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Research Article

A Note on the Regularity Criterion of Weak Solutions of Navier-Stokes Equations in Lorentz Space

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This paper is concerned with the regularity of Leray weak solutions to the 3D Navier-Stokes equations in Lorentz space. It is proved that the weak solution is regular if the horizontal velocity denoted by $\widetilde{u} = (u_1, u_2, 0)$ satisfies $\widetilde{u}(x,t) \in L_q(0,T; L_{p,\infty}(\mathbf{R}^3))$ for $2/q + 3/p = 1,\ 3 < p < \infty$. The result is obvious and improved that of Dong and Chen (2008) on the Lebesgue space.

1. Introduction and Main Results

In this note, we consider the regularity criterion of weak solutions of the Navier-Stokes equations in the whole space $\mathbf{R}^3$

$$\begin{gathered} \partial_t u + (u \cdot \nabla) u + \nabla \pi = \Delta u, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0. \end{gathered} \tag{1.1}$$

Here $u = (u_1, u_2, u_3)$ and π denote the unknown velocity field and the unknown scalar pressure field. u_0 is a given initial velocity. For simplicity, we assume that the external force is zero, but it is easy to extend our results to the nonzero external force case. Here and in what follows, we use the notations for vector functions u, v,

$$(u \cdot \nabla) v = \sum_{i=1}^{3} u_i \partial_i v_k \quad (k = 1, 2, 3), \qquad \nabla \cdot u = \sum_{i=1}^{3} \partial_i u_i. \tag{1.2}$$

For a given initial data $u_0 \in L^2(\mathbf{R}^3)$, Leary in the pioneer work, [1] constructed a global weak solution

$$u \in L^\infty\left(0,\infty; L^2\left(\mathbf{R}^3\right)\right) \cap L^2\left(0,T; H^1\left(\mathbf{R}^3\right)\right). \tag{1.3}$$

From that time on, although much effort has been made on the uniqueness and regularity of weak solutions, the question of global regularity or finite time singularity for weak solutions in $\mathbf{R}^3$ is still open. One important observation is that the regularity can be derived when certain growth conditions are satisfied. This is known as a regularity criterion problem. The investigation of the regularity criterion on the weak solution stems from the celebrated work of Serrin [2]. Namely, Serrin's regularity criterion can be described as follows. *A weak solution u of Navier-Stokes equations is regular if the growth condition on velocity field u*

$$u \in L_p\left(0,T; L_q\left(\mathbf{R}^3\right)\right) \equiv L_p L_q, \quad \text{for } \frac{2}{p} + \frac{3}{q} \le 1, \quad 3 < q \le \infty, \tag{1.4}$$

holds true.

It should be mentioned that the Serrin's condition (1.4) is important from the point of view of the relation between scaling invariance and regularity criteria of weak solutions; indeed, if a pair (u,p) solves (1.1), then so does (u_λ, p_λ) defined by

$$u_\lambda(x,t) = \lambda u\left(\lambda x, \lambda^2 t\right), \qquad p_\lambda(x,t) = \lambda^2 p\left(\lambda x, \lambda^2 t\right). \tag{1.5}$$

Scaling invariance means that

$$\|u\|_{L^p(0,T;L^q(\mathbf{R}^3))} = \|u_\lambda\|_{L^p(0,T;L^q(\mathbf{R}^3))} \tag{1.6}$$

holds for all $\lambda > 0$ and this happens if and only if p and q satisfy (1.4).

Actually, the condition described by (1.4) which involves all components of the velocity vector field $u = (u_1, u_2, u_3)$ is known as degree -1 growth condition (see Chen and Xin [3] for details), since

$$\left\|u\left(\lambda\cdot, \lambda^2\cdot\right)\right\|_{L_p L_q} = \|u\|_{L_p(0,\lambda^2 T; L_q(\mathbf{R}^3))} \lambda^{-2/p-3/q} = \|u\|_{L_p(0,\lambda^2 T; L_q(\mathbf{R}^3))} \lambda^{-1}. \tag{1.7}$$

The degree -1 growth condition is critical due to the scaling invariance property. That is, $(u(x,t), p(x,t))$ solves (1.1) if and only if $(u_\lambda(x,t), p_\lambda(x,t))$ is a solution of (1.1).

Moreover, this pioneer result [2] has been extended by many authors in terms of velocity $u(x,t)$, the gradient of velocity $\nabla u(x,t)$ or vorticity $w(x,t) = (w_1, w_2, w_3) = \nabla \times u$ in Lebesgue spaces or Besov spaces, respectively (refer to [4–7] and reference therein).

Actually, the weak solution remains regular when a part of the velocity components or vorticity is involved in a growth condition. On one hand, regularity of the weak solution was recently obtained by Dong and Chen [8] when two velocity components denoted by

$$\tilde{u} = (u_1, u_2, 0) \tag{1.8}$$

satisfy the critical growth condition

$$\widetilde{u} \in L_p L_q, \quad \text{for } \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty. \tag{1.9}$$

It should be mentioned that the weak solution remains regular if the single velocity component satisfies the higher (subcritical) growth conditions (see Zhou [9], Penel and Pokorný [10], Kukavica and Ziane [11], and Cao and Titi [12]). One may also refer to some interesting regularity criteria [13–15] for weak solutions of micropolar fluid flows. It seems difficult to show regularity of weak solutions by imposing Serrin's growth condition on only one component of velocity field for both Navier-Stokes equations and micropolar fluid flows.

However, whether or not the result (1.9) can be improved to the critical weak L_p spaces is an interesting and challenging problem, that is to say, when the weak critical growth condition is imposed to only two velocity components. The main difficulty lies in the lack of *a priori* estimates on two-velocity components $\widetilde{u}$ due to the special structure of the nonlinear convection term in monument equations.

The aim of the present paper is to improve the two-component regularity criterion (1.9) from Lebesgue space to the critical Lorentz space (see the definitions in Section 2) which satisfies the scaling invariance property.

Before stating the main results, we firstly recall the definition of the Leray weak solutions.

Definition 1.1 (Temam, [16]). Let $u_0 \in L_2(\mathbf{R}^3)$ and $\nabla \cdot u_0 = 0$. A vector field $u(x,t)$ is termed as a Leray weak solution of (1.1) on $(0,T)$ if u satisfies the following properties:

(i) $u \in L_\infty(0,T; L_2(\mathbf{R}^3)) \cap L_2(0,T; H^1(\mathbf{R}^3))$;

(ii) $\partial_t u + (u \cdot \nabla) u + \nabla \pi = \Delta u$ in the distribution space $\mathfrak{D}'((0,T) \times \mathbf{R}^3)$;

(iii) $\nabla \cdot u = 0$ in the distribution space $\mathfrak{D}'((0,T) \times \mathbf{R}^3)$;

(iv) u satisfies the energy inequality

$$\|u(t)\|_{L_2}^2 + 2\int_0^t \int_{\mathbf{R}^3} |\nabla u(x,s)|^2 \, dx ds \leq \|u_0\|_{L_2}^2, \quad \text{for } 0 \leq t \leq T. \tag{1.10}$$

The main results now read as follows.

Theorem 1.2. *Suppose $T > 0$, $u_0 \in H^1(\mathbf{R}^3)$ and $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that u is a Leray weak solution of the Navier-Stokes equations* (1.1) *in* $(0,T)$. *If the horizontal velocity denoted by $\widetilde{u} = (u_1, u_2, 0)$ satisfies the following growth condition:*

$$\int_0^T \|\widetilde{u}(t)\|_{L_{p,\infty}}^q dt < \infty, \quad \textit{for } \frac{2}{q} + \frac{3}{p} = 1, \quad 3 < p < \infty, \tag{1.11}$$

then u is a regular solution on $(0,T]$.

Remark 1.3. It is easy to verify that the spaces (1.11) satisfy the degree −1 growth conditions due to the scaling invariance property. Moreover, since the embedding relation $L_p \hookrightarrow L_{p,\infty}$, Theorem 1.2 is an important improvement of (1.9).

Remark 1.4. Unlike the previous investigations via two components of vorticity (see [17, 18]) in weak space, of which the approaches are mainly based on the vorticity equations and seem not available in our case here due to the special structure of convection term, the present examination is directly based on the momentum equations. In order to make use of the structure of the nonlinear convection term $(u\cdot\nabla)u$, we study every component of $((u\cdot\nabla)u, \Delta u)$ and estimate them one by one with the aid of the identities $\nabla \cdot u = 0$.

2. Preliminaries and *A Priori* Estimates

To start with, let us introduce the definitions of some functional spaces. $L_p(\mathbf{R}^3)$, $W^{k,p}(\mathbf{R}^3)$ with $k \in \mathbf{R}$, $1 \le p \le \infty$ are usual Lebesgue space and Sobolev space.

To define the Lorenz space $L_{p,q}(\mathbf{R}^3)$ with $1 \le p,\ q \le \infty$, we consider a measurable function f and define for $t \ge 0$ the Lebesgue measure

$$m(f,t) := m\left\{x \in \mathbf{R}^3 : \left|f(x)\right| > t\right\}, \tag{2.1}$$

of the set $\{x \in \mathbf{R}^3 : |f(x)| > t\}$.

Then $f \in L_{p,q}(\mathbf{R}^3)$ if and only if

$$\begin{gathered}\|f\|_{L_{p,q}} = \left(\int_0^\infty t^q (m(f,t))^{q/p} \frac{dt}{t}\right)^{1/q} < \infty \quad \text{for } 1 \le q < \infty, \\ \|f\|_{L_{p,\infty}} = \sup_{t\ge 0}\left(t(m(f,t))^{1/p}\right) < \infty \quad \text{for } q = \infty.\end{gathered} \tag{2.2}$$

Actually, Lorentz space $L_{p,q}(\mathbf{R}^3)$ may be alternatively defined by real interpolation (see Bergh and Löfström [19] and Triebel [20])

$$L_{p,q}\left(\mathbf{R}^3\right) = \left(L_{p_1}\left(\mathbf{R}^3\right), L_{p_2}\left(\mathbf{R}^3\right)\right)_{\theta,q}, \tag{2.3}$$

with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad 1 \le p_1 < p < p_2 \le \infty. \tag{2.4}$$

Especially, $\|f\|_{L_{q,\infty}}$ is equivalent to the norm

$$\sup_{0<|E|<\infty} |E|^{1/q-1} \int_E \left|f(x)\right| dx, \tag{2.5}$$

and thus it readily seen that

$$L_p\left(\mathbf{R}^3\right) = L_{p,p}\left(\mathbf{R}^3\right) \subset L_{p,q}\left(\mathbf{R}^3\right) \subset L_{p,\infty}\left(\mathbf{R}^3\right), \quad 1 < p < q < \infty. \tag{2.6}$$

In order to prove Theorem 1.2, it is sufficient to examine *a priori* estimates for smooth solutions of (1.1) described in the following.

Theorem 2.1. *Let $T > 0$, $u_0 \in H^1(\mathbf{R}^3)$ with $\nabla \cdot u_0 = 0$. Assume that $u(x,t)$ is a smooth solution of* (1.1) *on $\mathbf{R}^3 \times (0,T)$ and satisfies the growth conditions* (1.11)*. Then*

$$\sup_{0<t<T} \|\nabla u(t)\|_{L_2}^2 + \int_0^T \|\Delta u(t)\|_{L_2}^2 dt \le c\|\nabla u_0\|_{L_2}^2 \exp\left\{\int_0^T \|\tilde{u}(t)\|_{L_{p,\infty}}^q dt\right\} \tag{2.7}$$

holds true.

Proof of Theorem 2.1. Taking inner product of the momentum equations of (1.1) with Δu and integrating by parts, one shows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L_2}^2 + \|\Delta u(t)\|_{L_2}^2 \le -\sum_{i,j,k=1}^{3} \int_{\mathbf{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx. \tag{2.8}$$

In order to estimate the right-hand side of (2.8), with the aid of the divergence-free condition $\sum_{i=1}^3 \partial_i u_i = 0$ and integration by parts, observe that

$$\begin{aligned}
-\sum_{i,j,k=1}^{3} \int_{\mathbf{R}^3} u_i \partial_i u_j \partial_{kk} u_j dx &= \sum_{i,j,k=1}^{3} \int_{\mathbf{R}^3} \partial_k \left(u_i \partial_i u_j\right) \partial_k u_j dx \\
&= \sum_{i,j,k=1}^{3} \int_{\mathbf{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx + \frac{1}{2}\sum_{i,j,k=1}^{3} \int_{\mathbf{R}^3} u_i \partial_i \left(\partial_k u_j \partial_k u_j\right) dx \\
&= \sum_{i,j,k=1}^{3} \int_{\mathbf{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j \, dx \\
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3} \int_{\mathbf{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx + \sum_{j=1}^{2}\sum_{k=1}^{3} \int_{\mathbf{R}^3} \partial_k u_3 \partial_3 u_j \partial_k u_j dx \\
&\quad + \sum_{k=1}^{3} \int_{\mathbf{R}^3} \partial_k u_3 \partial_3 u_3 \partial_k u_3 dx = \sum_{m=1}^{3} I_m.
\end{aligned} \tag{2.9}$$

The estimation of the terms I_m is now estimated one by one.

In order to estimate I_1 and I_2, employing integration by parts deduces that

$$\begin{aligned}
I_1 &= \sum_{i=1}^{2}\sum_{j,k=1}^{3} \int_{\mathbf{R}^3} u_i \partial_k \left(\partial_i u_j \partial_k u_j\right) dx \le c \int_{\mathbf{R}^3} |\tilde{u}||\nabla u| \left|\nabla^2 u\right| dx, \\
I_2 &= \sum_{j=1}^{2}\sum_{k=1}^{3} \int_{\mathbf{R}^3} u_j \partial_3 \left(\partial_k u_3 \partial_k u_j\right) dx \le c \int_{\mathbf{R}^3} |\tilde{u}||\nabla u| \left|\nabla^2 u\right| dx.
\end{aligned} \tag{2.10}$$

For I_3, the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ and integration by parts imply

$$\begin{aligned} I_3 &= \sum_{k=1}^{3} \int_{\mathbf{R}^3} \partial_k u_3 (\partial_1 u_1 + \partial_2 u_2) \partial_k u_3 dx \\ &\le -\sum_{k=1}^{3} \int_{\mathbf{R}^3} (u_1 \partial_1 (\partial_k u_3 \partial_k u_3) + u_2 \partial_2 (\partial_k u_3 \partial_k u_3)) dx \\ &\le c \int_{\mathbf{R}^3} |\widetilde{u}| |\nabla u| \left| \nabla^2 u \right| dx. \end{aligned} \tag{2.11}$$

Thus, plugging the above inequalities into (2.8) to produce

$$\frac{d}{dt} \|\nabla u(t)\|_{L_2}^2 + 2\|\Delta u(t)\|_{L_2}^2 \le c \int_{\mathbf{R}^3} |\widetilde{u}| |\nabla u| \left| \nabla^2 u \right| dx := \mathrm{RHS}. \tag{2.12}$$

We now carry out the estimation of (2.12) based on the assumption described by (1.11).

Applying Hölder's inequality and Young's-inequality, we have for the right-hand side (RHS) of (2.12)

$$\begin{aligned} \mathrm{RHS} &\le c \| |\widetilde{u}| |\nabla u| \|_{L_2} \left\| \nabla^2 u \right\|_{L_2} \le c(\varepsilon) \| |\widetilde{u}| |\nabla u| \|_{L_2}^2 + \varepsilon \left\| \nabla^2 u \right\|_{L_2}^2 \\ &\le c \|\widetilde{u}\|_{L_{p,\infty}}^2 \|\nabla u\|_{L_{2p/(p-2),2}}^2 + \frac{1}{2} \|\Delta u\|_{L_2}^2, \end{aligned} \tag{2.13}$$

where we have used the following Hölder inequality's in Lorentz space in the last line (refer to O'Neil [21, Theorems 3.4 and 3.5])

$$\|fg\|_{L_{p_1,q_1}} \le c \|f\|_{L_{p_2,q_2}} \|g\|_{L_{p_3,q_3}}, \tag{2.14}$$

for

$$\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}, \qquad \frac{1}{q_1} \le \frac{1}{q_2} + \frac{1}{q_3}, \tag{2.15}$$

with

$$1 \le p_2, \qquad p_3 \le \infty, \qquad 1 \le q_2, \qquad q_3 \le \infty. \tag{2.16}$$

We now claim that the term $\|\nabla u\|_{L_{2p/(p-2),2}}$ in (2.13) can be estimated by applying the following Gagliardo-Nirenberg inequality in Lorentz space

$$\|\nabla f\|_{L_{2p/(p-2),2}} \le c \|\nabla f\|_{L_2}^{(p-3)/p} \|\Delta f\|_{L_2}^{3/p}. \tag{2.17}$$

Indeed, choosing p_1 and p_2 such that

$$3 < p_1 < p < p_2 < \infty, \quad \frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{2.18}$$

and then applying Gagliardo-Nirenberg inequality, it follows that

$$\|\nabla f\|_{2p_i/(p_i-2)} \leq c\|\nabla f\|_{L_2}^{(p_i-3)/p_i} \|\Delta f\|_{L_2}^{3/p_i}, \quad i = 1, 2. \tag{2.19}$$

Thus, applying the interpolation inequality (2.3), we have

$$L_{2p/(p-2),2}\left(\mathbf{R}^3\right) = \left(L_{2p_1/(p_1-2)}\left(\mathbf{R}^3\right), L_{2p_2/(p_2-2)}\left(\mathbf{R}^3\right)\right)_{1/2,2}, \tag{2.20}$$

that is to say,

$$\begin{aligned}
\|\nabla f\|_{L_{2p/(p-2),2}} &\leq c\|\nabla f\|_{L_{2p_1/(p_1-2)}}^{1/2} \|\nabla f\|_{L_{2p_2/(p_2-2)}}^{1/2} \\
&\leq c\left(\|\nabla f\|_{L_2}^{p_1-3/p_1} \|\Delta f\|_{L_2}^{3/p_2}\right)^{1/2} \left(\|\nabla f\|_{L_2}^{(p_2-3)/p_2} \|\Delta f\|_{L_2}^{3/p_2}\right)^{1/2} \\
&\leq c\|\nabla f\|_{L_2}^{(p-3)/p} \|\Delta f\|_{L_2}^{3/p},
\end{aligned} \tag{2.21}$$

and (2.17) is derived. Therefore, by employing (2.17) and Young's inequality, the inequality (2.13) becomes

$$\begin{aligned}
\text{RHS} &\leq c\|\widetilde{u}\|_{L_{p,\infty}}^2 \|\nabla u\|_{L_2}^{2(p-3)/p} \|\Delta u\|_{L_2}^{6/p} + \frac{1}{2}\|\Delta u\|_{L_2}^2 \\
&\leq c\|\widetilde{u}\|_{L_{p,\infty}}^{2p/(p-3)} \|\nabla u\|_{L_2}^2 + \|\Delta u\|_{L_2}^2.
\end{aligned} \tag{2.22}$$

Inserting (2.22) into (2.12) to produce

$$\frac{d}{dt}\|\nabla u(t)\|_{L_2}^2 + \|\Delta u(t)\|_{L_2}^2 \leq c\|\widetilde{u}\|_{L_{p,\infty}}^{2p/(p-3)} \|\nabla u\|_{L_2}^2. \tag{2.23}$$

Taking Gronwall's inequality into account yields the desired estimate,

$$\sup_{0<t<T} \|\nabla u(t)\|_{L_2}^2 + \int_0^T \|\Delta u(t)\|_{L_2}^2 dt \leq c\|\nabla u_0\|_{L_2}^2 \exp\left\{\int_0^T \|\widetilde{u}(t)\|_{L_{p,\infty}}^q dt\right\}, \tag{2.24}$$

note that

$$\frac{2p}{p-3} = q. \tag{2.25}$$

This completes the proof of Theorem 2.1. □

3. Proof of Theorem 1.2

According to *a priori* estimates of smooth solutions described in Theorem 2.1, the proofs of Theorem 1.2 are standard.

Since $u_0 \in H^1(\mathbf{R}^3)$ with $\nabla \cdot u_0 = 0$, by the local existence theorem of strong solutions to the Navier-Stokes equations (see, e.g., Fujita and Kato [22]), there exist a $T^* > 0$ and a smooth solution $\overline{u}$ of (1.1) satisfying

$$\overline{u} \in C\Big([0,T^*);H^1\Big) \cap C^1\Big((0,T^*);H^1\Big) \cap C\Big([0,T^*);H^3\Big), \quad \overline{u}(x,0) = u_0. \tag{3.1}$$

Note that the Leray weak solution satisfies the energy inequality (1.10). It follows from Serrin's weak-strong uniqueness criterion [2] that

$$\overline{u} \equiv u \quad \text{on } [0,T^*). \tag{3.2}$$

Thus, it is sufficient to show that

$$T^* = T. \tag{3.3}$$

Suppose that $T^* < T$. Without loss of generality, we may assume that T^* is the maximal existence time for $\overline{u}$. Since $\overline{u} \equiv u$ on $[0,T^*)$ and by the assumptions (1.11), it follows from *a priori* estimate (2.7) that the existence time of $\overline{u}$ can be extended after $t = T^*$ which contradicts with the maximality of $t = T^*$.

Thus, we complete the proof of Theorem 1.2.

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