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Research Article

A Note on the Growth of Meromorphic Functions with a Radially Distributed Value

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This note is to investigate the growth of transcendental meromorphic functions with radially distributed values. We generalize a more recent result of Chen et al. (2011). The paper is closely related to some previous results due to Fang and Zalcman (2008), and Xu et al. (2009).

1. Introduction and Results

Let $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, where $\mathbb{C}$ is the whole complex plane and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We shall use the basic results and notations of Nevanlinna's value distribution theory of meromorphic functions (see [1–3]), such as $T(r,f)$, $N(r,f)$, and $m(r,f)$. Meantime, the Nevanlinna's deficiency $\delta(a,f)$ of $f(z)$ with respect to $a \in \mathbb{C}$ is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,1/(f-a))}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,1/(f-a))}{T(r,f)} \tag{1.1}$$

and $\delta(\infty,f)$ is obtained by the above formula with $m(r,f)$ in place of $m(r,1/(f-a))$, and $N(r,f)$ in place of $N(r,1/(f-a))$, respectively. The lower order μ and order λ are defined in turn as follows:

$$\begin{aligned} \mu := \mu(f) &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}, \\ \lambda := \lambda(f) &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \end{aligned} \tag{1.2}$$

For an unbounded subset X of $\mathbb{C}$ and $a \in \widehat{\mathbb{C}}$, we denote by $n(r, X, f = a)$ the number of the zeros of $f - a$ counting multiplicities in $X \cap \{z : |z| \le r\}$.

In 1992, Yang [4], cf. [5] posed the following interesting conjecture.

Yang's Conjecture

Let P be a property (or a set of properties) such that any entire (or meromorphic in $\mathbb{C}$) function satisfying P must be a constant. Suppose that f is an entire (or meromorphic in $\mathbb{C}$) function of finite lower order μ, and that $L_j : \arg z = \theta_j$ $(j = 1, 2, \ldots, q; 0 \le \theta_1 < \theta_2 < \cdots < \theta_q < 2\pi; \theta_{q+1} = \theta_1 + 2\pi)$ are a finite number of rays issuing from the origin. If f satisfies P in $\mathbb{C} \setminus (\bigcup_{j=1}^{q} L_j)$, then the order λ of f has the following estimation:

$$\lambda \le \max_{1 \le j \le q} \left\{ \frac{\pi}{\theta_{j+1} - \theta_j}, \theta_{q+1} = \theta_1 + 2\pi \right\}. \tag{1.3}$$

Yang's conjecture implies that if meromorphic functions satisfy certain properties in the vicinities of a finite number of rays, then the growth of meromorphic functions will be restricted.

A well-known result of Clunie [6], cf. [7] is that an entire function f which satisfies $f(z)f'(z) \ne 1$ in $\mathbb{C}$ must be constant. In [5], L. Yang and C.-C. Yang chose the property P as $f(z)f'(z) \ne 1$ and verified the above conjecture. On the other hand, Fang and Zalcman [8] considered the value distribution of $f + a(f')^n$ in $\mathbb{C}$, where a is a nonzero finite complex number and $n \ge 2$ is a positive integer. Actually, they [8] gave an affirmative answer to a question suggested by Ye [9]. Later on, Xu et al. [10] further generalized $f + a(f')^n$ to $f + a(f^{(k)})^n$ in $\mathbb{C}$ for a positive integer k with $n \ge k + 1$ and investigated its value distribution.

More recently, Chen et al. [11] chose another property P as $f + a(f')^n$ to continue to study Yang's Conjecture and proved the following results.

Theorem A. *Let $f(z)$ be a transcendental meromorphic function with $\delta(\infty, f') > 0$ in $\mathbb{C}$ and let $L_j : \arg z = \theta_j$ $(j = 1, 2, \ldots, q)$ be a finite number of rays issued from the origin such that*

$$-\pi \le \theta_1 < \theta_2 < \cdots < \theta_q < \pi, \quad \theta_{q+1} = \theta_1 + 2\pi \tag{1.4}$$

with $\omega = \max\{\pi/(\theta_{j+1} - \theta_j) : 1 \le j \le q\}$. Set $Y = \mathbb{C} \setminus (\bigcup_{j=1}^{q} L_j)$. If $f(z)$ satisfies

$$\limsup_{r \to \infty} \frac{\log n(r, Y, f + a(f')^n = b)}{\log r} \le \rho \tag{1.5}$$

with a positive number ρ, finite complex numbers $a \ne 0$ and b, for any positive integer $n \ge 2$, then the order λ of $f(z)$ has the estimation $\lambda \le \max\{\omega, \rho\}$.

Theorem B. *Let $f(z)$ be a transcendental meromorphic function of finite lower order μ with $\delta(\infty, f') > 0$ in $\mathbb{C}$. For q pairs of real numbers $\{\alpha_j, \beta_j\}$ such that*

$$-\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_q < \beta_q \le \pi \tag{1.6}$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \le j \le q\}$, *suppose that*

$$\limsup_{r\to\infty} \frac{\log n(r, Y, f + a(f')^n = b)}{\log r} \le \rho \tag{1.7}$$

with a positive number ρ, *finite complex numbers* $a \neq 0$ *and* b, *and* $Y = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}$, *for any positive integer* $n \ge 2$, *and that*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta(\infty, f')}{2}}, \quad \alpha_{q+1} = \alpha_1 + 2\pi \tag{1.8}$$

with $\sigma = \max\{\omega, \rho, \mu\}$. *Then the order* λ *of* $f(z)$ *has the estimation* $\lambda \le \max\{\omega, \rho\}$.

Now there arises a natural question.

Question 1. What can be said if f' in Theorems A and B is replaced by the kth derivative $f^{(k)}$?

In this paper, we will prove the following results which generalize Theorems A and B.

Theorem 1.1. *Let* $f(z)$ *be a transcendental meromorphic function with* $\delta(\infty, f^{(k)}) > 0$ *for a positive integer* k *in* $\mathbb{C}$ *and let* $L_j : \arg z = \theta_j$ $(j = 1, 2, \ldots, q)$ *be a finite number of rays issued from the origin such that*

$$-\pi \le \theta_1 < \theta_2 < \cdots < \theta_q < \pi, \quad \theta_{q+1} = \theta_1 + 2\pi \tag{1.9}$$

with $\omega = \max\{\pi/(\theta_{j+1} - \theta_j) : 1 \le j \le q\}$. *Set* $Y = \mathbb{C} \setminus (\bigcup_{j=1}^{q} L_j)$. *If* $f(z)$ *satisfies*

$$\limsup_{r\to\infty} \frac{\log n\left(r, Y, f + a(f^{(k)})^n = b\right)}{\log r} \le \rho \tag{1.10}$$

with a positive number ρ, *finite complex numbers* $a \neq 0$ *and* b, *for any positive integer* $n \ge k + 1$, *then the order* λ *of* $f(z)$ *has the estimation* $\lambda \le \max\{\omega, \rho\}$.

Remark 1.2. Let $k = 1$. Then by Theorem 1.1 we get Theorem A.

Corollary 1.3. *Let* $f(z)$ *be a transcendental entire function, let the notations* θ_j $(j = 1, 2, \ldots, q + 1)$, ω, *and* Y *be defined as in Theorem 1.1, and suppose that the function* $f(z)$ *fulfills the same condition* (1.10) *as in Theorem 1.1. Then the order* λ *of* $f(z)$ *has the estimation* $\lambda \le \max\{\omega, \rho\}$.

Theorem 1.4. *Let* $f(z)$ *be a transcendental meromorphic function of finite lower order* μ *with* $\delta(\infty, f^{(k)}) > 0$ *for a positive integer* k *in* $\mathbb{C}$. *For* q *pairs of real numbers* $\{\alpha_j, \beta_j\}$ *such that*

$$-\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_q < \beta_q \le \pi \tag{1.11}$$

with $\omega = \max\{\pi/(\beta_j - \alpha_j) : 1 \le j \le q\}$, *suppose that*

$$\limsup_{r\to\infty} \frac{\log n\left(r, Y, f + a\left(f^{(k)}\right)^n = b\right)}{\log r} \le \rho \tag{1.12}$$

with a positive number ρ, *finite complex numbers* $a \neq 0$ *and* b, *and* $Y = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}$, *for any positive integer* $n \ge k+1$, *and that*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta\left(\infty, f^{(k)}\right)}{2}}, \quad \alpha_{q+1} = \alpha_1 + 2\pi \tag{1.13}$$

with $\sigma = \max\{\omega, \rho, \mu\}$. *Then the order* λ *of* $f(z)$ *has the estimation* $\lambda \le \max\{\omega, \rho\}$.

Remark 1.5. Let $k = 1$. Then by Theorem 1.4 we get Theorem B.

Corollary 1.6. *Let* $f(z)$ *be a transcendental entire function of finite lower order* μ, *let the notations* α_j, β_j $(j = 1, 2, \ldots, q)$, ω, σ, *and* Y *be defined as in Theorem 1.4, and suppose that the function* $f(z)$ *fulfills the same conditions* (1.12) *and* (1.13) *as in Theorem 1.4. Then the order* λ *of* $f(z)$ *has the estimation* $\lambda \le \max\{\omega, \rho\}$.

In order to prove our results, we require the Nevanlinna theory of meromorphic functions in an angular domain. For the sake of convenience, we recall some notations and definitions. Let f be a meromorphic function on the angular domain $\overline{\Omega}(\alpha, \beta) = \{z : \alpha \le \arg z \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. Nevanlinna et al. [12, 13] introduced the following notations:

$$\begin{aligned} A_{\alpha,\beta}(r, f) &= \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)\left\{\log^+\left|f\left(te^{i\alpha}\right)\right| + \log^+\left|f\left(te^{i\beta}\right)\right|\right\}\frac{dt}{t}, \\ B_{\alpha,\beta}(r, f) &= \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+\left|f\left(re^{i\theta}\right)\right| \sin\omega(\theta - \alpha)\,d\theta, \\ C_{\alpha,\beta}(r, f) &= 2\sum_{1<|b_m|<r}\left(\frac{1}{|b_m|^\omega} - \frac{|b_m|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_m - \alpha), \end{aligned} \tag{1.14}$$

where $\omega = \pi/(\beta - \alpha)$ and $b_m = |b_m|e^{i\theta_m}$ are the poles of f in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The function $C_{\alpha,\beta}(r, f)$ is called the angular counting function (counting multiplicities) of the poles of f in $\overline{\Omega}(\alpha, \beta)$, and $\overline{C}_{\alpha,\beta}(r, f)$ is called the angular reduced counting function (ignoring multiplicities) of the poles of f in $\overline{\Omega}(\alpha, \beta)$. Further, Nevanlinna's angular characteristic function $S_{\alpha,\beta}(r, f)$ is defined as follows:

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f). \tag{1.15}$$

Throughout the paper, we denote by $R(r, *)$ a quantity satisfying

$$R(r, *) = O\{\log(rT(r, *))\}, \quad \forall r \notin E, \tag{1.16}$$

where E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context.

2. Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 (see [5, 12–14]). *Let f be meromorphic in $\mathbb{C}$. Then in $\overline{\Omega}(\alpha,\beta)$ for an arbitrary finite complex number a, we have*

$$S_{\alpha,\beta}\left(r,\frac{1}{f-a}\right)=S_{\alpha,\beta}(r,f)+O(1), \tag{2.1}$$

and for each positive integer k, we have

$$A_{\alpha,\beta}\left(r,\frac{f^{(k)}}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f^{(k)}}{f}\right)=R(r,f). \tag{2.2}$$

Lemma 2.2 (see [10]). *Let $n\geq 2$ and k be positive integers, let a be a nonzero finite complex number, and let $p(z)$ be a polynomial. Then the solution of the differential equation $a[\omega^{(k)}(z)]^n+\omega(z)=p(z)$ must be polynomial.*

Lemma 2.3 (see [1, Theorem 3.1]; [1, page 33, (2.1)]). *Let f be meromorphic in $\mathbb{C}$, let $a_1,a_2,\ldots,a_q$, where $q\geq 2$, be distinct complex numbers, $\delta>0$, and suppose that $|a_\mu-a_\nu|\geq\delta$ for $1\leq\mu<\nu\leq q$. Then, for each positive integer k, we have*

$$\begin{gathered}T\left(r,f^{(k)}\right)\leq T(r,f)+k\overline{N}(r,f)+O(\log rT(r,f)),\\ m\left(r,\sum_{\nu=1}^{q}\frac{1}{f-a_\nu}\right)\geq\sum_{\nu=1}^{q}m\left(r,\frac{1}{f-a_\nu}\right)-q\log^{+}\frac{3q}{\delta}-\log 2,\quad\forall r\notin E.\end{gathered} \tag{2.3}$$

Next we slightly modify the proof of Lemma 2.4 in [10] to give the following key lemma, which is an important generalization of Lemma 3 in [11].

Lemma 2.4. *Let f be transcendental meromorphic in $\mathbb{C}$, let $a\neq 0$ and b be finite complex numbers, and let n and k be positive integers with $n\geq k+1$. Then in $\overline{\Omega}(\alpha,\beta)$,*

$$B_{\alpha,\beta}\left(r,f^{(k)}\right)\leq(k+1)C_{\alpha,\beta}\left(r,\frac{1}{f+a\left(f^{(k)}\right)^n-b}\right)+R(r,f). \tag{2.4}$$

Proof. Put

$$g=f+a\left(f^{(k)}\right)^n-b, \tag{2.5}$$

$$\phi=\frac{g^{(k)}}{g}. \tag{2.6}$$

Then $\phi \not\equiv 0$, for otherwise g would be a polynomial of degree at most $k-1$. This and (2.5) together with Lemma 2.2 imply that f must be a polynomial, a contradiction. By the Nevanlinna's basic reasoning, Lemmas 2.1, and 2.3, (2.5), and (2.6), we have

$$\left(A_{\alpha,\beta} + B_{\alpha,\beta}\right)(r,\phi) = R(r,g) = R(r,f). \tag{2.7}$$

Now a simple calculation for (2.5) shows that

$$g^{(k)} = f^{(k)} + a\left(\left(f^{(k)}\right)^n\right)^{(k)} = f^{(k)}\left(1 + Q\left(f^{(k)}\right)\right), \tag{2.8}$$

where $Q(f^{(k)})$ is a homogeneous differential polynomial in $f^{(k)}$ of degree $n-1$ and of the form

$$\begin{aligned} Q\left(f^{(k)}\right) &= a\left(f^{(k)}\right)^{n-k-1} \\ &\quad \times \left(\frac{n!\left(f^{(k+1)}\right)^k}{(n-k)!} + \frac{k(k-1)n!}{2!(n-k+1)!} f^{(k)}\left(f^{(k+1)}\right)^{k-2} f^{(k+2)} + \cdots + n\left(f^{(k)}\right)^{k-1} f^{(2k)}\right). \end{aligned} \tag{2.9}$$

Then by (2.5), (2.6), and (2.8), we get

$$f^{(k)}\left(1 + Q\left(f^{(k)}\right)\right) = \phi\left(f + a\left(f^{(k)}\right)^n - b\right). \tag{2.10}$$

From (2.5)–(2.10), and Lemma 2.1, it thus follows that

$$\begin{aligned} &C_{\alpha,\beta}\left(r, \frac{1}{f^{(k)}}\right) + C_{\alpha,\beta}\left(r, \frac{1}{Q(f^{(k)}) + 1}\right) \\ &\quad \le C_{\alpha,\beta}\left(r, \frac{1}{\phi}\right) + C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) \\ &\quad \le C_{\alpha,\beta}(r,\phi) + C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f) \\ &\quad \le k\overline{C}_{\alpha,\beta}(r,f) + (k+1)C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f) \\ &\quad \le \frac{k}{k+1} C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1)C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f). \end{aligned} \tag{2.11}$$

On the other hand, by the Nevanlinna's basic reasoning, Lemmas 2.1 and 2.3, we deduce that

$$\begin{aligned}
&(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(f^{(k)})^{n-1}}\right)+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{Q(f^{(k)})+1}\right)\\
&\quad\le (A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{Q(f^{(k)})}\right)+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{Q(f^{(k)})}{(f^{(k)})^{n-1}}\right)\\
&\qquad+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{Q(f^{(k)})+1}\right)\\
&\quad\le (A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{Q(f^{(k)})}+\frac{1}{Q(f^{(k)})+1}\right)+R(r,f)\\
&\quad\le (A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{(Q(f^{(k)}))'}{Q(f^{(k)})}+\frac{(Q(f^{(k)})+1)'}{Q(f^{(k)})+1}\right)\\
&\qquad+(A_{\alpha,\beta}+B_{\alpha,\beta})\left(r,\frac{1}{(Q(f^{(k)}))'}\right)+R(r,f)\\
&\quad\le S_{\alpha,\beta}\left(r,\left(Q\left(f^{(k)}\right)\right)'\right)-C_{\alpha,\beta}\left(r,\frac{1}{(Q(f^{(k)}))'}\right)+R(r,f)\\
&\quad\le S_{\alpha,\beta}\left(r,Q\left(f^{(k)}\right)\right)+\overline{C}_{\alpha,\beta}(r,f)-C_{\alpha,\beta}\left(r,\frac{1}{(Q(f^{(k)}))'}\right)+R(r,f)\\
&\quad\le S_{\alpha,\beta}\left(r,Q\left(f^{(k)}\right)\right)+\frac{1}{k+1}C_{\alpha,\beta}\left(r,f^{(k)}\right)-C_{\alpha,\beta}\left(r,\frac{1}{(Q(f^{(k)}))'}\right)+R(r,f).
\end{aligned}\tag{2.12}$$

This, together with Lemma 2.1, yields

$$\begin{aligned}
(n-1)S_{\alpha,\beta}\left(r,f^{(k)}\right)\le{}&\frac{1}{k+1}C_{\alpha,\beta}\left(r,f^{(k)}\right)+(n-1)C_{\alpha,\beta}\left(r,\frac{1}{f^{(k)}}\right)+C_{\alpha,\beta}\left(r,\frac{1}{Q(f^{(k)})+1}\right)\\
&-C_{\alpha,\beta}\left(r,\frac{1}{(Q(f^{(k)}))'}\right)+R(r,f).
\end{aligned}\tag{2.13}$$

Next we divide into two cases.

Case 1 ($n \geq k+2$). Suppose that z_0 is a zero of $f^{(k)}$ of multiplicity m. Then we can know that z_0 is a zero of $(Q(f^{(k)}))'$ of multiplicity at least $(n-k-1)m + k(m-1) - 1 = (n-1)m - k - 1$. Thus, we have

$$\begin{aligned}
&(n-1)C_{\alpha,\beta}\left(r, \frac{1}{f^{(k)}}\right) + C_{\alpha,\beta}\left(r, \frac{1}{Q(f^{(k)})+1}\right) - C_{\alpha,\beta}\left(r, \frac{1}{(Q(f^{(k)}))'}\right) \\
&\quad = C_{\alpha,\beta}\left(r, \frac{1}{(f^{(k)})^{n-1}}\right) + C_{\alpha,\beta}\left(r, \frac{1}{Q(f^{(k)})+1}\right) - C_{\alpha,\beta}\left(r, \frac{1}{(Q(f^{(k)}))'}\right) \\
&\quad \leq (k+1)\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f^{(k)}}\right) + \overline{C}_{\alpha,\beta}\left(r, \frac{1}{Q(f^{(k)})+1}\right).
\end{aligned} \tag{2.14}$$

Substituting this into (2.13) gives

$$\begin{aligned}
(n-1)S_{\alpha,\beta}\left(r, f^{(k)}\right) \leq{}& \frac{1}{k+1}C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1)\overline{C}_{\alpha,\beta}\left(r, \frac{1}{f^{(k)}}\right) \\
&+ \overline{C}_{\alpha,\beta}\left(r, \frac{1}{Q(f^{(k)})+1}\right) + R(r,f).
\end{aligned} \tag{2.15}$$

From this and (2.11) it follows that

$$(n-1)S_{\alpha,\beta}\left(r, f^{(k)}\right) \leq \frac{k^2+k+1}{k+1}C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1)^2 C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f), \tag{2.16}$$

implying that

$$S_{\alpha,\beta}\left(r, f^{(k)}\right) \leq \frac{k^2+k+1}{(n-1)(k+1)}C_{\alpha,\beta}\left(r, f^{(k)}\right) + \frac{(k+1)^2}{n-1}C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f). \tag{2.17}$$

Thereby, noting $n \geq k+2$, we get

$$\begin{aligned}
S_{\alpha,\beta}\left(r, f^{(k)}\right) &\leq \frac{k^2+k+1}{(k+1)^2}C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1)C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f) \\
&\leq C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1)C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f),
\end{aligned} \tag{2.18}$$

so that

$$B_{\alpha,\beta}\left(r, f^{(k)}\right) \leq A_{\alpha,\beta}\left(r, f^{(k)}\right) + B_{\alpha,\beta}\left(r, f^{(k)}\right) \leq (k+1)C_{\alpha,\beta}\left(r, \frac{1}{f + a(f^{(k)})^n - b}\right) + R(r,f). \tag{2.19}$$

This is the desired result.

Case 2 ($n = k+1$). Then, by (2.11) and (2.13), we have

$$kS_{\alpha,\beta}\left(r, f^{(k)}\right) \le \frac{k^2+1}{k+1} C_{\alpha,\beta}\left(r, f^{(k)}\right) + k(k+1) C_{\alpha,\beta}\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) + R(r, f), \tag{2.20}$$

which leads to

$$\begin{aligned} S_{\alpha,\beta}\left(r, f^{(k)}\right) &\le \frac{k^2+1}{k(k+1)} C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1) C_{\alpha,\beta}\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) + R(r, f) \\ &\le C_{\alpha,\beta}\left(r, f^{(k)}\right) + (k+1) C_{\alpha,\beta}\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) + R(r, f). \end{aligned} \tag{2.21}$$

This yields

$$B_{\alpha,\beta}\left(r, f^{(k)}\right) \le A_{\alpha,\beta}\left(r, f^{(k)}\right) + B_{\alpha,\beta}\left(r, f^{(k)}\right) \le (k+1) C_{\alpha,\beta}\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) + R(r, f), \tag{2.22}$$

obtaining the desired result.

This completes the proof of Lemma 2.4. □

The following auxiliary results regarding Pólya peaks and the spread relation are necessary in the proofs of our theorems.

Lemma 2.5 (see [14–16]). *Let f be a transcendental meromorphic function of finite lower order μ and order $\lambda (0 < \lambda \le \infty)$ in $\mathbb{C}$. Then, for an arbitrary positive number σ satisfying $\mu \le \sigma \le \lambda$ and any set E of finite linear measure, there exist Pólya peaks $\{r_n\}$ satisfying the following:*

(i) $r_n \notin E$, $\lim_{n\to\infty}(r_n/n) = \infty$;

(ii) $\liminf_{n\to\infty}(\log T(r_n, f))/\log r_n \ge \sigma$;

(iii) $T(t, f) < (1 + o(1))(t/r_n)^{\sigma} T(r_n, f)$, $t \in [r_n/n, nr_n]$.

A sequence of $\{r_n\}$ satisfying (i), (ii), and (iii) in Lemma 2.5 is called a Pólya peak of order σ of f outside E. Given a positive function $\Lambda = \Lambda(r)$ on $(0, \infty)$ with $\Lambda \to 0$ as $r \to \infty$, we define

$$\begin{aligned} D_{\Lambda}(r, a) &= \left\{\theta \in [-\pi, \pi) \mid \log^{+} \frac{1}{\left|f\left(re^{i\theta}\right) - a\right|} > \Lambda(r) T(r, f)\right\}, \\ D_{\Lambda}(r, \infty) &= \left\{\theta \in [-\pi, \pi) \mid \log^{+} \left|f\left(re^{i\theta}\right)\right| > \Lambda(r) T(r, f)\right\}. \end{aligned} \tag{2.23}$$

Lemma 2.6 (see [17]). *Let f be a transcendental meromorphic function of finite lower order μ and order $\lambda (0 < \lambda \le \infty)$ in $\mathbb{C}$. Suppose that $\delta = \delta(a, f) > 0$ for some $a \in \hat{\mathbb{C}}$. Then for an arbitrary*

Pólya peak $\{r_n\}$ *of order* $\sigma(\mu \le \sigma \le \lambda)$ *and an arbitrary positive function* $\Lambda = \Lambda(r)$ *with* $\Lambda \to 0$ *as* $r \to \infty$, *we have*

$$\liminf_{n\to\infty} \text{meas } D_\Lambda(r_n, a) \ge \min\left\{2\pi,\ \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\}. \tag{2.24}$$

Now a more precise estimation of $T(r, f)$ *in terms of* $T(r, f^{(k)})$ *is introduced as follows.*

Lemma 2.7 (see [18]). *Let* f *be transcendental meromorphic in* $\mathbb{C}$. *Then, for a positive integer* k *and a real number* $\tau > 1$, *we have*

$$T(r, f) < K_{\tau,k} T\left(\tau r, f^{(k)}\right) + \log(\tau r) + O(1), \tag{2.25}$$

where $K_{\tau,k}$ *is a positive number depending on only* τ *and* k.

At last, we state the following results due to Edrei, Hayman, and Miles, respectively.

Lemma 2.8 (see [19]). *Let* f *be a transcendental meromorphic function with* $\delta = \delta(\infty, f) > 0$ *in* $\mathbb{C}$. *Then, given* $\varepsilon > 0$, *we have*

$$\text{meas } E(r, f) > \frac{1}{T^{\varepsilon}(r, f)\left[\log r\right]^{1+\varepsilon}}, \quad \forall r \notin F, \tag{2.26}$$

where

$$E(r, f) = \left\{\theta \in [-\pi, \pi) : \log^+\left|f\left(re^{i\theta}\right)\right| > \frac{\delta}{4}T(r, f)\right\} \tag{2.27}$$

and F *is a set of positive real numbers with finite logarithmic measure (i.e.,* $\int_F (dt/t) < \infty$*) depending on* ε *only.*

Lemma 2.9 (see [20]). *Let* f *be a transcendental meromorphic function in* $\mathbb{C}$. *Then for each* $K > 1$ *there exists a set* $M(K)$ *of the lower logarithmic density at least* $d(K) = 1 - (2e^{K-1} - 1)^{-1} > 0$, *that is,*

$$\underline{\log \text{dens}}\, M(K) := \liminf_{r\to\infty} \frac{1}{\log r} \int_{M(K)\cap[1,r]} \frac{dt}{t} \ge d(K), \tag{2.28}$$

such that, for every positive integer p, *we have*

$$\limsup_{r\to\infty, r\in M(K)} \frac{T(r, f)}{T(r, f^{(p)})} \le 3eK. \tag{2.29}$$

3. Proofs of Theorems 1.1 and 1.4

In this section, we state the detailed proofs of Theorems 1.1 and 1.4 by using the method in [14]. To begin with, we give the proof of Theorem 1.4. Finally the proof of Theorem 1.1 can be derived from Theorem 1.4.

3.1. Proof of Theorem 1.4

Assume on the contrary that Theorem 1.4 does not hold. Then $\lambda(f) > \max\{\omega, \rho\}$. Now by (1.12), we have

$$n\left(r, \overline{\Omega}(\alpha_j, \beta_j), f + a\left(f^{(k)}\right)^n = b\right) \leq r^{\rho+\varepsilon}, \quad j = 1, 2, \ldots, q \tag{3.1}$$

for arbitrarily small $\varepsilon > 0$ and sufficiently large $r \geq r_0$. Let ξ_m be the zeros of $f + a(f^{(k)})^n - b$ on $\overline{\Omega}(\alpha_j, \beta_j)$ appearing according to their multiplicities, and set $\omega_j = \pi/(\beta_j - \alpha_j)$. By the definition of $C_{\alpha,\beta}(r, *)$, we deduce that

$$\begin{aligned}
&C_{\alpha_j,\beta_j}\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) \\
&\quad \leq 2 \sum_{1<|\xi_m|<r} \frac{1}{|\xi_m|^{\omega_j}} = 2\int_1^r \frac{dn\left(t, \overline{\Omega}(\alpha_j, \beta_j), f + a\left(f^{(k)}\right)^n = b\right)}{t^{\omega_j}} \\
&\quad \leq 2\frac{n\left(r, \overline{\Omega}(\alpha_j, \beta_j), f + a\left(f^{(k)}\right)^n = b\right)}{r^{\omega_j}} + 2\omega_j \int_1^r \frac{n\left(t, \overline{\Omega}(\alpha_j, \beta_j), f + a\left(f^{(k)}\right)^n = b\right)}{t^{\omega_j+1}} dt \\
&\quad \leq 2r^{\rho+\varepsilon-\omega_j} + O(1) + 2\omega_j \int_{r_0}^r \frac{t^{\rho+\varepsilon}}{t^{\omega_j+1}} dt \\
&\quad \leq K_{j,\varepsilon} r^{\rho+\varepsilon-\omega_j} + O(\log r),
\end{aligned} \tag{3.2}$$

where $K_{j,\varepsilon}$ is a positive number depending on only j and ε, which is not necessarily the same for every occurrence in the context. From Lemma 2.4, we have

$$B_{\alpha_j,\beta_j}\left(r, f^{(k)}\right) \leq (k+1) C_{\alpha_j,\beta_j}\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) + R(r, f). \tag{3.3}$$

Thus, it follows by (3.2) and (3.3) that

$$B_{\alpha_j,\beta_j}\left(r, f^{(k)}\right) \leq K_{j,\varepsilon} r^{\rho+\varepsilon-\omega_j} + O(\log rT(r, f)), \quad \forall r \notin E, \tag{3.4}$$

where the exceptional set E associated with $R(r, f)$ is of at most finite linear measure.

Now we discuss two cases separately.

Case 1 ($\lambda(f) > \mu(f)$). Then by the assumption $\sigma = \max\{\omega, \rho, \mu\}$ and $\lambda(f) > \max\{\omega, \rho\}$, we have $\lambda(f^{(k)}) = \lambda(f) > \sigma \geq \mu(f) = \mu(f^{(k)})$. Now from (1.13), we can find a real number $\varepsilon > 0$ such that

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\varepsilon) + 2\varepsilon < \frac{4}{\sigma + 2\varepsilon} \arcsin\sqrt{\frac{\delta(\infty, f^{(k)})}{2}}, \quad \alpha_{q+1} = \alpha_1 + 2\pi, \tag{3.5}$$

$$\lambda\left(f^{(k)}\right) > \sigma + 2\varepsilon > \mu\left(f^{(k)}\right). \tag{3.6}$$

Applying Lemma 2.5 to $f^{(k)}$ gives the existence of the Pólya peak $\{r_n\}$ of order $\sigma + 2\varepsilon$ of $f^{(k)}$ outside the set E. Then, noting that $\sigma + 2\varepsilon > \omega_j \geq 1/2$ and $0 < \delta(\infty, f^{(k)}) \leq 1$, by applying Lemma 2.6 to the Pólya peak $\{r_n\}$, for sufficiently large n we have

$$\operatorname{meas} D_\Lambda(r_n, \infty) > \frac{4}{\sigma + 2\varepsilon} \arcsin\sqrt{\frac{\delta(\infty, f^{(k)})}{2}} - \varepsilon. \tag{3.7}$$

Without loss of generality, we can assume that (3.7) holds for all the n. Set

$$K_n := \operatorname{meas}\left(D_\Lambda(r_n, \infty) \cap \bigcup_{j=1}^{q}(\alpha_j + \varepsilon, \beta_j - \varepsilon)\right). \tag{3.8}$$

It then follows from (3.5) and (3.7) that

$$\begin{aligned} K_n \geq\ & \operatorname{meas}(D_\Lambda(r_n, \infty)) - \operatorname{meas}\left([-\pi, \pi) \setminus \bigcup_{j=1}^{q}(\alpha_j + \varepsilon, \beta_j - \varepsilon)\right) \\ =\ & \operatorname{meas}(D_\Lambda(r_n, \infty)) - \operatorname{meas}\left(\bigcup_{j=1}^{q}(\beta_j - \varepsilon, \alpha_{j+1} + \varepsilon)\right) \\ =\ & \operatorname{meas}(D_\Lambda(r_n, \infty)) - \sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\varepsilon) > \varepsilon > 0. \end{aligned} \tag{3.9}$$

By (3.9), it is easy to see that there exists a j_0 such that, for infinitely many n, we have

$$\operatorname{meas}\left(D_\Lambda(r_n, \infty) \cap (\alpha_{j_0} + \varepsilon, \beta_{j_0} - \varepsilon)\right) \geq \frac{K_n}{q} > \frac{\varepsilon}{q}. \tag{3.10}$$

Without loss of generality, we can assume that (3.10) holds for all the n. Set $E_n = D_\Lambda(r_n, \infty) \cap (\alpha_{j_0} + \varepsilon, \beta_{j_0} - \varepsilon)$ and $\Lambda(r) = [\log r]^{-1}$. From the definition of $D_\Lambda(r_n, \infty)$, we deduce that

$$\begin{aligned}\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon} \log^+\left|f^{(k)}\left(r_n e^{i\theta}\right)\right| d\theta &\geq \int_{E_n} \log^+\left|f^{(k)}\left(r_n e^{i\theta}\right)\right| d\theta \\ &\geq \frac{\varepsilon}{q}\frac{T\left(r_n, f^{(k)}\right)}{\log r_n}.\end{aligned} \tag{3.11}$$

On the other hand, by the definition of $B_{\alpha,\beta}(r,*)$ and (3.4), it follows that

$$\begin{aligned}\int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon} \log^+\left|f^{(k)}\left(r_n e^{i\theta}\right)\right| d\theta &\leq \frac{\pi r_n^{\omega_{j_0}}}{2\omega_{j_0}\sin(\varepsilon\omega_{j_0})} B_{\alpha_{j_0},\beta_{j_0}}\left(r_n, f^{(k)}\right) \\ &\leq K_{j_0,\varepsilon}\left(r_n^{\rho+\varepsilon} + r_n^{\omega_{j_0}}\log\left(r_n T\left(r_n, f\right)\right)\right),\end{aligned} \tag{3.12}$$

where $r_n \notin E$, $\omega_{j_0} = \pi/(\beta_{j_0} - \alpha_{j_0})$, and $K_{j_0,\varepsilon}$ is a positive number depending on only j_0 and ε. Combining (3.11) with (3.12) gives

$$T\left(r_n, f^{(k)}\right) \leq \frac{qK_{j_0,\varepsilon}\log r_n}{\varepsilon}\left(r_n^{\rho+\varepsilon} + r_n^{\omega_{j_0}}\log\left(r_n T\left(r_n, f\right)\right)\right), \tag{3.13}$$

implying together with (iii) in Lemma 2.5 and Lemma 2.7 that

$$\log T\left(r_n, f^{(k)}\right) \leq 3\log\log r_n + \max\left\{\rho+\varepsilon, \omega_{j_0}\right\}\log r_n + \log\log T\left(r_n, f^{(k)}\right) + O(1). \tag{3.14}$$

Thus, from (ii) in Lemma 2.5 for $\sigma + 2\varepsilon$, we have

$$\sigma + 2\varepsilon \leq \limsup_{n\to\infty}\frac{\log T\left(r_n, f^{(k)}\right)}{\log r_n} \leq \max\left\{\rho+\varepsilon, \omega_{j_0}\right\} \leq \sigma + \varepsilon, \tag{3.15}$$

which is impossible.

Case 2 ($\lambda(f) = \mu(f)$). Then by the assumption $\sigma = \max\{\omega, \rho, \mu\}$ and $\lambda(f) > \max\{\omega, \rho\}$, we have $\sigma = \mu(f) = \lambda(f) = \lambda(f^{(k)}) = \mu(f^{(k)})$. By the same argument as in Case 1 with all the $\sigma + 2\varepsilon$ replaced by $\sigma = \mu(f)$, we can derive

$$\mu(f) = \sigma \leq \max\left\{\rho+\varepsilon, \omega_{j_0}\right\} < \lambda(f), \tag{3.16}$$

which is also impossible.

This completes the proof of Theorem 1.4.

3.2. Proof of Theorem 1.1

By Theorem 1.4, it suffices to prove that the lower order $\mu(f)$ of f is finite. As in the proof of Theorem 1.4, we have, for each $j \in \{1, 2, \ldots, q\}$,

$$B_{\theta_j,\theta_{j+1}}\left(r, f^{(k)}\right) \leq K_{j,\varepsilon} r^{\rho+\varepsilon-\omega_j} + O(\log rT(r, f)), \quad \omega_j = \frac{\pi}{\theta_{j+1} - \theta_j}, \ \forall r \notin E, \tag{3.17}$$

where the exceptional set E associated with $R(r, f)$ is of at most finite linear measure.

For F in Lemma 2.8 and E in (3.17), $\overline{\log \operatorname{dens}}(F \cup E) = 0$ and hence for $M(2)$ in Lemma 2.9 when $K = 2$, $\underline{\log \operatorname{dens}}(M(2) \setminus (F \cup E)) \geq d(2) > 0$. Applying Lemma 2.8 to $f^{(k)}$ gives the existence of a sequence $\{r_n\}$ of positive numbers such that $r_n \to \infty$ $(n \to \infty)$, $r_n \in M(2) \setminus (F \cup E)$, and

$$\operatorname{meas} E\left(r_n, f^{(k)}\right) > \frac{1}{T^{\varepsilon}\left(r_n, f^{(k)}\right)\left[\log r_n\right]^{1+\varepsilon}}. \tag{3.18}$$

Set

$$\varepsilon_n = \frac{1}{2q+1} \frac{1}{T^{\varepsilon}\left(r_n, f^{(k)}\right)\left[\log r_n\right]^{1+\varepsilon}}. \tag{3.19}$$

Then, from (3.18) and (3.19), it follows that

$$\begin{aligned} &\operatorname{meas}\left(E\left(r_n, f^{(k)}\right) \cap \bigcup_{j=1}^{q} (\theta_j + \varepsilon_n, \theta_{j+1} - \varepsilon_n)\right) \\ &\quad \geq \operatorname{meas} E\left(r_n, f^{(k)}\right) - \operatorname{meas}\left(\bigcup_{j=1}^{q} (\theta_j - \varepsilon_n, \theta_j + \varepsilon_n)\right) \\ &\quad \geq (2q+1)\varepsilon_n - 2q\varepsilon_n = \varepsilon_n > 0. \end{aligned} \tag{3.20}$$

Hence, there exists a $j \in \{1, 2, \ldots, q\}$ such that, for infinitely many n, we have

$$\operatorname{meas}\left(E\left(r_n, f^{(k)}\right) \cap (\theta_j + \varepsilon_n, \theta_{j+1} - \varepsilon_n)\right) \geq \frac{\varepsilon_n}{q}. \tag{3.21}$$

Without loss of generality, we can assume that this holds for all the n. Let $E_n = E(r_n, f^{(k)}) \cap (\theta_j + \varepsilon_n, \theta_{j+1} - \varepsilon_n)$. Thus, by the definition of $E(r, f)$ and (3.21), it follows that

$$\begin{aligned}\int_{\theta_j+\varepsilon_n}^{\theta_{j+1}-\varepsilon_n} \log^+\left|f^{(k)}\left(r_n e^{i\theta}\right)\right| d\theta &\geq \int_{E_n} \log^+\left|f^{(k)}\left(r_n e^{i\theta}\right)\right| d\theta \\ &\geq \operatorname{meas}(E_n)\frac{\delta(\infty, f^{(k)})}{4} T\left(r_n, f^{(k)}\right) \\ &\geq \frac{\varepsilon_n \delta(\infty, f^{(k)})}{4q} T\left(r_n, f^{(k)}\right).\end{aligned} \tag{3.22}$$

On the other hand, by the definition of $B_{\alpha,\beta}(r, *)$ and (3.17), we have

$$\begin{aligned}\int_{\theta_j+\varepsilon_n}^{\theta_{j+1}-\varepsilon_n} \log^+\left|f^{(k)}\left(r_n e^{i\theta}\right)\right| d\theta &\leq \frac{\pi r_n^{\omega_j}}{2\omega_j \sin(\varepsilon_n \omega_j)} B_{\theta_j,\theta_{j+1}}\left(r_n, f^{(k)}\right) \\ &\leq \frac{\pi^2 K_{j,\varepsilon}}{4\omega_j^2 \varepsilon_n}\left(r_n^{\rho+\varepsilon} + r_n^{\omega_j} \log\left(r_n T(r_n, f)\right)\right),\end{aligned} \tag{3.23}$$

where $r_n \notin E \cup F$, $\omega_j = \pi/(\theta_{j+1} - \theta_j)$, and $K_{j,\varepsilon}$ is a positive number depending on only j and ε. Combining (3.22) with (3.23) now yields

$$\varepsilon_n^2 T\left(r_n, f^{(k)}\right) \leq \frac{q\pi^2 K_{j,\varepsilon}}{\omega_j^2 \delta(\infty, f^{(k)})}\left(r_n^{\rho+\varepsilon} + r_n^{\omega_j} \log\left(r_n T(r_n, f)\right)\right), \tag{3.24}$$

so that, together with (3.19) and Lemma 2.9, we have

$$T^{1-2\varepsilon}\left(r_n, f^{(k)}\right) \leq \frac{q\pi^2 (2q+1)^2 [\log r_n]^{2+2\varepsilon} K_{j,\varepsilon}}{\omega_j^2 \delta(\infty, f^{(k)})}\left(r_n^{\rho+\varepsilon} + r_n^{\omega_j}\left(\log r_n + \log T\left(r_n, f^{(k)}\right) + \log(6e)\right)\right). \tag{3.25}$$

Thus $\mu(f) = \mu(f^{(k)}) \leq \max\{\rho + \varepsilon, \omega_j\}/(1 - 2\varepsilon) < \infty$ and so Theorem 1.1 follows from Theorem 1.4.

This completes the proof of Theorem 1.1.

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