Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 150571, 10 pages
doi:10.1155/2012/150571

Research Article

On Subclass of k-Uniformly Convex Functions of Complex Order Involving Multiplier Transformations

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Received 30 December 2011; Revised 28 February 2012; Accepted 13 March 2012

Academic Editor: Ondřej Došlý

We introduce a subclass of k-uniformly convex functions of order α with negative coefficients by using the multiplier transformations in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We obtain coefficient estimates, radii of convexity and close-to-convexity, extreme points, and integral means inequalities for the function f that belongs to the class $\mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$.

1. Introduction

Let $\mathcal{N}$ denote the class of functions of the form:

$$f(z)^{\beta} = z^{\beta} + \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1}, \quad \beta > 0, \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ (see [1]). Also denote by $\mathcal{M}$ the subclass of $\mathcal{N}$ consisting of functions of the form:

$$f(z)^{\beta} = z^{\beta} - \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1}, \quad (a_n \geq 0,\ \beta > 0). \tag{1.2}$$

For any integer m, we define the multiplier transformations I_m^ℓ (see [2, 3]) of functions $f \in \mathcal{N}(n)$ by

$$\begin{aligned} I_m^\ell f(z)^\beta &= z^\beta - \sum_{n=2}^{\infty} \beta \left(\frac{\beta+\ell}{\beta+\ell+n-1} \right)^m a_n z^{\beta+n-1} \\ &= z^\beta - \sum_{n=2}^{\infty} \beta Q(n,\beta,\ell) a_n z^{\beta+n-1}, \quad (\ell \geq 0,\ z \in U), \end{aligned} \tag{1.3}$$

where $Q(n,\beta,\ell) = ((\beta+\ell)/(\beta+\ell+n-1))^m$.

A function $f \in \mathcal{M}$ is said to be in the class USL(α,k) (k-uniformly starlike Functions of order α) if it satisfies the condition:

$$\operatorname{Re}\left\{ \frac{zf'(z)^\beta}{f(z)^\beta} - \alpha \right\} > k \left| \frac{zf'(z)^\beta}{f(z)^\beta} - 1 \right|, \quad (0 \leq \alpha < 1,\ k \geq 0),\ z \in U \tag{1.4}$$

and is said to be in the class UCV(α,k) (k-uniformly convex Functions of order α) if it satisfies the condition:

$$\operatorname{Re}\left\{ 1 + \frac{zf''(z)^\beta}{f'(z)^\beta} - \alpha \right\} > k \left| \frac{zf''(z)^\beta}{f'(z)^\beta} \right|, \quad (0 \leq \alpha < 1,\ k \geq 0),\ z \in U. \tag{1.5}$$

Indeed it follows from (1.4) and (1.5) that

$$f \in \mathrm{UCV}(\alpha,k) \Longleftrightarrow zf' \in \mathrm{USL}(\alpha,k). \tag{1.6}$$

The interesting geometric properties of these function classes were extensively studied by Kanas et al., in [4, 5], motivated by Altintas et al. [6], Murugusundaramoorthy and Srivastava [7], and Murugusundaramoorthy and Magesh [8, 9], Atshan and Kulkarni [10] and Atshan and Buti [11].

Now, we define a new subclass of uniformly convex functions of complex order.

For $0 \leq \alpha < 1$, $k \geq 0$, $\upsilon \in \mathbb{C} \setminus \{0\}$, we let $\mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$ be the class of functions f satisfying (1.2) with the analytic criterion:

$$\operatorname{Re}\left\{ 1 + \frac{1}{\upsilon}\left(1 + \frac{z\left(I_m^\ell f(z)^\beta\right)''}{\left(I_m^\ell f(z)^\beta\right)'} - \alpha \right) \right\} > k \left| 1 + \frac{1}{\upsilon}\left(\frac{z\left(I_m^\ell f(z)^\beta\right)''}{\left(I_m^\ell f(z)^\beta\right)'} \right) \right|, \quad z \in U, \tag{1.7}$$

where $I_m^\ell f(z)^\beta$ is given by (1.3).

2. Main Results

First, we obtain the necessary and sufficient condition for functions f in the class $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$.

Theorem 2.1. *The necessary and sufficient condition for f of the form of* (1.2) *to be in the class $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$ is*

$$\sum_{n=2}^{\infty}(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)a_n \le (k-\alpha)+(1-k)(\beta+|\upsilon|), \tag{2.1}$$

where $0\le\alpha<1$, $k\ge 0$, $\upsilon\in\mathbb{C}\setminus\{0\}$.

Proof. Suppose that (2.1) is true for $z\in U$. Then

$$\operatorname{Re}\left\{1+\frac{1}{\upsilon}\left(1+\frac{z\left(I_m^{\ell}f(z)^{\beta}\right)''}{\left(I_m^{\ell}f(z)^{\beta}\right)'}-\alpha\right)\right\}-k\left|1+\frac{1}{\upsilon}\left(\frac{z\left(I_m^{\ell}f(z)^{\beta}\right)''}{\left(I_m^{\ell}f(z)^{\beta}\right)'}\right)\right|>0, \tag{2.2}$$

if

$$\begin{aligned}&1+\frac{1}{|\upsilon|}\left(\frac{(\beta-\alpha)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-\alpha-1)Q(n,\beta,\ell)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n|z|^{n-1}}\right)\\&\quad-k\left[1+\frac{1}{|\upsilon|}\left(\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)Q(n,\beta,\ell)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n|z|^{n-1}}\right)\right]>0,\end{aligned} \tag{2.3}$$

that is, if

$$\sum_{n=2}^{\infty}(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)a_n \le (k-\alpha)+(1-k)(\beta+|\upsilon|). \tag{2.4}$$

Conversely, assume that $f \in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$, then

$$\begin{aligned}
&\operatorname{Re}\left\{1+\frac{1}{\upsilon}\left(1+\frac{z\left(I_m^{\ell}f(z)^{\beta}\right)''}{\left(I_m^{\ell}f(z)^{\beta}\right)'}-\alpha\right)\right\} > k\left|1+\frac{1}{\upsilon}\left(\frac{z\left(I_m^{\ell}f(z)^{\beta}\right)''}{\left(I_m^{\ell}f(z)^{\beta}\right)'}\right)\right|,\\
&\operatorname{Re}\left\{1+\frac{1}{\upsilon}\left(\frac{(\beta-\alpha)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-\alpha-1)Q(n,\beta,\ell)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n z^{n-1}}\right)\right\}\\
&\quad > k\left|1+\frac{1}{\upsilon}\left(\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)Q(n,\beta,\ell)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n z^{n-1}}\right)\right|.
\end{aligned} \tag{2.5}$$

Letting $z \to 1^-$ along the real axis, we have

$$\begin{aligned}
&1+\frac{1}{|\upsilon|}\left(\frac{(\beta-\alpha)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-\alpha-1)Q(n,\beta,\ell)a_n}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n}\right)\\
&\quad > k\left[1+\frac{1}{|\upsilon|}\left(\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)Q(n,\beta,\ell)a_n}{1-\sum_{n=2}^{\infty}(\beta+n-1)Q(n,\beta,\ell)a_n}\right)\right].
\end{aligned} \tag{2.6}$$

Hence, by maximum modulus theorem, the simple computation leads to the desired inequality

$$\sum_{n=2}^{\infty}(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)a_n \le (k-\alpha)+(1-k)(\beta+|\upsilon|), \tag{2.7}$$

which completes the proof. □

Corollary 2.2. *Let the function f defined by* (1.2) *belong to $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then,*

$$a_n \le \frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}, \tag{2.8}$$

where $0 \le \alpha < 1$, $k \ge 0$, $\upsilon \in \mathbb{C} \setminus \{0\}$, with equality for

$$f(z)^{\beta} = z^{\beta} - \beta\frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}z^{\beta+n-1}. \tag{2.9}$$

3. Radii of Convexity and Close-to-Convexity

We obtain the radii of convexity and close-to-convexity results for f functions in the class $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$ in the following theorems.

Theorem 3.1. *Let $f\in\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then f is convex of order $\delta(0\le\delta<1)$ in the disk $|z|<r=r_1(\alpha,\beta,k,\upsilon,n,\delta)$, where*

$$r_1=\inf_{n\ge2}\left[\frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(3-\delta-\beta-n)\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}\right]^{1/n-1}. \tag{3.1}$$

Proof. Let $f\in\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}Q(n,\beta,\ell)a_n\le1. \tag{3.2}$$

For $0\le\delta<1$, we need to show that

$$\left|\frac{zf''(z)^{\beta}}{f'(z)^{\beta}}\right|\le1-\delta, \tag{3.3}$$

and we have to show that

$$\left|\frac{zf''(z)^{\beta}}{f'(z)^{\beta}}\right|\le\frac{(\beta-1)-\sum_{n=2}^{\infty}(\beta+n-1)(\beta+n-2)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(\beta+n-1)a_n|z|^{n-1}}\le1-\ \delta. \tag{3.4}$$

Hence,

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)(3-\delta-\beta-n)}{(2-\ \delta-\beta)}a_n|z|^{n-1}\le1. \tag{3.5}$$

This is enough to consider

$$|z|^{n-1}\le\frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(3-\delta-\beta-n)\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}. \tag{3.6}$$

Therefore,

$$|z|\le\left\{\frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(3-\delta-\beta-n)\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}\right\}^{1/n-1}. \tag{3.7}$$

Setting $z=r_1(\alpha,\beta,k,\upsilon,n,\delta)$ in (3.7), we get the radius of convexity, which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $f \in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then f is close-to-convex of order $\delta(0 \le \delta < 1)$ in the disk $|z| < r = r_2(\alpha,\beta,k,\upsilon,n,\delta)$, where*

$$r_2 = \inf_{n\ge 2}\left[\frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}\right]^{1/n-1}. \tag{3.8}$$

Proof. Let $f \in \mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$. Then by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]}{(k-\alpha)+(1-k)(\beta+|\upsilon|)}Q(n,\beta,\ell)a_n \le 1. \tag{3.9}$$

For $0 \le \delta < 1$, we need to show that

$$\left|\frac{f'(z)^{\beta}}{z^{\beta-1}}-1\right| \le 1-\delta, \tag{3.10}$$

and we have to show that

$$\left|\frac{f'(z)^{\beta}}{z^{\beta-1}}-1\right| \le (\beta-1)+\sum_{n=2}^{\infty}\beta(\beta+n-1)a_n|z|^{n-1} \le 1-\ \delta. \tag{3.11}$$

Hence,

$$\sum_{n=2}^{\infty}\frac{\beta(\beta+n-1)}{(2-\ \delta-\beta)}a_n|z|^{n-1} \le 1. \tag{3.12}$$

This is enough to consider

$$|z|^{n-1} \le \frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{\beta\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}. \tag{3.13}$$

Therefore,

$$|z| \le \left\{\frac{(2-\ \delta-\beta)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{\beta\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}\right\}^{1/n-1}. \tag{3.14}$$

Setting $z = r_2(\alpha,\beta,k,\upsilon,n,\delta)$ in (3.14), we get the radius of close-to-convexity, which completes the proof of Theorem 3.2. □

4. Extreme Points

The extreme points of the class $\mathcal{N}_m^{\ell}(\alpha,\beta,k,\upsilon)$ are given by the following theorem.

Theorem 4.1. *Let*

$$f_1(z)^\beta = z^\beta,$$
$$f_n(z)^\beta = z^\beta - \beta \frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)} z^{\beta+n-1}, \tag{4.1}$$

for $n = 2, 3, 4, \ldots$.

Then, $f \in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$ *if and only if it can be expressed in the form:*

$$f(z)^\beta = \sum_{n=1}^{\infty} \Upsilon_n f_n(z)^\beta, \tag{4.2}$$

where $\Upsilon_n \geq 0$ *and*

$$\sum_{n=1}^{\infty} \Upsilon_n = 1. \tag{4.3}$$

Proof. Suppose that f can be expressed as in (4.2). Our goal is to show that $f \in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$. By (4.2), we have that

$$\begin{aligned}
f(z)^\beta &= \sum_{n=1}^{\infty} \Upsilon_n f_n(z)^\beta = \Upsilon_1 f_1(z)^\beta + \sum_{n=2}^{\infty} \Upsilon_n f_n(z)^\beta \\
&= \Upsilon_1 f_1(z)^\beta \\
&\quad + \sum_{n=2}^{\infty} \Upsilon_n \left(z^\beta - \beta \frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)} z^{\beta+n-1} \right) \\
&= \sum_{n=1}^{\infty} \Upsilon_n z^\beta - \sum_{n=2}^{\infty} \beta \Upsilon_n \frac{(k-\alpha)+(1-k)(\beta+|\upsilon|)}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)} z^{\beta+n-1} \\
&= z^\beta - \sum_{n=2}^{\infty} \beta \frac{\Upsilon_n\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)} z^{\beta+n-1}.
\end{aligned} \tag{4.4}$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)} \\
&\qquad \times \frac{\Upsilon_n\left[(k-\alpha)+(1-k)(\beta+|\upsilon|)\right]}{(\beta+n-1)\left[(\beta+n-1+|\upsilon|)(1-k)+(k-\alpha)\right]Q(n,\beta,\ell)} \\
&\qquad = \sum_{n=2}^{\infty} \Upsilon_n = 1 - \Upsilon_1 \leq 1.
\end{aligned} \tag{4.5}$$

Thus, $f \in \mathcal{N}_m^\ell(\alpha,\beta,k,\upsilon)$.

Conversely, assume that $f \in \mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$. Since

$$a_n \leq \frac{(k-\alpha) + (1-k)(\beta + |\upsilon|)}{(\beta + n - 1)\left[(\beta + n - 1 + |\upsilon|)(1-k) + (k-\alpha)\right] Q(n, \beta, \ell)} \quad (n \geq 2), \tag{4.6}$$

we can set

$$\begin{gathered} \Upsilon_n = \frac{(\beta + n - 1)\left[(\beta + n - 1 + |\upsilon|)(1-k) + (k-\alpha)\right] Q(n, \beta, \ell)}{(k-\alpha) + (1-k)(\beta + |\upsilon|)} a_n \quad (n \geq 2), \\ \Upsilon_1 = 1 - \sum_{n=2}^{\infty} \Upsilon_n. \end{gathered} \tag{4.7}$$

Then,

$$\begin{aligned} f(z)^{\beta} &= z^{\beta} - \sum_{n=2}^{\infty} \beta a_n z^{\beta+n-1} \\ &= z^{\beta} - \sum_{n=2}^{\infty} \beta \frac{\Upsilon_n \left[(k-\alpha) + (1-k)(\beta + |\upsilon|)\right]}{(\beta + n - 1)\left[(\beta + n - 1 + |\upsilon|)(1-k) + (k-\alpha)\right] Q(n, \beta, \ell)} z^{\beta+n-1} \\ &= z^{\beta} - \sum_{n=2}^{\infty} \Upsilon_n \left(z^{\beta} - f_n(z)^{\beta}\right) \\ &= z^{\beta}\left(1 - \sum_{n=2}^{\infty} \Upsilon_n\right) + \sum_{n=2}^{\infty} \Upsilon_n f_n(z)^{\beta} \\ &= \Upsilon_1 f_1(z)^{\beta} + \sum_{n=2}^{\infty} \Upsilon_n f_n(z)^{\beta} \\ &= \sum_{n=1}^{\infty} \Upsilon_n f_n(z)^{\beta}. \end{aligned} \tag{4.8}$$

This completes the proof of Theorem 4.1. □

5. Integral Means

In order to find the integral means inequality and to verify the Silverman Conjuncture [12] for $f \in \mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$, we need the following definition of subordination and subordination result according to Littlewood [13].

Definition 5.1 (see [13]). Let f and g be analytic in U. Then, we say that the function f is subordinate to g if there exists a Schwarz function w, analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$. We denote this subordination $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$, $f(U) \subset g(U)$.

Lemma 5.2 (see [13]). *If the functions f and g are analytic in U with $g \prec f$, then*

$$\int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\eta} d\theta, \quad \eta > 0,\ z = re^{i\theta},\ 0 < r < 1. \tag{5.1}$$

Applying Theorem 2.1 with the extremal function and Lemma 5.2, we prove the following theorem.

Theorem 5.3. *Let $\eta > 0$. If $f \in \mathcal{N}_m^{\ell}(\alpha, \beta, k, \upsilon)$ and $\{\Phi(\alpha, \beta, k, \upsilon, n)\}_{n=2}^{\infty}$ are nondecreasing sequences, then, for $z = re^{i\theta}$ and $0 < r < 1$, one has*

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)^{\beta}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f_2\left(re^{i\theta}\right)^{\beta}\right|^{\eta} d\theta, \tag{5.2}$$

where

$$\begin{gathered} f_2(z)^{\beta} = z^{\beta} - \beta \frac{(k-\alpha) + (1-k)(\beta + |\upsilon|)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z^{\beta+1}, \\ \Phi(\alpha, \beta, k, \upsilon, n) = (\beta + n - 1)\left[(\beta + n - 1 + |\upsilon|)(1-k) + (k-\alpha)\right] Q(n, \beta, \ell). \end{gathered} \tag{5.3}$$

Proof. Let f of the form of (1.2) and

$$f_2(z)^{\beta} = z^{\beta} - \beta \frac{(k-\alpha) + (1-k)(\beta + |\upsilon|)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z^{\beta+1}, \tag{5.4}$$

then we must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} \beta a_n z^{n-1}\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|1 - \beta \frac{(k-\alpha) + (1-k)(\beta + |\upsilon|)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z\right|^{\eta} d\theta. \tag{5.5}$$

By Lemma 5.2, it suffices to show that

$$1 - \sum_{n=2}^{\infty} \beta a_n z^{n-1} \prec 1 - \beta \frac{(k-\alpha) + (1-k)(\beta + |\upsilon|)}{\Phi(\alpha, \beta, k, \upsilon, 2)} z. \tag{5.6}$$

Setting

$$1 - \sum_{n=2}^{\infty} \beta a_n z^{n-1} = 1 - \beta \frac{(k-\alpha) + (1-k)(\beta + |\upsilon|)}{\Phi(\alpha, \beta, k, \upsilon, 2)} w(z), \tag{5.7}$$

from (5.7) and (2.1) we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,k,\upsilon,2)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\alpha,\beta,k,\upsilon,n)}{(k-\alpha)+(1-k)(\beta+|\upsilon|)} a_n \\ &\leq |z| < 1. \end{aligned} \tag{5.8}$$

This completes the proof of Theorem 5.3. □

References

[1] K. Al-Shaqsi, M. Darus, and O. A. Fadipe-Joseph, "A new subclass of Sălăgean-type harmonic univalent functions," *Abstract and Applied Analysis*, vol. 2010, Article ID 821531, 12 pages, 2010.

[2] N. E. Cho and T. H. Kim, "Multiplier transformations and strongly close-to-convex functions," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 3, pp. 399–410, 2003.

[3] R. M. El-Ashwah and M. K. Aouf, "New classes of p-valent harmonic functions," *Bulletin of Mathematical Analysis and Applications*, vol. 2, no. 3, pp. 53–64, 2010.

[4] S. Kanas and A. Wisniowska, "Conic regions and k-uniform convexity," *Journal of Computational and Applied Mathematics*, vol. 105, no. 1-2, pp. 327–336, 1999.

[5] S. Kanas and T. Yaguchi, "Subclasses of k-uniformly convex and starlike functions defined by generalized derivative. II," *Publications Institute Mathematique*, vol. 69, no. 83, pp. 91–100, 2001.

[6] O. Altintas, O. Ozkan, and H. M. Srivastava, "Neighborhoods of a class of analytic functions with negative coefficients," *Applied Mathematics Letters*, vol. 13, no. 3, pp. 63–67, 2000.

[7] G. Murugusundaramoorthy and H. M. Srivastava, "Neighborhoods of certain classes of analytic functions of complex order," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 2, article 24, pp. 1–8, 2004.

[8] G. Murugusundaramoorthy and N. Magesh, "Starlike and convex functions of complex order involving the Dziok-Srivastava operator," *Integral Transforms and Special Functions*, vol. 18, no. 5-6, pp. 419–425, 2007.

[9] G. Murugusundaramoorthy and N. Magesh, "Certain subclasses of starlike functions of complex order involving generalized hypergeometric functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 178605, 12 pages, 2010.

[10] W. G. Atshan and S. R. Kulkarni, "Neighborhoods and partial sums of subclass of k-uniformly convex functions and related class of k-starlike functions with negative coefficients based on integral operator," *Southeast Asian Bulletin of Mathematics*, vol. 33, no. 4, pp. 623–637, 2009.

[11] W. G. Atshan and R. H. Buti, "On generalized hypergeometric functions and associated classes of k-uniformly convex and k-starlike p-valent functions," *Advances and Applications in Mathematical Sciences*, vol. 6, no. 2, pp. 149–160, 2010.

[12] H. Silverman, "Integral means for univalent functions with negative coefficients," *Houston Journal of Mathematics*, vol. 23, no. 1, pp. 169–174, 1997.

[13] J. E. Littlewood, "On inequalities in theory of functions," *Proceedings of the London Mathematical Society*, vol. 23, pp. 481–519, 1925.