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Research Article

Asymptotic Formulae via a Korovkin-Type Result

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We present a sort of Korovkin-type result that provides a tool to obtain asymptotic formulae for sequences of linear positive operators.

1. Introduction

This paper deals with the approximation of continuous functions by sequences of positive linear operators. In this setting, on studying a sequence of operators, say L_n, one usually intends to prove firstly that the sequence defines an approximation process; that is, for each function f of some space, $L_n f$ converges to f, in a certain sense as n tends to infinity; afterwards, one searches for quantitative results that estimate the degree of convergence, and finally one measures the goodness of the estimates, mainly through inverse and saturation results. An outstanding tool to achieve these saturation results is given by the asymptotic formulae that provide information about the so-called optimal degree of convergence. The most representative expression of this type was stated for the classical Bernstein operators by Voronovskaja in [1] and reads as follows: for $f \in C[0,1]$ and $x \in (0,1)$, if $f''(x)$ exists, then

$$\lim_{n\to\infty} n\big(B_n f(x) - f(x)\big) = \frac{x(1-x)}{2} f''(x). \tag{1.1}$$

This work dwells upon this type of expressions.

A classical key ingredient to prove an asymptotic formula for a sequence of positive linear operators L_n is Taylor's theorem. From it, the formula appears after some minor work if one is able to find easy-to-use expressions for the first moments of the operators, namely,

$$L_n e_i^x(x) = L_n\Big((e_1 - xe_0)^i\Big)(x), \qquad i = 0, 1, 2, 4, \tag{1.2}$$

(e_i and e_i^x denote the monomials $e_i(t) = t^i$ and $e_i^x(t) = (t - x)^i$). On the contrary, if the calculation of the moments gets complicated, then a drawback appears.

The main purpose of this paper is to present a tool that helps to overcome these situations. The basic ideas behind the main result (Theorem 2.1 below) lies in [2, Chapter 5], the novelty here being that instead of using the Taylor formula of the function f to be approximated, we consider the Taylor formula of $f \circ \varphi^{-1}$ for a certain function φ.

It is the intention of the authors that the paper offers a clear and quick procedure to obtain asymptotic expressions for a wide variety of sequences of linear positive operators.

2. The Main Result

Let φ be any ∞-times continuously differentiable function on $[0, 1]$, such that $\varphi(0) = 0$, $\varphi(1) = 1$, and $\varphi'(x) > 0$ for $x \in (0, 1)$. We denote by $e_{\varphi,i}^x$ the function

$$e_{\varphi,i}^x(t) = \big(\varphi(t) - \varphi(x)\big)^i, \tag{2.1}$$

and D^i denotes the usual ith differential operator, though we keep on using the common notation f' and f'' for the first and second derivatives of a function f.

Although we restrict our attention to the space $C[0, 1]$ of all continuous functions defined on $[0, 1]$, the following main result of the paper remains valid for any compact real interval with the obvious modifications.

Theorem 2.1. *Let $L_n : C[0, 1] \to C[0, 1]$ be a sequence of linear positive operators, and let $x \in (0, 1)$ be fixed. Let us assume that there exist a sequence of positive real numbers $\lambda_n \to +\infty$ (as $n \to +\infty$) and two functions $p, q \in C[0, 1]$, p being strictly positive on $(0, 1)$, such that for all $i \in \{0, 1, 2, 4\}$*

$$\lim_{n\to+\infty} \lambda_n\Big(L_n e_{\varphi,i}^x(x) - e_{\varphi,i}^x(x)\Big) = p(x)D^2 e_{\varphi,i}^x(x) + q(x)D^1 e_{\varphi,i}^x(x). \tag{2.2}$$

Then for each $f \in C[0, 1]$, twice differentiable at the point x,

$$\lim_{n\to+\infty} \lambda_n\big(L_n f(x) - f(x)\big) = p(x)f''(x) + q(x)f'(x). \tag{2.3}$$

Remark 2.2. Notice that for $i = 0$, identity (2.2) becomes $\lim_{n\to+\infty}\lambda_n(L_n e_0(x) - 1) = 0$, which is obviously fulfilled if the operators L_n preserve the constants. This property is satisfied by most classical sequences of linear operators (see, e.g., [3]), among them being the ones that we study in the present paper.

Moreover, for $i = 1, 2, 4$, identity (2.2) becomes, respectively,

$$\begin{aligned}
&\lim_{n\to+\infty} \lambda_n L_n e^x_{\varphi,1}(x) = p(x)\varphi''(x) + q(x)\varphi'(x),\\
&\lim_{n\to+\infty} \lambda_n L_n e^x_{\varphi,2}(x) = 2p(x)\varphi'(x)^2,\\
&\lim_{n\to+\infty} \lambda_n L_n e^x_{\varphi,4}(x) = 0.
\end{aligned} \tag{2.4}$$

These are, actually, the identities that we will explicitly use throughout the paper. However, we have decided to write the hypotheses of the theorem as in (2.2) for the sake of brevity and to put across that we can think of the result as if it were of Korovkin type, since we can guarantee the convergence of $\lambda_n(L_n f(x) - f(x))$ and obtain its limit for any $f \in C[0,1]$, whenever we have it for four test functions.

Proof. The classical Taylor theorem, applied to the function $f \circ \varphi^{-1}$, yields for $t \in [0,1]$ that

$$\begin{aligned}
f(t) = \left(f\circ\varphi^{-1}\right)(\varphi(t)) &= \left(f\circ\varphi^{-1}\right)(\varphi(x)) + D^1\left(f\circ\varphi^{-1}\right)(\varphi(x))(\varphi(t)-\varphi(x))\\
&+ \frac{D^2(f\circ\varphi^{-1})(\varphi(x))}{2}(\varphi(t)-\varphi(x))^2 + h(t-x)(\varphi(t)-\varphi(x))^2,
\end{aligned} \tag{2.5}$$

where h is a continuous function which vanishes at 0. Equivalently we can write for $t \in [0,1]$

$$f(t) = f(x) + D^1\left(f\circ\varphi^{-1}\right)(\varphi(x))e^x_{\varphi,1}(t) + \frac{D^2(f\circ\varphi^{-1})(\varphi(x))}{2}e^x_{\varphi,2}(t) + h_x(t)e^x_{\varphi,2}(t), \tag{2.6}$$

where $h_x(t) := h(t-x)$. Applying the operator L_n and then evaluating at the fixed point x, we obtain the equality

$$\begin{aligned}
L_n f(x) = f(x)L_n e^x_{\varphi,0}(x) &+ D^1\left(f\circ\varphi^{-1}\right)(\varphi(x))L_n e^x_{\varphi,1}(x) + \frac{D^2(f\circ\varphi^{-1})(\varphi(x))}{2}L_n e^x_{\varphi,2}(x)\\
&+ L_n\left(h_x e^x_{\varphi,2}\right)(x).
\end{aligned} \tag{2.7}$$

Now we subtract $f(x)$ from both sides and multiply by λ_n to get

$$\begin{aligned}
\lambda_n(L_n f(x) - f(x)) - \lambda_n L_n\left(h_x e^x_{\varphi,2}\right)(x) = f(x)\lambda_n&\left(L_n e^x_{\varphi,0}(x) - 1\right)\\
&+ D^1\left(f\circ\varphi^{-1}\right)(\varphi(x))\lambda_n L_n e^x_{\varphi,1}(x)\\
&+ \frac{D^2(f\circ\varphi^{-1})(\varphi(x))}{2}\lambda_n L_n e^x_{\varphi,2}(x).
\end{aligned} \tag{2.8}$$

Taking into account the basic identities

$$D^1\left(f\circ\varphi^{-1}\right)(\varphi(x)) = \frac{f'(x)}{\varphi'(x)},$$
$$D^2\left(f\circ\varphi^{-1}\right)(\varphi(x)) = \frac{1}{\varphi'(x)}\left(\frac{f''(x)\varphi'(x)-f'(x)\varphi''(x)}{(\varphi'(x))^2}\right) \tag{2.9}$$

and using (2.4), we derive that

$$\lim_{n\to+\infty}\left(\lambda_n(L_nf(x)-f(x))-\lambda_nL_n\left(h_xe^x_{\varphi,2}\right)(x)\right) = p(x)f''(x)+q(x)f'(x). \tag{2.10}$$

The proof will be over once we prove that $\lambda_nL_n(h_xe^x_{\varphi,2})(x)\to 0$ as $n\to+\infty$.

To this purpose let $\epsilon>0$ and let θ_x be an open set containing x such that for $t\in\theta_x$, $|h_x(t)|<\epsilon$. Then if we define $w(t):=\max\{0,|h_x(t)|-\epsilon\}e^x_{\varphi,2}(t)$, we have that for all $t\in[0,1]$

$$\left|h_x(t)e^x_{\varphi,2}(t)\right| \le \epsilon e^x_{\varphi,2}(t)+\max\{0,|h_x(t)|-\epsilon\}e^x_{\varphi,2}(t) = \epsilon e^x_{\varphi,2}(t)+w(t). \tag{2.11}$$

On the other hand, w vanishes on θ_x, so there is a constant M such that for all $t\in[0,1]$,

$$|w(t)|\le Me^x_{\varphi,4}(t). \tag{2.12}$$

Finally, the linearity and positivity of L_n allow us to write, from (2.11) and (2.12),

$$\left|\lambda_nL_n\left(h_xe^x_{\varphi,2}\right)(x)\right| \le \epsilon\lambda_nL_ne^x_{\varphi,2}(x)+\lambda_nML_ne^x_{\varphi,4}(x), \tag{2.13}$$

from where taking limits and using again (2.4),

$$\limsup_{n\to+\infty}\left|\lambda_nL_n\left(h_xe^x_{\varphi,2}\right)(x)\right| \le 2\epsilon p(x)\varphi'(x)^2. \tag{2.14}$$

This ends the proof, as $p(x)$ and $\varphi'(x)$ are strictly positive and ϵ was arbitrary. □

3. Applications

The first two applications correspond to sequences of operators recently introduced in [4, 5]. They represent respective modifications of the classical Bernstein operators and the well-known modified Meyer-König and Zeller operators (see [6]) which, instead of preserving the linear functions, hold fixed e_0 and e_2. Thus they are inside this new line of work which originated with the paper [7] and found further development in a long list of papers (see, e.g., [8–14]).

The aforementioned preserving property usually makes it quite difficult to compute the first moments and consequently to obtain asymptotic formulae. Here our result, applied with $\lambda_n=n$ and $\varphi=e_2$, enters the scene.

Two further applications with different values of φ are presented afterwards in less detail.

3.1. Modified Bernstein Operators Which Preserve x^2

This section deals with the following sequence of operators presented in [4] (we use the same notation) as a byproduct of some interesting results, defined for $f \in C[0,1]$ and $n > 1$ as

$$B_{n,0,2}f(x) = \sum_{k=0}^{n} f\left(\sqrt{\frac{k(k-1)}{n(n-1)}}\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{3.1}$$

As pointed out in [13], this provides an example of a sequence of positive linear polynomial operators that preserve e_0 and e_2 and represents an approximation process for functions $f \in C[0,1]$.

The presence of the square root in the definition makes it difficult to obtain easy-to-handle expressions for the moments $B_{n,0,2}e_i^x(x)$, $i = 0,1,2,4$, and consequently to obtain an asymptotic formula. We will apply our theorem to get it, though we first prove a quantitative result missed in [4].

Proposition 3.1. *Let $f \in C[0,1]$, $x \in [0,1]$, and let $\delta > 0$. Then*

$$\left|B_{n,0,2}f(x) - f(x)\right| \le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{2x(1-x)\frac{1-(1-x)^{n-1}}{n-1}}\right). \tag{3.2}$$

Proof. From the usual quantitative estimate in terms of $\omega(f,\cdot)$ stated in [15], we can write

$$\left|B_{n,0,2}f(x) - f(x)\right| \le \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{2x(x - B_{n,0,2}e_1(x))}\right). \tag{3.3}$$

Now, for $n > 1$, $(k-1)/(n-1) \le k/n$, so $\sqrt{k(k-1)/n(n-1)} \le k/n$ and $B_{n,0,2}e_1 \le B_n e_1 = e_1$. Thus

$$x - B_{n,0,2}e_1(x) \ge 0. \tag{3.4}$$

On the other hand, for $n > 1$, $(1/2)(k/n+(k-1)/(n-1)) \ge \sqrt{k(k-1)/n(n-1)} \ge (k-1)/(n-1)$, so

$$\frac{1}{2}\sum_{k=1}^{n}\left(\frac{k}{n} + \frac{k-1}{n-1}\right)\binom{n}{k}x^k(1-x)^{n-k} \ge B_{n,0,2}e_1(x) \ge \sum_{k=1}^{n}\frac{k-1}{n-1}\binom{n}{k}x^k(1-x)^{n-k}, \tag{3.5}$$

and using the equalities

$$\sum_{k=1}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k}=x,$$
$$\sum_{k=1}^{n}\frac{k-1}{n-1}\binom{n}{k}x^k(1-x)^{n-k}=\frac{n}{n-1}\sum_{k=1}^{n}\left(\frac{k}{n}-\frac{1}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}=\frac{nx-1}{n-1}+\frac{(1-x)^n}{n-1}, \tag{3.6}$$

we get

$$\frac{1-x}{2(n-1)}\left(1-(1-x)^{n-1}\right)\le x-B_{n,0,2}e_1(x)\le\frac{1-x}{n-1}\left(1-(1-x)^{n-1}\right), \tag{3.7}$$

from which the result follows. □

Corollary 3.2. *For all $f\in C[0,1]$ and $x\in(0,1)$, whenever $f''(x)$ exists,*

$$\lim_{n\to\infty}n\left(B_{n,0,2}f(x)-f(x)\right)=-\frac{1-x}{2}f'(x)+\frac{(1-x)x}{2}f''(x). \tag{3.8}$$

Proof. We will apply the theorem with $\varphi=e_2$, $\lambda_n=n$, $q(x)=-(1-x)/2$, and $p(x)=(1-x)x/2$. As the operators $B_{n,0,2}$ preserve the constants, it suffices to check that (2.4) holds true.

First computations yield

$$\begin{aligned}
B_{n,0,2}e_0(x)&=1,\\
B_{n,0,2}e_2(x)&=x^2,\\
B_{n,0,2}e_4(x)&=\frac{2}{n(n-1)}x^2+\frac{4(n-2)}{n(n-1)}x^3+\frac{(n-3)(n-2)}{n(n-1)}x^4,\\
B_{n,0,2}e_6(x)&=\frac{4}{n^2(n-1)^2}x^2+\frac{32(n-2)}{n^2(n-1)^2}x^3+\frac{38(n-3)(n-2)}{n^2(n-1)^2}x^4\\
&\quad+\frac{12(n-4)(n-3)(n-2)}{n^2(n-1)^2}x^5+\frac{(n-5)(n-4)(n-3)(n-2)}{n^2(n-1)^2}x^6,\\
B_{n,0,2}e_8(x)&=\frac{8}{n^3(n-1)^3}x^2+\frac{208(n-2)}{n^3(n-1)^3}x^3+\frac{652(n-2)(n-3)}{n^3(n-1)^3}x^4\\
&\quad+\frac{576(n-4)(n-3)(n-2)}{n^3(n-1)^3}x^5+\frac{188(n-5)(n-4)(n-3)(n-2)}{n^3(n-1)^3}x^6\\
&\quad+\frac{24(n-6)(n-5)(n-4)\ (n-3)(n-2)}{n^3(n-1)^3}x^7\\
&\quad+\frac{(n-7)(n-6)(n-5)(n-4)(n-3)(n-2)}{n^3(n-1)^3}x^8.
\end{aligned} \tag{3.9}$$

Thus for $i = 1, 2, 4$ the quantities $B_{n,0,2}e^x_{\varphi,i}$ appear after some calculations using the following identities:

$$
\begin{aligned}
e^x_{\varphi,1} &= e^x_{e_2,1} = e_2 - x^2 e_0, \\
e^x_{\varphi,2} &= e^x_{e_2,2} = e_4 - 2x^2 e_2 + x^4 e_0, \\
e^x_{\varphi,4} &= e^x_{e_2,4} = e_8 - 4x^2 e_6 + 6x^4 e_4 - 4x^6 e_2 + x^8 e_0, \\
B_{n,0,2}e^x_{\varphi,1}(x) &= 0, \\
B_{n,0,2}e^x_{\varphi,2}(x) &= \frac{2(1-x)x^2}{(n-1)n}(1 - 3x + 2nx), \\
B_{n,0,2}e^x_{\varphi,4}(x) &= \frac{4(1-x)x^2}{(n-1)^3 n^3}\Big(n^4(12-12x)x^4 + n^3\left(112 - 324x + 216x^2\right)x^3 \\
&\quad + n^2\left(159 - 1041x + 1881x^2 - 1011x^3\right)x^2 \\
&\quad + n\left(52 - 759x + 2921x^2 - 4089x^3 + 1887x^4\right)x \\
&\quad + \left(2 - 102x + 876x^2 - 2580x^3 + 3060x^4 - 1260x^5\right)\Big).
\end{aligned}
\tag{3.10}
$$

Finally we are in a position to apply Theorem 2.1 and then prove the corollary, since we show below that assumptions in (2.4) are fulfilled:

$$
\begin{aligned}
&\lim_{n\to\infty} nB_{n,0,2}e^x_{\varphi,1}(x) = 0 = 2xq(x) + 2p(x) = q(x)\varphi'(x) + p(x)\varphi''(x), \\
&\lim_{n\to\infty} nB_{n,0,2}e^x_{\varphi,2}(x) = 4(1-x)x^3 = 8x^2 p(x) = 2p(x)\varphi'(x)^2, \\
&\lim_{n\to\infty} nB_{n,0,2}e^x_{\varphi,4}(x) = 0.
\end{aligned}
\tag{3.11}
$$

□

3.2. The Modified Meyer-König and Zeller Operators

For $f \in C[0,1]$, $x \in [0,1]$, and $n > 1$, we consider the operators defined as

$$
R_n f(x) = \sum_{k=0}^{\infty} f\left(\sqrt{\frac{k(k-1)}{(n+k)(n+k-1)}}\right)\binom{n+k}{k} x^k (1-x)^{n+1}, \tag{3.12}
$$

for $x < 1$ and $R_n f(1) = f(1)$.

They were introduced in [5] as a modification of the well-known modified Meyer-König and Zeller operators (see [6]).

It turns to be another example of a sequence of positive linear operators that preserve e_0 and e_2 and represents an approximation process for functions $f \in C[0,1]$.

Here again, the presence of the square root in the definition makes it difficult to obtain easy-to-handle expressions for $R_n e_i^x(x)$, $i = 0,1,2,4$, and consequently to obtain by usual means an asymptotic formula. With this aim we shall apply our theorem.

Corollary 3.3. *For all $f \in C[0,1]$ and $x \in (0,1)$, whenever $f''(x)$ exists,*

$$\lim_{n\to\infty} n\left(R_n f(x) - f(x)\right) = -\frac{(1-x)^2}{2} f'(x) + \frac{x(1-x)^2}{2} f''(x). \tag{3.13}$$

Proof. We will apply the theorem with $\varphi = e_2$, $\lambda_n = n$, $q(x) = -(1-x)^2/2$ and $p(x) = x(1-x)^2/2$. As the operators R_n preserve the constants, it suffices to check that (2.4) holds true.

Direct computations with the use of mathematical software (Mathematica) give

$$\begin{aligned}
R_n e_0(x) &= 1,\\
R_n e_2(x) &= x^2,\\
R_n e_4(x) &= \frac{4nx^3}{(2+n)(3+n)} + \frac{n^2(1+n)x^4}{(2+n)(3+n)(4+n)}\\
&\quad + \frac{4n^3x^5}{(2+n)(3+n)(4+n)(5+n)} + \mathcal{O}\left(n^{-2}\right),\\
R_n e_6(x) &= \frac{12n^8x^5}{(1+n)(2+n)^2(3+n)^2(4+n)^2(5+n)^2}\\
&\quad + \frac{n^{10}(17+n)x^6}{(1+n)(2+n)^2(3+n)^2(4+n)^2(5+n)^2(6+n)^2}\\
&\quad + \frac{12n^{12}x^7}{(1+n)(2+n)^2(3+n)^2(4+n)^2(5+n)^2(6+n)^2(7+n)^2} + \mathcal{O}\left(n^{-2}\right),\\
R_n e_8(x) &= \frac{24n^{19}x^7}{(1+n)^2(2+n)^3(3+n)^3(4+n)^3(5+n)^3(6+n)^3(7+n)^3}\\
&\quad + \frac{n^{22}(59+n)x^8}{(1+n)^2(2+n)^3(3+n)^3(4+n)^3(5+n)^3(6+n)^3(7+n)^3(8+n)^3}\\
&\quad + \frac{24n^{25}x^9}{(1+n)^2(2+n)^3(3+n)^3(4+n)^3(5+n)^3(6+n)^3(7+n)^3(8+n)^3(9+n)^3} + \mathcal{O}\left(n^{-2}\right).
\end{aligned} \tag{3.14}$$

Now, using again (3.10) we can write

$$\begin{aligned}
R_n e_{\varphi,1}^x(x) &= 0,\\
R_n e_{\varphi,2}^x(x) &= \frac{4x^3n^3(1-x)^2}{(2+n)(3+n)(4+n)(5+n)} + \mathcal{O}\left(n^{-2}\right),
\end{aligned}$$

$$\begin{aligned}R_n e_{\varphi,4}^x(x) = {} & \frac{n^{25}x^7(24+59x+24x^2)}{(1+n)^2(2+n)^3(3+n)^3(4+n)^3(5+n)^3(6+n)^3(7+n)^3(8+n)^3(9+n)^3} \\ & + \frac{3n^{22}x^8}{(1+n)^2(2+n)^3(3+n)^3(4+n)^3(5+n)^3(6+n)^3(7+n)^3(8+n)^3} \\ & - \frac{4n^{12}x^7(12+17x+12x^2)}{(1+n)(2+n)^2(3+n)^2(4+n)^2(5+n)^2(6+n)^2(7+n)^2} \\ & + \frac{6n^3x^7(4+x+4x^2)}{(2+n)(3+n)(4+n)(5+n)} + \mathcal{O}\left(n^{-2}\right)\end{aligned} \tag{3.15}$$

and then the following identities which prove the corollary:

$$\begin{aligned}&\lim_{n\to\infty} nR_n e_{\varphi,1}^x(x) = 0 = 2xq(x) + 2p(x) = q(x)\varphi'(x) + p(x)\varphi''(x), \\ &\lim_{n\to\infty} nR_n e_{\varphi,2}^x(x) = 4x^3(1-x)^2 = 8x^2p(x) = 2p(x)\varphi'(x)^2, \\ &\lim_{n\to\infty} nM_{n,4} e_{\varphi,4}^x(x) = 0.\end{aligned} \tag{3.16}$$

□

3.3. The Modified Bernstein Operators Which Preserve a General Function τ

Let τ be any function fulfilling the same properties as the general function φ considered in the paper. For $f \in C[0,1]$, $x \in [0,1]$, and $n > 0$, we consider the following operators defined from the classical Bernstein operators B_n:

$$B_n^\tau f(x) = \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right). \tag{3.17}$$

This sequence of linear operators was studied by the authors in [13]. Here we show a nice way to obtain its asymptotic formula.

Corollary 3.4. *For all $f \in C[0,1]$ and $x \in (0,1)$, whenever $f''(x)$ exists,*

$$\begin{aligned}\lim_{n\to+\infty} n\left(B_n^\tau f(x) - f(x)\right) = {} & -\frac{\tau(x)(1-\tau(x))\tau''(x)}{2\tau'(x)^3} f'(x) \\ & + \frac{\tau(x)(1-\tau(x))}{2\tau'(x)^2} f''(x).\end{aligned} \tag{3.18}$$

Proof. It follows the same pattern as the proof of the previous corollaries. It suffices to make use of the following identities which one can obtain directly from the corresponding ones for the Bernstein operators (i.e., $B_n e_i^x(x),\ i = 1, 2, 4$; see, e.g., [2]):

$$\begin{aligned} B_n^\tau e_{\tau,1}^x(x) &= 0, \\ B_n^\tau e_{\tau,2}^x(x) &= \frac{\tau(x)(1-\tau(x))}{n}, \\ B_n^\tau e_{\tau,4}^x(x) &= \frac{3(1-\tau(x))^2\tau(x)^2}{n^2} \\ &\quad + \frac{\tau(x)(1-\tau(x))\left(1-6\tau(x)+6\tau(x)^2\right)}{n^3}. \end{aligned} \tag{3.19}$$

□

3.4. The Modified Bernstein Operators Which Preserve x^j

This section deals with the family of sequences of positive linear polynomial operators $B_{n,0,j},\ j = 1, 2, \ldots$, presented in [4] and defined for $f \in C[0,1]$ and $n \geq j$ as

$$B_{n,0,j} f(x) = \sum_{k=0}^{n} f\left(\left(\frac{k(k-1)\cdots(k-j+1)}{n(n-1)\cdots(n-j+1)}\right)^{1/j}\right)\binom{n}{k} x^k (1-x)^{n-k}. \tag{3.20}$$

The first two elements $B_{n,0,1}$ and $B_{n,0,2}$ are, respectively, the Bernstein operators and those ones studied in Section 3.1. The operator $B_{n,0,j}$ holds fixed the functions e_0 and e_j.

The next corollary deals with $B_{n,0,3}$ and shows an application of Theorem 2.1 with $\varphi = e_3$.

Corollary 3.5. *For all $f \in C[0,1]$ and $x \in (0,1)$, whenever $f''(x)$ exists,*

$$\lim_{n\to+\infty} n\left(B_{n,0,3} f(x) - f(x)\right) = \frac{x(1-x)}{2} f''(x) - (1-x) f'(x). \tag{3.21}$$

Proof. It follows the same pattern as Corollary 3.2 although some more cumbersome calculations, which we have carried out with the use of Mathematica, are required. We detail

below the identities that allow us to end the proof. We make use of the notation

$$
\begin{aligned}
(n-3)^{(i)} &= (n-3)(n-4)\cdots(n-i-2),\\
B_{n,0,3}e_0(x) &= 1,\\
B_{n,0,3}e_3(x) &= x^3,\\
B_{n,0,3}e_6(x) &= \frac{(n-3)^{(3)}}{n(n-1)(n-2)}x^6 + \frac{9(n-3)^{(2)}}{n(n-1)(n-2)}x^5 + \frac{18(n-3)}{n(n-1)(n-2)}x^4 + \frac{6}{n(n-1)(n-2)}x^3,\\
B_{n,0,3}e_9(x) &= \frac{(n-3)^{(6)}}{n^2(n-1)^2(n-2)^2}x^9 + \frac{27(n-3)^{(5)}}{n^2(n-1)^2(n-2)^2}x^8 + \frac{243(n-3)^{(4)}}{n^2(n-1)^2(n-2)^2}x^7\\
&\quad + \frac{862(n-3)^{(3)}}{n^2(n-1)^2(n-2)^2}x^6 + \frac{1242(n-3)^{(2)}}{n^2(n-1)^2(n-2)^2}x^5 + \frac{540(n-3)}{n^2(n-1)^2(n-2)^2}x^4\\
&\quad + \frac{36}{n^2(n-1)^2(n-2)^2}x^3,\\
B_{n,0,3}e_{12}(x) &= \frac{(n-3)^{(9)}}{n^3(n-1)^3(n-2)^3}x^{12} + \frac{54(n-3)^{(8)}}{n^3(n-1)^3(n-2)^3}x^{11} + \frac{1107(n-3)^{(7)}}{n^3(n-1)^3(n-2)^3}x^{10}\\
&\quad + \frac{11025(n-3)^{(6)}}{n^3(n-1)^3(n-2)^3}x^9 + \frac{56808(n-3)^{(5)}}{n^3(n-1)^3(n-2)^3}x^8 + \frac{149580(n-3)^{(4)}}{n^3(n-1)^3(n-2)^3}x^7\\
&\quad + \frac{186876(n-3)^{(3)}}{n^3(n-1)^3(n-2)^3}x^6 + \frac{94284(n-3)^{(2)}}{n^3(n-1)^3(n-2)^3}x^5 + \frac{13608(n-3)}{n^3(n-1)^3(n-2)^3}x^4\\
&\quad + \frac{216}{n^3(n-1)^3(n-2)^3}x^3.
\end{aligned}
\tag{3.22}
$$

For $i = 1, 2, 4$ the quantities $B_{n,0,3}e^x_{\varphi,i} = B_{n,0,3}e^x_{e_3,i}$, required to apply Theorem 2.1, appear after some calculations using the following identities:

$$
\begin{aligned}
e^x_{\varphi,1} &= e^x_{e_3,1} = e_3 - x^3 e_0,\\
e^x_{\varphi,2} &= e^x_{e_3,2} = e_6 - 2x^3 e_3 + x^6 e_0,\\
e^x_{\varphi,4} &= e^x_{e_3,4} = e_{12} - 4x^3 e_9 + 6x^6 e_6 - 4x^9 e_3 + x^{12} e_0.
\end{aligned}
\tag{3.23}
$$

□

Finally, motivated by the well-known Voronovskaja formula for the classical Bernstein operators and by the results in Corollaries 3.2 and 3.5, we close this section and the paper stating the following conjecture.

Conjecture 3.6. *For all $f \in C[0,1]$, $x \in (0,1)$, and $j \in \{4, 5, \ldots\}$, whenever $f''(x)$ exists,*

$$
\lim_{n\to+\infty} n\left(B_{n,0,j}f(x) - f(x)\right) = \frac{x(1-x)}{2}f''(x) - (j-1)\frac{(1-x)}{2}f'(x). \tag{3.24}
$$

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References

[1] E. Voronovskaja, "Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de S. Bernstein," *Doklady Akademii Nauk SSSR*, pp. 79–85, 1932.
[2] R. A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, vol. 293 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1972.
[3] F. Altomare and M. Campiti, *Korovkin-Type Approximation Theory and Its Applications*, vol. 17, Walter de Gruyter, Berlin, Germany, 1994.
[4] J. M. Aldaz, O. Kounchev, and H. Render, "Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces," *Numerische Mathematik*, vol. 114, no. 1, pp. 1–25, 2009.
[5] M. A. Özarslan and O. Duman, "MKZ type operators providing a better estimation on (1/2, 3)," *Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques*, vol. 50, no. 3, pp. 434–439, 2007.
[6] E. W. Cheney and A. Sharma, "Bernstein power series," *Canadian Journal of Mathematics. Journal Canadien de Mathématiques*, vol. 16, pp. 241–252, 1964.
[7] J. P. King, "Positive linear operators which preserve x^2," *Acta Mathematica Hungarica*, vol. 99, no. 3, pp. 203–208, 2003.
[8] H. Gonska and P. Piţul, "Remarks on an article of J. P. King," *Commentationes Mathematicae Universitatis Carolinae*, vol. 46, no. 4, pp. 645–652, 2005.
[9] D. Cárdenas-Morales, P. Garrancho, and F. J. Muñoz-Delgado, "Shape preserving approximation by Bernstein-type operators which fix polynomials," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1615–1622, 2006.
[10] O. Agratini, "Linear operators that preserve some test functions," *International Journal of Mathematics and Mathematical Sciences*, Article ID 94136, 11 pages, 2006.
[11] O. Duman and M. A. Özarslan, "Szász-Mirakjan type operators providing a better error estimation," *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 20, no. 12, pp. 1184–1188, 2007.
[12] H. Gonska, P. Piţul, and I. Raşa, "General King-type operators," *Results in Mathematics*, vol. 53, no. 3-4, pp. 279–286, 2009.
[13] D. Cárdenas-Morales, P. Garrancho, and I. Raşa, "Bernstein-type operators which preserve polynomials," *Computers & Mathematics with Applications. An International Journal*, vol. 62, no. 1, pp. 158–163, 2011.
[14] Z. Finta, "Estimates for Bernstein type operators," *Mathematical Inequalities & Applications*, vol. 15, no. 1, pp. 127–135, 2012.
[15] O. Shisha and B. Mond, "The degree of convergence of sequences of linear positive operators," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 60, pp. 1196–1200, 1968.