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Research Article

Existence of Periodic Solutions for a Class of Difference Systems with p-Laplacian

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By applying the least action principle and minimax methods in critical point theory, we prove the existence of periodic solutions for a class of difference systems with p-Laplacian and obtain some existence theorems.

1. Introduction

Consider the following p-Laplacian difference system:

$$\Delta\left(|\Delta u(t-1)|^{p-2}\Delta u(t-1)\right) = \nabla F(t, u(t)), \quad t \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $p \in (1, +\infty)$ such that $1/p + 1/q = 1$, $t \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, and $F(t,x)$ is continuously differentiable in x for every $t \in \mathbb{Z}$ and T-periodic in t for all $x \in \mathbb{R}^N$.

When $p = 2$, (1.1) reduces to the following second-order discrete Hamiltonian system:

$$\Delta^2 u(t-1) = \nabla F(t, u(t)), \quad t \in \mathbb{Z}. \tag{1.2}$$

Difference equations provide a natural description of many discrete models in real world. Since discrete models exist in various fields of science and technology such as statistics, computer science, electrical circuit analysis, biology, neural network, and optimal control, it is of practical importance to investigate the solutions of difference equations. For more details about difference equations, we refer the readers to the books [1–3].

In some recent papers [4–18], the authors studied the existence of periodic solutions and subharmonic solutions of difference equations by applying critical point theory. These papers show that the critical point theory is an effective method to the study of periodic solutions for difference equations. Motivated by the above papers, we consider the existence of periodic solutions for problem (1.1) by using the least action principle and minimax methods in critical point theory.

2. Preliminaries

Now, we first present our main results.

Theorem 2.1. *Suppose that F satisfies the following conditions:*

(F1) *there exists an integer $T > 1$ such that $F(t+T, x) = F(t, x)$ for all $(t, x) \in \mathbb{Z} \times \mathbb{R}^N$;*

(F2) *there exist $f, g \in l^1([1, T], \mathbb{R}^+)$ and $\alpha \in [0, p-1)$ such that*

$$|\nabla F(t, x)| \leq f(t)|x|^{\alpha} + g(t), \quad \forall (t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N, \tag{2.1}$$

where $\mathbb{Z}[a, b] := \mathbb{Z} \cap [a, b]$ for every $a, b \in \mathbb{Z}$ with $a \leq b$,

(F3)

$$\liminf_{|x| \to +\infty} |x|^{-q\alpha} \sum_{t=1}^{T} F(t, x) > \frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{qT} \sum_{t=1}^{T} f^q(t), \quad \forall t \in \mathbb{Z}[1, T]. \tag{2.2}$$

Then problem (1.1) *has at least one periodic solution with period T.*

Theorem 2.2. *Suppose that F satisfies (F1) and the following conditions:*

$$\sum_{t=1}^{T} f(t) < \frac{T^p}{2^{p-1}(T-1)^{p(1+q)/q}}; \tag{2.3}$$

(F2)′ there exist $f, g \in l^1([1, T], \mathbb{R}^+)$ such that

$$|\nabla F(t, x)| \leq f(t)|x|^{p-1} + g(t), \quad \forall (t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N, \tag{2.4}$$

where $\mathbb{Z}[a, b] := \mathbb{Z} \cap [a, b]$ for every $a, b \in \mathbb{Z}$ with $a \leq b$;

(F4)

$$\liminf_{|x| \to +\infty} |x|^{-p} \sum_{t=1}^{T} F(t, x) > \frac{2^p T^{q/p}(T-1)^{q(2p-1)/p}}{\left[T^p - 2^{p-1}(T-1)^{p(1+q)/q} \sum_{t=1}^{T} f(t)\right]^{q/p}} \sum_{t=1}^{T} f^q(t), \quad \forall t \in \mathbb{Z}[1, T]. \tag{2.5}$$

Then problem (1.1) *has at least one periodic solution with period T.*

Theorem 2.3. *Suppose that F satisfies (F1), (F2), and the following condition:*

(F5)

$$\limsup_{|x|\to+\infty}|x|^{-q\alpha}\sum_{t=1}^{T}F(t,x) < -\left[\frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{pT}+\frac{2^{q\alpha}(T-1)^{(q-1)^2(2p-1)/p}}{qT^{(q-1)^2/q}}+\frac{2^{q\alpha}(T-1)^{2p-1+(2p-1)/p}}{pT^{(p+1)/q}}\right]\sum_{t=1}^{T}f^{q}(t) \quad \forall t\in\mathbb{Z}[1,T]. \tag{2.6}$$

Then problem (1.1) *has at least one periodic solution with period* T*.*

Theorem 2.4. *Suppose that F satisfies (F1),* (2.3)*, (F2)′, and the following condition:*

(F6)

$$\begin{aligned}\limsup_{|x|\to+\infty}&|x|^{-p}\sum_{t=1}^{T}F(t,x)\\ &< -\left[\frac{2^{p}(pT)^{q/p}(T-1)^{q(2p-1)/p}\left(T^{p}+2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right)}{\left[pT^{p}-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{q}}\right.\\ &\quad+\frac{2^{p}(pT)^{1/p}T(T-1)^{2p-1+(2p-1)/p}}{\left[pT^{p}-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{1+1/p}}\\ &\quad\left.+\frac{2^{p}(pT)^{(q-1)^2/p}(T-1)^{(q-1)^2(2p-1)/p}}{q\left[pT^{p}-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{(q-1)^2/p}}\right]\sum_{t=1}^{T}f^{q}(t),\quad \forall t\in\mathbb{Z}[1,T].\end{aligned} \tag{2.7}$$

Then problem (1.1) *has at least one periodic solution with period* T*.*

Remark 2.5. The lower bounds and the upper bounds of our theorems are more accurate than the existing results in the literature. Moreover, there are functions satisfying our results but not satisfying the existing results in the literature.

Let the Sobolev space E_T be defined by

$$E_T=\left\{u:\mathbb{Z}\longrightarrow\mathbb{R}^N \mid u(t+T)=u(t),\ t\in\mathbb{Z}\right\}. \tag{2.8}$$

For $u\in E_T$, let $\overline{u}=(1/T)\sum_{t=1}^{T}u(t)$, $u=\overline{u}+\widetilde{u}$, and $\widetilde{E}_T=\{u\in E_T \mid \overline{u}=0\}$, then $E_T=\mathbb{R}^N\oplus\widetilde{E}_T$. Let

$$\|u\|=\left(|\overline{u}|^{p}+\sum_{t=1}^{T}|\Delta\widetilde{u}(t)|^{p}\right)^{1/p},\quad u\in E_T. \tag{2.9}$$

As usual, let

$$\|u\|_{\infty}=\sup\{|u(t)| : t\in\mathbb{Z}[1,T]\},\quad \forall u\in l^{\infty}\left(\mathbb{Z}[1,T],\mathbb{R}^N\right). \tag{2.10}$$

For any $u \in E_T$, let

$$\varphi(u) = \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}F(t,u(t)) = \frac{1}{p}\sum_{t=1}^{T}|\Delta \tilde{u}(t)|^p + \sum_{t=1}^{T}F(t,u(t)). \tag{2.11}$$

To prove our results, we need the following lemma.

Lemma 2.6 (see [18]). *Let $u \in E_T$. If $\sum_{t=1}^{T} u(t) = 0$, then*

$$\|u\|_\infty \leq \frac{(T-1)^{(1+q)/q}}{T}\|\tilde{u}\|, \tag{2.12}$$

$$\|u\|_p^p = \sum_{t=1}^{T}|u(t)|^p \leq \frac{(T-1)^{2p-1}}{T^{p-1}}\|\tilde{u}\|^p. \tag{2.13}$$

3. Proofs

For the sake of convenience, we denote

$$M_1 = \left(\sum_{t=1}^{T}f^q(t)\right)^{1/q}, \qquad M_2 = \sum_{t=1}^{T}f(t), \qquad M_3 = \sum_{t=1}^{T}g(t). \tag{3.1}$$

Proof of Theorem 2.1. From (F3), we can choose $a_1 > (T-1)^{(2p-1)/p}/T^{(p-1)/p}$ such that

$$\liminf_{|x|\to+\infty}|x|^{-q\alpha}\sum_{t=1}^{T}F(t,x) > \frac{a_1^q 2^{q\alpha}}{q}M_1^q. \tag{3.2}$$

It follows from (F2), (2.12), and (2.13) that

$$\begin{aligned}
&\left|\sum_{t=1}^{T}[F(t,u(t)) - F(t,\overline{u})]\right| \\
&= \left|\sum_{t=1}^{T}\int_0^1(\nabla F(t,\overline{u}+s\tilde{u}(t)),\tilde{u}(t))ds\right| \\
&\leq \sum_{t=1}^{T}\int_0^1 f(t)|\overline{u}+s\tilde{u}(t)|^\alpha|\tilde{u}(t)|ds + \sum_{t=1}^{T}\int_0^1 g(t)|\tilde{u}(t)|ds \\
&\leq 2^\alpha\sum_{t=1}^{T}f(t)\left(|\overline{u}|^\alpha + |\tilde{u}(t)|^\alpha\right)|\tilde{u}(t)| + \sum_{t=1}^{T}g(t)|\tilde{u}(t)| \\
&\leq 2^\alpha|\overline{u}|^\alpha\left(\sum_{t=1}^{T}f^q(t)\right)^{1/q}\left(\sum_{t=1}^{T}|\tilde{u}(t)|^p\right)^{1/p} + 2^\alpha\|\tilde{u}\|_\infty^{1+\alpha}\sum_{t=1}^{T}f(t) + \|\tilde{u}\|_\infty\sum_{t=1}^{T}g(t) \\
&= 2^\alpha|\overline{u}|^\alpha \mathrm{M}_1\|\tilde{u}\|_p + 2^\alpha M_2\|\tilde{u}\|_\infty^{1+\alpha} + M_3\|\tilde{u}\|_\infty
\end{aligned}$$

$$\leq \frac{1}{pa_1^p}\|\tilde{u}\|_p^p + \frac{a_1^q 2^{q\alpha}}{q}|\overline{u}|^{q\alpha} M_1^q + 2^\alpha M_2 \|\tilde{u}\|_\infty^{1+\alpha} + M_3\|\tilde{u}\|_\infty$$
$$\leq \frac{(T-1)^{2p-1}}{pa_1^p T^{p-1}}\|\tilde{u}\|^p + \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\tilde{u}\|^{1+\alpha} + \frac{a_1^q 2^{q\alpha}}{q}|\overline{u}|^{q\alpha} M_1^q + \frac{M_3 (T-1)^{(1+q)/q}}{T}\|\tilde{u}\|. \tag{3.3}$$

Hence, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}F(t,u(t)) \\
&= \frac{1}{p}\sum_{t=1}^{T}|\Delta \tilde{u}(t)|^p + \sum_{t=1}^{T}[F(t,u(t)) - F(t,\overline{u})] + \sum_{t=1}^{T}F(t,\overline{u}) \\
&\geq \frac{1}{p}\sum_{t=1}^{T}|\Delta \tilde{u}(t)|^p - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\tilde{u}\|^{1+\alpha} + \sum_{t=1}^{T}F(t,\overline{u}) \\
&\quad - \frac{(T-1)^{2p-1}}{pa_1^p T^{p-1}}\|\tilde{u}\|^p - \frac{a_1^q 2^{q\alpha}}{q}|\overline{u}|^{q\alpha} M_1^q - \frac{M_3 (T-1)^{(1+q)/q}}{T}\|\tilde{u}\| \\
&= \left(\frac{1}{p} - \frac{(T-1)^{2p-1}}{pa_1^p T^{p-1}}\right)\|\tilde{u}\|^p - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\tilde{u}\|^{1+\alpha} \\
&\quad + |\overline{u}|^{q\alpha}\left(|\overline{u}|^{-q\alpha}\sum_{t=1}^{T}F(t,\overline{u}) - \frac{a_1^q 2^{q\alpha}}{q}M_1^q\right) - \frac{M_3 (T-1)^{(1+q)/q}}{T}\|\tilde{u}\|.
\end{aligned} \tag{3.4}$$

The above inequality and (3.2) imply that $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. Hence, by the least action principle, problem (1.1) has at least one periodic solution with period T. □

Proof of Theorem 2.2. From (2.3) and (F4), we can choose a constant $a_3 \in \mathbb{R}$ such that

$$\begin{gathered}
a_3 > \frac{T^{1/p}(T-1)^{(2p-1)/p}}{\left[T^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}} > 0, \\
\liminf_{|x|\to+\infty}|x|^{-p}\sum_{t=1}^{T}F(\mathrm{t},x) > \frac{a_3^q 2^p}{q}M_1^q.
\end{gathered} \tag{3.5}$$

It follows from (F2)′ and Lemma 2.6 that

$$\begin{aligned}
\left|\sum_{\mathrm{t}=1}^{T}[F(t,u(t)) - F(t,\overline{u})]\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,\overline{u} + s\tilde{u}(t)), \tilde{u}(t))ds\right| \\
&\leq \sum_{t=1}^{T}\int_0^1 f(t)|\overline{u} + s\tilde{u}(t)|^{p-1}|\tilde{u}(t)|ds + \sum_{t=1}^{T}g(t)|\tilde{u}(t)|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{t=1}^{T}\int_0^1 2^{p-1}f(t)\left(|\overline{u}|^{p-1}+s^{p-1}|\widetilde{u}(t)|^{p-1}\right)|\widetilde{u}(t)|ds+\sum_{t=1}^{T}g(t)|\widetilde{u}(t)| \\
&=\sum_{t=1}^{T}2^{p-1}f(t)\left(|\overline{u}|^{p-1}+\frac{1}{p}|\widetilde{u}(t)|^{p-1}\right)|\widetilde{u}(t)|+\sum_{t=1}^{T}g(t)|\widetilde{u}(t)| \\
&\leq 2^{p-1}|\overline{u}|^{p-1}\left(\sum_{t=1}^{T}f^q(t)\right)^{1/q}\left(\sum_{t=1}^{T}|\widetilde{u}(t)|^p\right)^{1/p}+\frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p+M_3\|\widetilde{u}\|_\infty \\
&=2^{p-1}M_1|\overline{u}|^{p-1}\|\widetilde{u}\|_p+\frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p+M_3\|\widetilde{u}\|_\infty \\
&\leq \frac{1}{pa_3^p}\|\widetilde{u}\|_p^p+\frac{a_3^qM_1^q2^p}{q}|\overline{u}|^p+\frac{2^{p-1}}{p}M_2\|\widetilde{u}\|_\infty^p+M_3\|\widetilde{u}\|_\infty \\
&\leq \left(\frac{(T-1)^{2p-1}}{pa_3^pT^{p-1}}+\frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}\|^p+\frac{a_3^qM_1^q2^p}{q}|\overline{u}|^p \\
&\quad+\frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\|,
\end{aligned} \tag{3.6}$$

which implies that

$$\begin{aligned}
\varphi(u)&=\frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p+\sum_{t=1}^{T}[F(t,u(t))-F(t,\overline{u})]+\sum_{t=1}^{T}F(t,\overline{u}) \\
&\geq \left(\frac{1}{p}-\frac{(T-1)^{2p-1}}{pa_3^pT^{p-1}}-\frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}\|^p \\
&\quad-\frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\|+|\overline{u}|^p\left(|\overline{u}|^{-p}\sum_{t=1}^{T}F(t,\overline{u})-\frac{a_3^qM_1^q2^p}{q}\right).
\end{aligned} \tag{3.7}$$

The above inequality and (3.5) imply that $\varphi(u)\to+\infty$ as $\|u\|\to\infty$. Hence, by the least action principle, problem (1.1) has at least one periodic solution with period T. □

Proof of Theorem 2.3. First we prove that φ satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of φ; that is, $\varphi'(u_n)\to 0$ as $n\to\infty$ and $\{\varphi(u_n)\}$ is bounded. By (F5), we can choose $a_2>(T-1)^{(2p-1)/p}/T^{(p-1)/p}$ such that

$$\limsup_{|x|\to+\infty}|x|^{-q\alpha}\sum_{t=1}^{T}F(\mathrm{t},x)<-\left(\frac{2^{q\alpha}a_2^q}{p}+\frac{2^{q\alpha}a_2^{(q-1)^2}}{q}+\frac{2^{q\alpha}a_2(T-1)^{2p-1}}{pT^{p-1}}\right)M_1^q. \tag{3.8}$$

In a similar way to the proof of Theorem 2.1, we have

$$\begin{aligned}
\left|\sum_{t=1}^{T}(\nabla F(t,u_n(t)),\widetilde{u}_n(t))\right|&\leq\frac{(T-1)^{2p-1}}{pa_2^pT^{p-1}}\|\widetilde{u}_n\|^p+\frac{2^\alpha M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\widetilde{u}_n\|^{1+\alpha} \\
&\quad+\frac{a_2^q2^{q\alpha}}{q}|\overline{u}_n|^{q\alpha}M_1^q+\frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}_n\|.
\end{aligned} \tag{3.9}$$

Hence, we have

$$\begin{aligned}
\|\tilde{u}_n\|_p &\geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\
&= \sum_{t=1}^{T} |\Delta u_n(t)|^p + \sum_{t=1}^{T} (\nabla F(t, u_n(t)), \tilde{u}_n(t)) \\
&\geq \left(1 - \frac{(T-1)^{2p-1}}{p a_2^p T^{p-1}}\right) \|\tilde{u}_n\|^p - \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\| \\
&\quad - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}} \|\tilde{u}_n\|^{1+\alpha} - \frac{a_2^q 2^{q\alpha}}{q} |\overline{u}_n|^{q\alpha} M_1^q.
\end{aligned} \tag{3.10}$$

From (2.13), we have

$$\|\tilde{u}_n\|_p = \left(\sum_{t=1}^{T} |\tilde{u}_n(t)|^p\right)^{1/p} \leq \frac{(T-1)^{(2p-1)/p}}{T^{(p-1)/p}} \|\tilde{u}_n\|. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\begin{aligned}
\frac{a_2^q 2^{q\alpha} M_1^q}{q} |\overline{u}_n|^{q\alpha} &\geq \left(1 - \frac{(T-1)^{2p-1}}{p a_2^p T^{p-1}}\right) \|\tilde{u}_n\|^p - \frac{(T-1)^{(2p-1)/p}}{T^{(p-1)/p}} \|\tilde{u}_n\| \\
&\quad - \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}} \|\tilde{u}_n\|^{1+\alpha} - \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\| \\
&\geq \frac{p-1}{p} \|\tilde{u}_n\|^p + C_1 \\
&= \frac{1}{q} \|\tilde{u}_n\|^p + C_1,
\end{aligned} \tag{3.12}$$

where $C_1 = \min_{s \in [0,+\infty)} \{(1/p - (T-1)^{2p-1}/p a_2^p T^{p-1}) s^p - (2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q} / T^{1+\alpha}) s^{1+\alpha} - [(T-1)^{(2p-1)/p}/T^{(p-1)/p} + M_3 (T-1)^{(1+q)/q}/T] s\}$. Notice that $a_2 > (T-1)^{(2p-1)/p}/T^{(p-1)/p}$ implies $-\infty < C_1 < 0$. Hence, it follows from (3.12) that

$$\|\tilde{u}_n\|^p \leq 2^{q\alpha} a_2^q M_1^q |\overline{u}_n|^{q\alpha} - q C_1, \tag{3.13}$$

$$\|\tilde{u}_n\| \leq 2^{q\alpha/p} a_2^{q/p} M_1^{q/p} |\overline{u}_n|^{q\alpha/p} + C_2, \tag{3.14}$$

where $C_2 > 0$. By the proof of Theorem 2.1, we have

$$\begin{aligned}
\left|\sum_{t=1}^{T} [F(t, u_n(t)) - F(t, \overline{u}_n)]\right| &\leq 2^\alpha M_1 |\overline{u}_n|^\alpha \|\tilde{u}_n\|_p + 2^\alpha M_2 \|\tilde{u}_n\|_\infty^{1+\alpha} + M_3 \|\tilde{u}_n\|_\infty \\
&\leq \frac{(T-1)^{2p-1}}{p a_2^{q-1} T^{p-1}} \|\tilde{u}_n\|^p + \frac{2^\alpha M_2 (T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}} \|\tilde{u}_n\|^{1+\alpha} \\
&\quad + \frac{a_2^{(q-1)^2} 2^{q\alpha}}{q} |\overline{u}_n|^{q\alpha} M_1^q + \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\|.
\end{aligned} \tag{3.15}$$

It follows from the boundedness of $\varphi(u_n)$, (3.13)–(3.15) that

$$\begin{aligned}
C_3 &\le \varphi(u_n) \\
&= \frac{1}{p}\sum_{t=1}^{T}|\Delta u_n(t)|^p + \sum_{t=1}^{T}[F(t,u_n(t)) - F(t,\overline{u}_n)] + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\le \left(\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_2^{q-1}T^{p-1}}\right)\|\widetilde{u}_n\|^p + \frac{2^{\alpha}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\|\widetilde{u}_n\|^{1+\alpha} \\
&\quad + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q}|\overline{u}_n|^{q\alpha}M_1^q + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}_n\| + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\le \left(\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_2^{q-1}T^{p-1}}\right)\left(2^{q\alpha}a_2^qM_1^q|\overline{u}_n|^{q\alpha} - qC_1\right) + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q}|\overline{u}_n|^{q\alpha}M_1^q + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\quad + \frac{2^{\alpha}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\left(2^{q\alpha/p}a_2^{q/p}M_1^{q/p}|\overline{u}_n|^{q\alpha/p} + C_2\right)^{1+\alpha} \\
&\quad \times \frac{M_3(T-1)^{(1+q)/q}}{T}\left(2^{q\alpha/p}a_2^{q/p}M_1^{q/p}|\overline{u}_n|^{q\alpha/p} + C_2\right) \\
&\le \left(\frac{2^{q\alpha}a_2^q}{p} + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q} + \frac{a_2 2^{q\alpha}(T-1)^{2p-1}}{pT^{p-1}}\right)M_1^q|\overline{u}_n|^{q\alpha} - \left(\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_2^{q-1}T^{p-1}}\right)qC_1 \\
&\quad + \frac{2^{2\alpha}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}\left(2^{q\alpha(1+\alpha)/p}a_2^{q(1+\alpha)/p}M_1^{q(1+\alpha)/p}|\overline{u}_n|^{q\alpha(1+\alpha)/p} + C_2^{1+\alpha}\right) \\
&\quad + \sum_{t=1}^{T}F(t,\overline{u}_n) + \frac{M_3(T-1)^{(1+q)/q}}{T}\left(2^{q\alpha/p}a_2^{q/p}M_1^{q/p}|\overline{u}_n|^{q\alpha/p} + C_2\right) \\
&= |\overline{u}_n|^{q\alpha}\Bigg[|\overline{u}_n|^{-q\alpha}\sum_{t=1}^{T}F(t,\overline{u}_n) + \left(\frac{2^{q\alpha}a_2^q}{p} + \frac{a_2^{(q-1)^2}2^{q\alpha}}{q} + \frac{a_2 2^{q\alpha}(T-1)^{2p-1}}{pT^{p-1}}\right)M_1^q \\
&\qquad + \frac{2^{2\alpha+q\alpha(1+\alpha)/p}a_2^{q(1+\alpha)/p}M_1^{q(1+\alpha)/p}M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{1+\alpha}}|\overline{u}_n|^{\alpha(1+\alpha-p)(q-1)} \\
&\qquad + \frac{2^{q\alpha/p}a_2^{q/p}M_1^{q/p}M_3(T-1)^{(1+q)/q}}{T}|\overline{u}_n|^{-\alpha}\Bigg] + C_4,
\end{aligned} \tag{3.16}$$

where C_3 is a positive constant and C_4 is a constant. The above inequality and (3.8) imply that $\{\overline{u}_n\}$ is bounded. Hence $\{u_n\}$ is bounded by (2.13) and (3.13). Since E_T is finite dimensional, we conclude that φ satisfies (PS) condition.

In order to use the saddle point theorem ([19], Theorem 4.6), we only need to verify the following conditions:

(I1) $\varphi(u) \to -\infty$ as $|u| \to \infty$ in $\mathbb{R}^N$,

(I2) $\varphi(u) \to +\infty$ as $|u| \to \infty$ in $\widetilde{E}_T$, In fact, from (F5), we have

$$\sum_{t=1}^{T} F(t,u) \longrightarrow -\infty \quad \text{as } |u| \longrightarrow \infty \text{ in } \mathbb{R}^N, \tag{3.17}$$

which together with (2.11) implies that

$$\varphi(u) = \sum_{t=1}^{T} F(t,u) \longrightarrow -\infty \quad \text{as } |u| \longrightarrow \infty \text{ in } \mathbb{R}^N. \tag{3.18}$$

Hence, (I1) holds.

Next, for all $u \in \widetilde{E}_T$, by (F2) and (2.12), we have

$$\begin{aligned} \left|\sum_{t=1}^{T}[F(t,u(t)) - F(t,0)]\right| &= \left|\sum_{t=1}^{T}\int_0^1 (\nabla F(t,su(t)),u(t))ds\right| \\ &\le \sum_{t=1}^{T} f(t)|u(t)|^{1+\alpha} + \sum_{t=1}^{T} g(t)|u(t)| \\ &\le M_2\|u\|_\infty^{1+\alpha} + M_3\|u\|_\infty \\ &\le \frac{M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{(1+\alpha)}}\|\widetilde{u}\|^{1+\alpha} + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\|, \end{aligned} \tag{3.19}$$

which implies that

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}[F(t,u(t)) - F(t,0)] + \sum_{t=1}^{T} F(t,0) \\ &\ge \frac{1}{p}\|\widetilde{u}\|^p - \frac{M_2(T-1)^{(1+q)(1+\alpha)/q}}{T^{(1+\alpha)}}\|\widetilde{u}\|^{1+\alpha} \\ &\quad - \frac{M_3(T-1)^{(1+q)/q}}{T}\|\widetilde{u}\| + \sum_{t=1}^{T} F(t,0), \end{aligned} \tag{3.20}$$

for all $u \in \widetilde{E}_T$. By Lemma 2.6, $\|u\| \to \infty$ in $\widetilde{E}_T$ if and only if $\|\widetilde{u}\| \to \infty$, so from (3.20), we obtain $\varphi(u) \to +\infty$ as $\|u\| \to \infty$ in $\widetilde{E}_T$; that is, (I2) is verified. Hence, the proof of Theorem 2.3 is complete. □

Proof of Theorem 2.4. First we prove that φ satisfies the (PS) condition. Assume that $\{u_n\}$ is a (PS) sequence of φ; that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded. By (2.3) and (F6), we can choose $a_4 \in \mathbb{R}$ such that

$$a_4 > \frac{p^{1/p} T^{1/p} (T-1)^{(2p-1)/p}}{\left[pT^p - 2^{p-1} M_2 (T-1)^{p(1+q)/q}\right]^{1/p}}, \tag{3.21}$$

$$\begin{aligned} &\limsup_{|x| \to +\infty} |x|^{-p} \sum_{t=1}^{T} F(t,x) \\ &\quad < -\left[\frac{2^p a_4^q \left(T^p + 2^{p-1} M_2 (T-1)^{p(1+q)/q}\right) + 2^p T a_4 (T-1)^{2p-1}}{pT^p - 2^{p-1} M_2 (T-1)^{p(1+q)/q}} + \frac{2^p a_4^{(q-1)^2}}{q}\right] M_1^q. \end{aligned} \tag{3.22}$$

In a similar way to the proof of Theorem 2.2, we obtain

$$\begin{aligned} \left|\sum_{t=1}^{T} (\nabla F(t, u_n(t)), \tilde{u}_n(t))\right| \le{}& \left(\frac{(T-1)^{2p-1}}{p a_4^p T^{p-1}} + \frac{2^{p-1} M_2 (T-1)^{p(1+q)/q}}{pT^p}\right) \|\tilde{u}_n\|^p \\ &+ \frac{a_4^q M_1^q 2^p}{q} |\overline{u}_n|^p + \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\|. \end{aligned} \tag{3.23}$$

Hence, we have

$$\begin{aligned} \|\tilde{u}_n\|_p &\ge \langle \varphi'(u_n), \tilde{u}_n \rangle \\ &= \frac{1}{p} \sum_{t=1}^{T} |\Delta u_n(t)|^p + \sum_{t=1}^{T} (\nabla F(t, u_n(t)), \tilde{u}_n(t)) \\ &\ge \left(1 - \frac{(T-1)^{2p-1}}{p a_4^p T^{p-1}} - \frac{2^{p-1} M_2 (T-1)^{p(1+q)/q}}{pT^p}\right) \|\tilde{u}_n\|^p - \frac{a_4^q M_1^q 2^p}{q} |\overline{u}_n|^p \\ &\quad - \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\|, \end{aligned} \tag{3.24}$$

which together with (3.11) implies that

$$\begin{aligned} \frac{a_4^q M_1^q 2^p}{q} |\overline{u}_n|^p \ge{}& \left(1 - \frac{(T-1)^{2p-1}}{p a_4^p T^{p-1}} - \frac{2^{p-1} M_2 (T-1)^{p(1+q)/q}}{pT^p}\right) \|\tilde{u}_n\|^p \\ &- \frac{M_3 (T-1)^{(1+q)/q}}{T} \|\tilde{u}_n\| - \frac{(T-1)^{(2p-1)/p}}{T^{1/q}} \|\tilde{u}_n\| \\ \ge{}& \frac{1}{q}\left(1 - \frac{2^{p-1} M_2 (T-1)^{p(1+q)/q}}{pT^p}\right) \|\tilde{u}_n\|^p + C_5, \end{aligned} \tag{3.25}$$

where $C_5 = \min_{s\in[0,+\infty)}\{(1/p - (T-1)^{2p-1}/pa_4^p T^{p-1} - 2^{p-1}M_2(T-1)^{p(1+q)/q}/p^2T^p)s^p - [M_3(T-1)^{(1+q)/q}/T + (T-1)^{(2p-1)/p}/T^{1/q}]s\}$. It follows from (3.21) that $-\infty < C_5 < 0$, so, we obtain

$$\|\tilde{u}_n\|^p \le \frac{pT^p a_4^q M_1^q 2^p}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}}|\overline{u}_n|^p - \frac{pT^p qC_5}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}}, \tag{3.26}$$

$$\|\tilde{u}_n\| \le \frac{2p^{1/p}Ta_4^{q/p}M_1^{q/p}}{\left[pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}|\overline{u}_n| + C_6, \tag{3.27}$$

where C_6 is a positive constant. By the proof of Theorem 2.2, we have

$$\begin{aligned}
&\left|\sum_{t=1}^{T}(\mathrm{F}(t,u_n(t)) - F(t,\overline{u}_n))\right| \\
&\quad\le 2^{p-1}M_1|\overline{u}|^{p-1}\|\tilde{u}\|_p + \frac{2^{p-1}}{p}M_2\|\tilde{u}\|_\infty^p + M_3\|\tilde{u}\|_\infty \\
&\quad\le \left(\frac{(T-1)^{2p-1}}{pa_4^{q-1}T^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\tilde{u}_n\|^p + \frac{a_4^{(q-1)^2}M_1^q 2^p}{q}|\overline{u}_n|^p \\
&\qquad + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\tilde{u}_n\|.
\end{aligned} \tag{3.28}$$

It follows from the boundedness of $\varphi(u_n)$, (3.26), (3.27), and the above inequality that

$$\begin{aligned}
C_7 &\le \varphi(u_n) \\
&= \frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p + \sum_{t=1}^{T}[F(t,u(t)) - F(t,\overline{u})] + \sum_{t=1}^{T}F(t,\overline{u}) \\
&\le \left[\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_4^{q-1}T^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right]\|\tilde{u}_n\|^p + \sum_{t=1}^{T}F(t,\overline{u}_n) \\
&\quad + \frac{M_3(T-1)^{(1+q)/q}}{T}\|\tilde{u}_n\| + \frac{a_4^{(q-1)^2}M_1^q 2^p}{q}|\overline{u}_n|^p \\
&\le \left[\frac{1}{p} + \frac{(T-1)^{2p-1}}{pa_4^{q-1}T^{p-1}} + \frac{2^{p-1}M_2(T-1)^{p(1+q)/q}}{pT^p}\right] \\
&\quad \times \left(\frac{pT^p a_4^q M_1^q 2^p}{pT^p - 2^{p-1}M_2(T-1)^{p(1+q)/q}}|\overline{u}_n|^p - \frac{pT^p qC_5}{pT^p - 2^{p-1}\mathrm{M}_2(T-1)^{p(1+q)/q}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t=1}^{T}F(t,\overline{u})+\frac{a_4^{(q-1)^2}M_1^q2^p}{q}|\overline{u}_n|^p\\
&+\frac{M_3(T-1)^{(1+q)/q}}{T}\left(\frac{2p^{1/p}Ta_4^{q/p}M_1^{q/p}}{\left[pT^p-2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}|\overline{u}_n|+C_6\right)\\
=|\overline{u}_n|^p&\left\{\left[\frac{2^pa_4^q\left(T^p+2^{p-1}M_2(T-1)^{p(1+q)/q}\right)+2^pTa_4(T-1)^{2p-1}}{pT^p-2^{p-1}M_2(T-1)^{p(1+q)/q}}+\frac{2^pa_4^{(q-1)^2}}{q}\right]M_1^q\right.\\
&\left.+|\overline{u}_n|^{-p}\sum_{t=1}^{T}F(t,\overline{u}_n)+\frac{2p^{1/p}Ta_4^{q/p}M_1^{q/p}M_3(T-1)^{(1+q)/q}}{T\left[pT^p-2^{p-1}M_2(T-1)^{p(1+q)/q}\right]^{1/p}}|\overline{u}_n|^{-p+1}\right\}+C_8,
\end{aligned}
\tag{3.29}
$$

where C_7 is a positive constant and C_8 is a constant. The above inequality and (3.22) imply that $\{\overline{u}_n\}$ is bounded. Hence, $\{u_n\}$ is bounded by (2.13) and (3.26).

Similar to the proof of Theorem 2.3, we only need to verify (I1) and (I2). It is easy to verify (I1) by (F6). Now, we verify that (I2) holds. For $u\in\widetilde{E}_T$, by (F2)′ and (2.12), we have

$$
\begin{aligned}
\left|\sum_{t=1}^{T}(F(t,u(t))-F(t,0))\right|&=\left|\sum_{t=1}^{T}\int_0^1(\nabla F(t,su(t)),u(t))ds\right|\\
&\le\sum_{t=1}^{T}\int_0^1 f(t)s^{p-1}|u(t)|^pds+\sum_{t=1}^{T}g(t)|u(t)|\\
&\le\frac{M_2}{p}\|u\|_\infty^p+M_3\|u\|_\infty\\
&\le\frac{M_2(T-1)^{p(1+q)/q}}{pT^p}\|\widetilde{u}\|^p+\frac{M_3(T-1)^{(1+q)/q}}{pT}\|\widetilde{u}\|.
\end{aligned}
\tag{3.30}
$$

Thus, we have

$$
\begin{aligned}
\varphi(u)&=\frac{1}{p}\sum_{t=1}^{T}|\Delta u(t)|^p+\sum_{t=1}^{T}(F(t,u(t))-F(t,0))+\sum_{t=1}^{T}F(t,0)\\
&\ge\left(\frac{1}{p}-\frac{M_2(T-1)^{p(1+q)/q}}{pT^p}\right)\|\widetilde{u}\|^p-\frac{M_3(T-1)^{(1+q)/q}}{pT}\|\widetilde{u}\|+\sum_{t=1}^{T}F(t,0),
\end{aligned}
\tag{3.31}
$$

for all $u\in\widetilde{E}_T$. By Lemma 2.6, $\|u\|\to\infty$ in $\widetilde{E}_T$ if and only $\|\widetilde{u}\|\to\infty$. So from the above inequality, we have $\varphi(u)\to+\infty$ as $\|u\|\to\infty$, that is (I2) is verified. Hence, the proof of Theorem 2.4 is complete. □

4. Example

In this section, we give four examples to illustrate our results.

Example 4.1. Let $p = 5/2$ and

$$F(t,x) = \sin\left(\frac{2\pi t}{T}\right)|x|^{5/3} + \left(\sin\frac{2\pi t}{T} + 1\right)|x|^{4/3} + (h(t), x), \tag{4.1}$$

where $h \in l^1(\mathbb{Z}[1,T], \mathbb{R}^N)$ and $h(t+T) = h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned} |\nabla F(t,x)| &\le \frac{5}{3}\left|\sin\frac{2\pi t}{T}\right||x|^{2/3} + \frac{4}{3}\left|\sin\frac{2\pi t}{T} + 1\right||x|^{1/3} + |h(t)| \\ &\le \frac{5}{3}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right)|x|^{2/3} + a(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \end{aligned} \tag{4.2}$$

where $\varepsilon > 0$, and $a(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2) holds with $\alpha = 2/3$ and

$$f(t) = \frac{5}{3}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right), \qquad g(t) = a(\varepsilon) + |h(t)|. \tag{4.3}$$

Moreover, we have

$$\begin{gathered} \liminf_{|x|\to+\infty} |x|^{-2\alpha}\sum_{t=1}^{T} F(t,x) = T, \\ \frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{qT}\sum_{t=1}^{T} f^q(t) = \frac{3\times 2^{10/9}(T-1)^{8/3}}{5}\left(\frac{5}{3}\varepsilon\right)^{5/3}. \end{gathered} \tag{4.4}$$

We can choose ε suitable such that

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha}\sum_{t=1}^{T} F(t,x) = T > \frac{3\times 2^{10/9}(T-1)^{8/3}}{5}\left(\frac{5}{3}\varepsilon\right)^{5/3} = \frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{qT}\sum_{t=1}^{T} f^q(t), \tag{4.5}$$

which shows that (F3) holds. Then from Theorem 2.1, problem (1.1) has at least one periodic solution with period T.

Example 4.2. Let $p = 2$, then $q = 2$. Let

$$F(t,x) = \frac{1}{6}\left(\frac{1}{2} + \sin\frac{2\pi t}{T}\right)|x|^2 + |x|^{3/2} + (h(t), x), \tag{4.6}$$

where $h \in l^1(\mathbb{Z}[1,T],\mathbb{R}^N)$ and $h(t+T)=h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned}|\nabla F(t,x)| &\leq \frac{1}{3}\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right||x|+\frac{3}{2}|x|^{1/2}+|h(t)| \\ &\leq \frac{1}{3}\left(\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right|+\varepsilon\right)|x|+b(\varepsilon)+|h(t)|, \quad \forall(t,x)\in\mathbb{Z}[1,T]\times\mathbb{R}^N,\end{aligned} \tag{4.7}$$

where $\varepsilon>0$, and $b(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2)′ holds with

$$f(t)=\frac{1}{3}\left(\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right|+\varepsilon\right), \qquad g(t)=b(\varepsilon)+|h(t)|. \tag{4.8}$$

Observe that

$$\begin{aligned}|x|^{-p}\sum_{t=1}^{T}F(t,x) &= |x|^{-2}\sum_{t=1}^{T}\left[\frac{1}{6}\left(\frac{1}{2}+\sin\frac{2\pi t}{T}\right)|x|^2+|x|^{3/2}+(h(t),x)\right] \\ &= \frac{T}{12}+T|x|^{-1/2}+\left(\sum_{t=1}^{T}h(t),|x|^{-2}x\right).\end{aligned} \tag{4.9}$$

On the other hand, if we let $T=2$, then we have

$$\begin{aligned}&\sum_{t=1}^{T}f(t)=\frac{2}{3}\left(\frac{1}{2}+\varepsilon\right), \qquad \sum_{t=1}^{T}f^2(t)=\frac{1}{9}\sum_{t=1}^{T}\left(\left|\frac{1}{2}+\sin\frac{2\pi t}{T}\right|+\varepsilon\right)^2=\frac{2}{9}\left(\frac{1}{2}+\varepsilon\right)^2, \\ &\frac{2^pT^{q/p}(T-1)^{q(2p-1)/p}}{\left[T^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{q/p}}\sum_{t=1}^{T}f^q(t)=\frac{2+8\varepsilon+8\varepsilon^2}{15-6\varepsilon}.\end{aligned} \tag{4.10}$$

We can choose ε sufficiently small such that

$$\begin{aligned}\sum_{t=1}^{T}f(t)&=\frac{2}{3}\left(\frac{1}{2}+\varepsilon\right)<2=\frac{T^p}{2^{p-1}(T-1)^{p-1}}, \\ \liminf_{|x|\to+\infty}|x|^{-p}\sum_{t=1}^{T}F(t,x) &= \frac{1}{6}>\frac{2+8\varepsilon+8\varepsilon^2}{15-6\varepsilon} \\ &= \frac{2^pT^{q/p}(T-1)^{q(2p-1)/p}}{\left[T^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^{T}f(t)\right]^{q/p}}\sum_{t=1}^{T}f^q(t),\end{aligned} \tag{4.11}$$

which shows that (2.3) and (F4) hold. Then from Theorem 2.2, problem (1.1) has at least one periodic solution with period T.

Example 4.3. Let $p=2$, then $q=2$. Let

$$F(t,x)=\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4}+\left(\sin\frac{2\pi t}{T}-1\right)|x|^{3/2}+(h(t),x), \tag{4.12}$$

where $h \in l^1(\mathbb{Z}[1,T],\mathbb{R}^N)$ and $h(t+T)=h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned} |\nabla F(t,x)| &\le \frac{7}{4}\left|\sin\frac{2\pi t}{T}\right||x|^{3/4} + \frac{3}{2}\left|\sin\frac{2\pi t}{T} - 1\right||x|^{1/2} + |h(t)| \\ &\le \frac{7}{4}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right)|x|^{3/4} + c(\varepsilon) + |h(t)|, \quad \forall (t,x)\in\mathbb{Z}[1,T]\times\mathbb{R}^N, \end{aligned} \tag{4.13}$$

where $\varepsilon > 0$ and $c(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2) holds with $\alpha = 3/4$ and

$$f(t) = \frac{7}{4}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right), \qquad g(t) = c(\varepsilon) + |h(t)|. \tag{4.14}$$

Observe that

$$\begin{aligned} |x|^{-q\alpha}\sum_{t=1}^{T} F(t,x) &= |x|^{-3/2}\sum_{t=1}^{T}\left[\sin\left(\frac{2\pi t}{T}\right)|x|^{7/4} + \left(\sin\frac{2\pi t}{T} - 1\right)|x|^{3/2} + (h(t),x)\right] \\ &= -T + \left(\sum_{t=1}^{T} h(t), |x|^{-3/2}x\right). \end{aligned} \tag{4.15}$$

On the other hand, we have

$$\begin{aligned} &\left[\frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{pT} + \frac{2^{q\alpha}(T-1)^{(q-1)^2(2p-1)/p}}{qT^{(q-1)^2/q}} + \frac{2^{q\alpha}(T-1)^{2p-1+(2p-1)/p}}{pT^{(p+1)/q}}\right]\sum_{t=1}^{T} f^q(t) \\ &= \left[\frac{\sqrt{2}(T-1)^3}{T} + \frac{\sqrt{2}(T-1)^{3/2}}{T^{1/2}} + \frac{\sqrt{2}(T-1)^{9/2}}{T^{3/2}}\right]\sum_{t=1}^{T}\frac{49}{16}\left(\left|\sin\frac{2\pi t}{T}\right| + \varepsilon\right)^2 \\ &= \frac{49\sqrt{2}\varepsilon^2(T-1)^{3/2}\left[T^{1/2}(T-1)^{3/2} + T + (T-1)^3\right]}{16T^{1/2}}. \end{aligned} \tag{4.16}$$

We can choose ε suitable such that

$$\begin{aligned} &\limsup_{|x|\to+\infty} |x|^{-q\alpha}\sum_{t=1}^{T} F(t,x) \\ &= -T \\ &< -\frac{49\sqrt{2}\varepsilon^2(T-1)^{3/2}\left[T^{1/2}(T-1)^{3/2} + T + (T-1)^3\right]}{16T^{1/2}} \\ &= -\left[\frac{2^{q\alpha}(T-1)^{q(2p-1)/p}}{pT} + \frac{2^{q\alpha}(T-1)^{(q-1)^2(2p-1)/p}}{qT^{(q-1)^2/q}} + \frac{2^{q\alpha}(T-1)^{2p-1+(2p-1)/p}}{pT^{(p+1)/q}}\right]\sum_{t=1}^{T} f^q(t), \end{aligned} \tag{4.17}$$

which shows that (F5) holds. Then from Theorem 2.3, problem (1.1) has at least one periodic solution with period T.

Example 4.4. Let $p = 2$, then $q = 2$. Let

$$F(t,x) = \frac{1}{3}\left(\sin\frac{2\pi t}{T} - \frac{1}{8}\right)|x|^2 + |x|^{3/2} + (h(t),x), \tag{4.18}$$

where $h \in l^1(\mathbb{Z}[1,T],\mathbb{R}^N)$ and $h(t+T) = h(t)$. It is easy to see that $F(t,x)$ satisfies (F1) and

$$\begin{aligned} |\nabla F(t,x)| &\le \frac{2}{3}\left|\sin\frac{2\pi t}{T} - \frac{1}{8}\right||x| + \frac{3}{2}|x|^{1/2} + |h(t)| \\ &\le \frac{2}{3}\left(\left|\sin\frac{2\pi t}{T} - \frac{1}{8}\right| + \varepsilon\right)|x| + d(\varepsilon) + |h(t)|, \quad \forall (t,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \end{aligned} \tag{4.19}$$

where $\varepsilon > 0$, $d(\varepsilon)$ is a positive constant and is dependent on ε. The above shows that (F2)′ holds with

$$f(t) = \frac{2}{3}\left(\left|\sin\frac{2\pi t}{T} - \frac{1}{8}\right| + \varepsilon\right), \quad g(t) = d(\varepsilon) + |h(t)|. \tag{4.20}$$

Observe that

$$\begin{aligned} |x|^{-p}\sum_{t=1}^{T} F(t,x) &= |x|^{-2}\sum_{t=1}^{T}\left[\frac{1}{3}\left(\sin\frac{2\pi t}{T} - \frac{1}{8}\right)|x|^2 + |x|^{3/2} + (h(t),x)\right] \\ &= -\frac{T}{24} + T|x|^{-1/2} + \left(\sum_{t=1}^{T} h(t), |x|^{-2}x\right). \end{aligned} \tag{4.21}$$

On the other hand, if we let $T = 2$, then we have

$$\sum_{t=1}^{T} f(t) = \frac{4}{3}\left(\frac{1}{8} + \varepsilon\right), \qquad \sum_{t=1}^{T} f^2(t) = \frac{4}{9}\sum_{t=1}^{T}\left(\left|\frac{1}{8} + \sin\frac{2\pi t}{T}\right| + \varepsilon\right)^2 = \frac{8}{9}\left(\frac{1}{8} + \varepsilon\right)^2, \tag{4.22}$$

$$
\begin{aligned}
&-\left[\frac{2^p(pT)^{q/p}(T-1)^{q(2p-1)/p}\left(T^p+2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right)}{\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right]^q}\right.\\
&\quad+\frac{2^p(pT)^{1/p}T(T-1)^{2p-1+(2p-1)/p}}{\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right]^{1+1/p}}\\
&\quad\left.+\frac{2^p(pT)^{(q-1)^2/p}(T-1)^{(q-1)^2(2p-1)/p}}{q\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right]^{(q-1)^2/p}}\right]\sum_{t=1}^T f^q(t)\\
&=\left[\frac{192+128\times(1/8+\varepsilon)}{3\times(8-(8/3)(1/8+\varepsilon))^2}+\frac{16}{(8-(8/3)(1/8+\varepsilon))^{3/2}}\right.\\
&\quad\left.+\frac{8}{(8-(8/3)(1/8+\varepsilon))^{1/2}}\right]\times\frac{8}{9}\left(\frac{1}{8}+\varepsilon\right)^2.
\end{aligned}
\tag{4.23}
$$

We can choose ε sufficiently small such that

$$
\sum_{t=1}^T f(t)=\frac{4}{3}\left(\frac{1}{8}+\varepsilon\right)<2=\frac{T^p}{2^{p-1}(T-1)^{p-1}},
\tag{4.24}
$$

$$
\begin{aligned}
\limsup_{|x|\to+\infty}|x|^{-p}\sum_{t=1}^T F(t,x)=-\frac{1}{12}
&<\left[\frac{192+128\times(1/8+\varepsilon)}{3\times(8-(8/3)(1/8+\varepsilon))^2}+\frac{16}{(8-(8/3)(1/8+\varepsilon))^{3/2}}\right.\\
&\quad\left.+\frac{8}{(8-(8/3)(1/8+\varepsilon))^{1/2}}\right]\times\frac{8}{9}\left(\frac{1}{8}+\varepsilon\right)^2\\
&=-\left[\frac{2^p(pT)^{q/p}(T-1)^{q(2p-1)/p}\left(T^p+2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right)}{\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right]^q}\right.\\
&\quad+\frac{2^p(pT)^{1/p}T(T-1)^{2p-1+(2p-1)/p}}{\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right]^{1+1/p}}\\
&\quad\left.+\frac{2^p(pT)^{(q-1)^2/p}(T-1)^{(q-1)^2(2p-1)/p}}{q\left[pT^p-2^{p-1}(T-1)^{p(1+q)/q}\sum_{t=1}^T f(t)\right]^{(q-1)^2/p}}\right]\sum_{t=1}^T f^q(t),
\end{aligned}
\tag{4.25}
$$

which shows that (F6) holds. Then from Theorem 2.4, problem (1.1) has at least one periodic solution with period T.

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References

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Chapman & Hall/ CRC Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.
[2] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, vol. 16 of *Kluwer Texts in the Mathematical Sciences*, Kluwer Academic Publishers Group, Dordrecht, The Netherlands, 1996.
[3] S. N. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 2nd edition, 1999.
[4] R. P. Agarwal and J. Popenda, "Periodic solutions of first order linear difference equations," *Mathematical and Computer Modelling*, vol. 22, no. 1, pp. 11–19, 1995.
[5] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular discrete p-Laplacian problems via variational methods," *Advances in Difference Equations*, vol. 2005, no. 2, pp. 93–99, 2005.
[6] Z. Guo and J. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China Series A*, vol. 46, no. 4, pp. 506–515, 2003.
[7] Z. Guo and J. Yu, "Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 7-8, pp. 969–983, 2003.
[8] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society*, vol. 68, no. 2, pp. 419–430, 2003.
[9] P. Jebelean and C. Şerban, "Ground state periodic solutions for difference equations with discrete p-Laplacian," *Applied Mathematics and Computation*, vol. 217, no. 23, pp. 9820–9827, 2011.
[10] H. Liang and P. Weng, "Existence and multiple solutions for a second-order difference boundary value problem via critical point theory," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 511–520, 2007.
[11] J. Mawhin, "Periodic solutions of second order nonlinear difference systems with ϕ-Laplacian: a variational approach," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 12, pp. 4672–4687, 2012.
[12] J. Rodriguez and D. L. Etheridge, "Periodic solutions of nonlinear second-order difference equations," *Advances in Difference Equations*, no. 2, pp. 173–192, 2005.
[13] Y.-F. Xue and C.-L. Tang, "Existence of a periodic solution for subquadratic second-order discrete Hamiltonian system," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 7, pp. 2072–2080, 2007.
[14] J. Yu, Z. Guo, and X. Zou, "Periodic solutions of second order self-adjoint difference equations," *Journal of the London Mathematical Society*, vol. 71, no. 1, pp. 146–160, 2005.
[15] J. Yu, Y. Long, and Z. Guo, "Subharmonic solutions with prescribed minimal period of a discrete forced pendulum equation," *Journal of Dynamics and Differential Equations*, vol. 16, no. 2, pp. 575–586, 2004.
[16] J. Yu, X. Deng, and Z. Guo, "Periodic solutions of a discrete Hamiltonian system with a change of sign in the potential," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1140–1151, 2006.
[17] Z. Zhou, J. Yu, and Z. Guo, "Periodic solutions of higher-dimensional discrete systems," *Proceedings of the Royal Society of Edinburgh A*, vol. 134, no. 5, pp. 1013–1022, 2004.
[18] Z. M. Luo and X. Y. Zhang, "Existence of nonconstant periodic solutions for a nonlinear discrete system involving the p-Laplacian," *Bulletin of the Malaysian Mathematical Science Society*, vol. 35, no. 2, pp. 373–382, 2012.
[19] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, USA, 1986.