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Research Article

The Dirichlet Problem on the Upper Half-Space

Jinjin Huang[1] and Lei Qiao[2]

[1] *Department of Economics and Management, Zhoukou Normal University, Zhoukou 466001, China*
[2] *Department of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450002, China*

Correspondence should be addressed to Lei Qiao, qiaocqu@163.com

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A solution of the Dirichlet problem on the upper half-space is constructed by the generalized Dirichlet integral with a fast-growing continuous boundary function.

1. Introduction and Results

Let $\mathbf{R}^n$ $(n \geq 3)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set D of $\mathbf{R}^n$ are denoted by ∂D and $\overline{D}$, respectively. The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points x, $y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting

$$x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n, \quad |x| = \sqrt{x \cdot x}, \quad \left|x'\right| = \sqrt{x' \cdot x'}. \tag{1.1}$$

Let $B(r)$ denote the open ball with center at the origin and radius r, and let σ denote $(n-1)$-dimensional surface area measure. Let $[d]$ denote the integer part of the positive real number d. In the sense of Lebesgue measure, $dy' = dy_1 \cdots dy_{n-1}$ and $dy = dy' dy_n$.

Given a continuous function f on ∂H, we say that h is a solution of the (classical) Dirichlet problem on H with f if $\Delta h = 0$ in H and $\lim_{x \in H, x \to z'} h(x) = f(z')$ for every $z' \in \partial H$.

The classical Poisson kernel for H is defined by $P(x, y') = 2x_n \omega_n^{-1} |x - y'|^{-n}$, where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbf{R}^n$.

To solve the Dirichlet problem on H, as in [1–6], we use the following modified Poisson kernel of order m defined by

$$P_m(x,y') = \begin{cases} P(x,y') & \text{when } |y'| \le 1, \\ P(x,y') - \displaystyle\sum_{k=0}^{m-1} \frac{2x_n|x|^k}{\omega_n |y'|^{n+k}} C_k^{n/2}\left(\frac{x\cdot y'}{|x||y'|}\right) & \text{when } |y'| > 1, \end{cases} \tag{1.2}$$

where m is a nonnegative integer, and $C_k^{n/2}(t)$ is the ultraspherical (Gegenbauer) polynomials [7]. The expression arises from the generating function for Gegenbauer polynomials

$$\left(1-2tr+r^2\right)^{-n/2} = \sum_{k=0}^{\infty} C_k^{n/2}(t) r^k, \tag{1.3}$$

where $|r| < 1$ and $|t| \le 1$. The coefficient $C_k^{n/2}(t)$ is called the ultraspherical (Gegenbauer) polynomial of degree k associated with $n/2$, and the function $C_k^{n/2}(t)$ is a polynomial of degree k in t.

Put

$$U_m(f)(x) = \int_{\partial H} P_m(x,y') f(y') dy', \tag{1.4}$$

where $f(y')$ is a continuous function on ∂H.

Using the modified Poisson kernel $P_m(x,y')$, Yoshida (cf. [6, Theorem 1]) and Siegel and Talvila (cf. [5, Corollary 2.1]) gave classical solutions of the Dirichlet problem on H, respectively. Motivated by their results, we consider the Dirichlet problem for harmonic functions of infinite order (e.g., see [8, Definition 4.1, page 2, Line 12] for the definition of harmonic functions).

To do this, we define a nondecreasing and continuously differentiable function $\rho(r) \ge 1$ on the interval $[0,+\infty)$. We assume further that

$$\varepsilon_0 = \limsup_{r\to\infty} \frac{\rho'(r) r \log r}{\rho(r)} < 1. \tag{1.5}$$

Let $F(\rho,\beta)$ be the set of continuous functions f on ∂H such that

$$\int_{\partial H} \frac{|f(y')| dy'}{1+|y'|^{\rho(|y'|)+n+\beta-1}} < \infty, \tag{1.6}$$

where β is a positive real number.

Now, we have the following.

Theorem 1.1. *If $f \in F(\rho,\beta)$, then the integral $U_{[\rho(|y'|)+\beta]}(f)(x)$ is a solution of the Dirichlet problem on H with f.*

If one puts $[\rho(|y'|)+\beta] = m$ in Theorem 1.1, one immediately obtains the following (cf. [6, Theorem 1] and [5, Corollary 2.1]).

Corollary 1.2. *If f is a continuous function on ∂H satisfying $\int_{\partial H} |f(y')|(1+|y'|)^{-n-m} dy' < \infty$, then $U_m(f)(x)$ is a solution of the Dirichlet problem on H with f.*

Theorem 1.3. *Let u be harmonic in H and continuous on $\overline{H}$. If $u \in F(\rho, \beta)$, then one has*

$$u(x) = U_{[\rho(|y'|)+\beta]}(u)(x) + h(x), \tag{1.7}$$

for all $x \in \overline{H}$, where $h(x)$ is harmonic in H and vanishes continuously on ∂H.

2. Proof of Theorem 1.1

We need to use the following inequality (see [5, page 3]):

$$|P_m(x, y')| \le M x_n |x|^m |y'|^{-n-m}, \tag{2.1}$$

for any $x \in H$ and $y' \in \partial H$ satisfying $|y'| \ge \max\{1, 2|x|\}$, where M is a positive constant.

For any ϵ $(0 < \epsilon < 1 - \epsilon_0)$, there exists a sufficiently large positive number R such that $r > R$, and by (1.5), we have

$$\rho(r) < \rho(e)(\ln r)^{(\epsilon_0+\epsilon)}, \tag{2.2}$$

which yields that there exists a positive constant $M(r)$ dependent only on r such that

$$k^{-\beta/2}(2r)^{\rho(k+1)+\beta+1} \le M(r), \tag{2.3}$$

for any $k > k_r = [2r] + 1$.

For any $x \in H$ and $|x| \le r$, we have by (1.6), (2.1), (2.3), $1/p + 1/q = 1$, and Hölder's inequality

$$\begin{aligned}
&M \sum_{k=k_r}^{\infty} \int_{\{y' \in \partial H: k \le |y'| < k+1\}} \frac{(2|x|)^{[\rho(|y'|)+\beta]+1}}{|y'|^{[\rho(|y'|)+\beta]+n}} |f(y')| dy' \\
&\le M \sum_{k=k_r}^{\infty} (2r)^{\rho(k+1)+\beta+1} \left(\int_{\{y' \in \partial H: k \le |y'| < k+1\}} \frac{|f(y')|^p}{|y'|^{\rho(|y'|)+n+p\beta/2-1}} dy' \right)^{1/p} \\
&\quad \times \left(\int_{\{y' \in \partial H: k \le |y'| < k+1\}} |y'|^{-q\{[\rho(|y'|)+\beta]+n-(\rho(|y'|)+n-1)/p-\beta/2\}} dy' \right)^{1/q} \\
&\le M \sum_{k=k_r}^{\infty} \frac{(2r)^{\rho(k+1)+\beta+1}}{k^{\beta/2}} \int_{\{y' \in \partial H: k \le |y'| < k+1\}} \frac{|f(y')|}{|y'|^{\rho(|y'|)+n+\beta/2-1}} dy' \\
&\le 2MM(r) \int_{\{y' \in \partial H: |y'| \ge k_r\}} \frac{|f(y')|}{1 + |y'|^{\rho(|y'|)+n+\beta/2-1}} dy' < \infty.
\end{aligned} \tag{2.4}$$

Thus, $U_{[\rho(|y'|)+\beta]}(f)(x)$ is finite for any $x \in H$. Since $P_{[\rho(|y'|)+\beta]}(x, y')$ is a harmonic function of $x \in H$ for any fixed $y' \in \partial H$, $U_{[\rho(|y'|)+\beta]}(f)(x)$ is also a harmonic function of $x \in H$.

To verify the boundary behavior of $U_{[\rho(|y'|)+\beta]}(f)(x)$, we fix a boundary point $z' \in \partial H$, choose a large $t > |z'| + 1$, and write

$$U_{[\rho(|y'|)+\beta]}(f)(x) = X(x) - Y(x) + Z(x), \tag{2.5}$$

where

$$\begin{aligned} X(x) &= \int_{\{y' \in \partial H : |y'| \le t\}} P(x, y') f(y') dy', \\ Y(x) &= \sum_{k=0}^{[\rho(|y'|+\beta)]-1} \frac{2x_n |x|^k}{\omega_n} \int_{\{y' \in \partial H : 1 < |y'| \le t\}} \frac{1}{|y'|^{n+k}} C_k^{n/2}\left(\frac{x' \cdot y'}{|x||y'|}\right) f(y') dy', \\ Z(x) &= \int_{\{y' \in \partial H : |y'| > t\}} P_{[\rho(|y'|+\beta)]}(x, y') f(y') dy'. \end{aligned} \tag{2.6}$$

Notice that $X(x)$ is the Poisson integral of $f(y')\chi_{B(t)}(y')$, where $\chi_{B(t)}$ is the characteristic function of the ball $B(t)$. So it tends to $f(z')$ as $x \to z'$. Since $Y(x)$ are polynomial times x_n and $Z(x) = O(x_n)$, both of them tend to zero as $x \to z'$. Thus, the function $U_{[\rho(|y'|)+\beta]}(f)(x)$ can be continuously extended to $\overline{H}$ such that $U_{[\rho(|y'|)+\beta]}(f)(z') = f(z')$, for any $z' \in \partial H$. Theorem 1.1 is proved.

3. Proof of Theorem 1.3

Consider that the function $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$, which is harmonic in H, can be continuously extended to $\overline{H}$ and vanishes on ∂H.

The Schwarz reflection principle [9, page 68] applied to $u(x) - U_{[\rho(|y'|)+\beta]}(u)(x)$ shows that there exists a harmonic function $h(x)$ in H such that $h(x^*) = -h(x) = -(u(x) - U_{[\rho(|y'|)+\beta]}(u)(x))$ for $x \in \overline{H}$, where $*$ denotes reflection in ∂H just as $x^* = (x', -x_n)$.

Thus, $u(x) = h(x) + U_{[\rho(|y'|)+\beta]}(u)(x)$ for all $x \in \overline{H}$, where $h(x)$ is a harmonic function on H vanishing continuously on ∂H. We complete the proof of Theorem 1.3.

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