t)H_0f(x(t)) \\
&\quad \le y^T(t)W_0D_0W_0^Ty(t) \\
&\qquad + y^T(t)PFD_0^{-1}FPy(t) + \epsilon_2y^T(t)E_0^TE_0y(t) + \epsilon_2^{-1}y^T(t)PFH_0^TH_0FPy(t), \\
&2y^T(t)(W_1 + E_1F_1(t)H_1)g(x(t-h(t))) \\
&\quad = 2y^T(t)W_1g(x(t-h(t))) + 2y^T(t)E_1F_1(t)H_1g(x(t-h(t))) \le y^T(t)W_1D_1W_1^Ty(t) \\
&\qquad + y^T(t-h(t))PGD_1^{-1}GPy(t-h(t)) + \epsilon_3y^T(t)E_1^TE_1y(t) \\
&\qquad + \epsilon_3^{-1}y^T(t-h(t))PGH_1^TH_1GPy^T(t-h(t)), \\
&2y^T(t)(W_2 + E_2F_2(t)H_2)\int_{t-k(t)}^{t} h(x(s))ds \\
&\quad = 2y^T(t)W_2\int_{t-k(t)}^{t} h(x(s))ds + 2y^T(t)E_2F_2(t)H_2\int_{t-k(t)}^{t} h(x(s))ds \\
&\quad \le ke^{2\alpha k}y^T(t)W_2D_2W_2^Ty(t) + e^{-2\alpha k}\int_{t-k}^{t} h^T(x(s))D_2^{-1}h(x(s))ds \\
&\qquad + ke^{2\alpha k}y^T(t)E_2^TH_2D_2H_2^TE_2y(t) + k^{-1}e^{-2\alpha k}\left[\int_{t-k(t)}^{t} h(x(s))ds\right]^T H_2^TH_2^{-T}D_2^{-1}H_2^{-1}H_2 \\
&\qquad \times \left[\int_{t-k(t)}^{t} h(x(s))ds\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq ke^{2\alpha k}y^{\mathrm{T}}(t)W_2D_2W_2^Ty(t)+e^{-2\alpha k}\int_{t-k}^{t}h^T(x(s))D_2^{-1}h(x(s))ds\\
&\quad+ke^{2\alpha k}y^T(t)E_2^TH_2D_2H_2^TE_2y(t)+e^{-2\alpha k}\int_{t-k}^{t}h^T(x(s))D_2^{-1}h(x(s))ds\\
&\quad-2\dot{y}^T(t)(A+E_aF_a(t)H_a)Py(t)\leq -2\dot{y}^T(t)APy(t)-2\dot{y}^T(t)E_aF_a(t)H_aPy(t)\\
&\leq -2\dot{y}^T(t)APy(t)+\epsilon_4\dot{y}^T(t)E_a^TE_a\dot{y}(t)+\epsilon_4^{-1}y^T(t)PH_a^TH_aPy(t),\\
&2\dot{y}^T(t)(W_0+E_0F_0(t)H_0)f(x(t))\\
&\quad=2\dot{y}^T(t)[W_0+E_0F_0(t)H_0]f(x(t))\\
&\quad\leq \dot{y}^T(t)W_0D_0W_0^T\dot{y}(t)\\
&\quad\quad+y^T(t)PFD_0^{-1}FPy(t)+\epsilon_5\dot{y}^T(t)E_0^TE_0\dot{y}(t)+\epsilon_5^{-1}y^T(t)PFH_0^TH_0FPy(t),\\
&2\dot{y}^T(t)(W_1+E_1F_1(t)H_1)g(x(t-h(t)))\\
&\quad\leq \dot{y}^T(t)W_1D_1W_1^T\dot{y}(t)+y^T(t-h(t))PGD_1^{-1}G\\
&\quad\quad\times Py(t-h(t))+\epsilon_6\dot{y}^T(t)E_1^TE_1\dot{y}(t)+\epsilon_6^{-1}y^T(t-h(t))PGH_1^TH_1GPy^T(t-h(t)),\\
&2\dot{y}^T(t)(W_2+E_2F_2(t)H_2)\\
&\quad\leq ke^{2\alpha k}\dot{y}^T(t)W_2D_2W_2^T\dot{y}(t)+2e^{-2\alpha k}\int_{t-k}^{t}h^T(x(s))D_2^{-1}h(x(s))ds\\
&\quad\quad+ke^{2\alpha k}\dot{y}^T(t)E_2^TH_2^TD_2H_2E_2\dot{y}(t).
\end{aligned}
\tag{3.42}
$$

□

Remark 3.4. In [10, 13, 14], exponential stability of neutral-type neural networks with time-varying delays were investigated. However, the distributed delays have not been considered. Stability conditions in [13, 26–28] are not applicable to our work, since we consider more activation functions than them. Therefore, our stability conditions are less conservative than some other existing results.

Remark 3.5. In this paper, the restriction that the state delay is differentiable is not required which allows the state delay to be fast time-varying. Meanwhile, this restriction is required in some existing result, see [13, 14, 26–28].

4. Numerical Examples

In this section, we now provide an example to show the effectiveness of the result in Theorem 3.2.

Example 4.1. Consider the neural networks with interval time-varying delay and control input with the following parameters:

$$\dot{x}(t)=-Ax(t)+W_0f(x(t))+W_1g(x(t-h(t)))+Bu(t), \tag{4.1}$$

where

$$A = \begin{bmatrix} -0.2 & 0 \\ 1 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.4 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.5 & 0.2 \end{bmatrix}, \quad F = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix},$$
$$G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}. \tag{4.2}$$

It is worth noting that, the delay functions $h(t) = 0.1 + 0.1|\sin t|$. Therefore, the methods used in [2, 9] are not applicable to this system. We have $h_1 = 0.1$, $h_2 = 0.2$. Given $\alpha = 0.2$ and any initial function $\phi(t) = C^1([-0.2, 0], \mathbb{R}^2)$. Using the Matlab LMI toolbox, we obtain

$$P = \begin{bmatrix} 0.0370 & 0.0010 \\ 0.0010 & 0.2938 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0008 & 0.0029 \\ 0.0029 & 0.0250 \end{bmatrix}, \quad U = \begin{bmatrix} 0.0153 & 0.0080 \\ 0.008 & 0.6201 \end{bmatrix},$$
$$R = \begin{bmatrix} 0.0377 & 0.0055 \\ 0.0055 & 0.8173 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.0353 & 0 \\ 0 & 0.2833 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.0215 & 0 \\ 0 & 0.5025 \end{bmatrix}. \tag{4.3}$$

Thus, the system (4.1) is 0.2-exponentially stabilizable and the value $\sqrt{\lambda_2/\lambda_1} = 1.6469$, so the solution of the closed-loop system satisfies

$$\|x(t,\phi)\| \le 1.6469 e^{-0.2t} \|\phi\|, \quad \forall t \in \mathbb{R}^+. \tag{4.4}$$

Example 4.2. Consider the neural networks with mixed interval time-varying delays and control input with the following parameters:

$$\begin{aligned} \dot{x}(t) = {} & -(A + \Delta A(t))x(t) + (W_0 + \Delta W_0) f(x(t)) + (W_1 + \Delta W_1) g(x(t - h(t))) \\ & + (W_2 + \Delta W_2) \int_{t-k(t)}^{t} h(x(s)) ds + B_0 \dot{x}(t - \eta(t)) + Bu(t), \end{aligned} \tag{4.5}$$

where

$$A = \begin{bmatrix} 0.15 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.5 & 0.12 \\ 0.1 & -0.3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.15 \end{bmatrix}, \quad F = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad H = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \tag{4.6}$$
$$B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad H_a = H_0 = H_1 = H_2 = E_a = E_0 = E_1 = E_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

It is worth noting that, the delay functions $h(t) = 0.2 + 0.2|\sin t|$, $k(t) = |\cos t|$ are nondifferentiable and $\eta(t) = 0.2\sin^2(t)$. Therefore, the methods used in [13, 14] are not

applicable to this system. We have $h_1 = 0.2$, $h_2 = 0.4$, $k = 0.1$, $\delta = 0.1$, $\eta = 0.2$. Given $\alpha = 0.1$ and any initial function $\phi(t) = C^1([-0.4, 0], \mathbb{R}^2)$. Using the Matlab LMI toolbox, we obtain $e_1 = 0.0173$, $e_2 = 0.0128$, $e_3 = 0.0111$, $e_4 = 0.0263$, $e_5 = 0.0209$, $e_6 = 0.0192$,

$$P = \begin{bmatrix} 0.0061 & 0.0002 \\ 0.0002 & 0.0228 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0003 & 0.0005 \\ 0.0001 & 0.0031 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0005 & 0.0001 \\ 0.0001 & 0.0024 \end{bmatrix},$$
$$U = \begin{bmatrix} 0.0028 & 0.0004 \\ 0.0004 & 0.00382 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0052 & 0.0008 \\ 0.0008 & 0.0543 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.0068 & 0 \\ 0 & 0.0304 \end{bmatrix}, \tag{4.7}$$
$$D_1 = \begin{bmatrix} 0.0038 & 0 \\ 0 & 0.0145 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.0433 & 0 \\ 0 & 0.0275 \end{bmatrix}.$$

Thus, the system (4.1) is 0.1-exponentially stabilizable and the value $\sqrt{\lambda_2/\lambda_1} = 2.2939$, so the solution of the closed-loop system satisfies

$$\|x(t,\phi)\| \le 2.2939 e^{-0.1t} \|\phi\|, \quad \forall t \in \mathbb{R}^+. \tag{4.8}$$

5. Conclusions

In this paper, we have investigated the exponential stabilization of neutral-type neural networks with various activation functions and interval nondifferentiable and distributed time-varying delays. The interval time-varying delay function is not necessary to be differentiable which allows time-delay function to be a fast time-varying function. By constructing a set of improved Lyapunov-Krasovskii functional combined with Leibniz-Newton's formula, the proposed stability criteria have been formulated in the form of a linear matrix inequalities. Numerical examples illustrate the effectiveness of the results.

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